ARCHIVUM MATHEMATICUM (BRNO)
Tomus 41 (2005), 37 – 58

STABILITY OF HYDRODYNAMIC MODEL FOR SEMICONDUCTOR

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Abstract. In this paper we study the stability of transonic strong shock solutions of the steady state one-dimensional unipolar hydrodynamic model for semiconductors in the isentropic case. The approach is based on the construction of a pseudo-local symmetrizer and on the paradifferential calculus with parameters, which combines the work of Bony-Meyer and the introduction of a large parameter.

Contents

2000 *Mathematics Subject Classification*: 35B35,82D37,35L50.
Key words and phrases: transonic shock waves, stability, hydrodynamic models, semiconductors.
Received April 1, 2003.

1. Introduction

This article is devoted to the study of strong transonic shock waves for the stationary solutions of the one-dimensional hydrodynamic model for unipolar semiconductors in the isentropic case. The analysis of these solutions is based on the construction of their orbits in the electron density-electric field phase plane and on the representation of discontinuous solutions by union of trajectory pieces. This approach is performed by U. M. Asher, P. A. Markowich, P. Pietra, C. Schmeiser in [1] and by P. A. Markowich in [17].

We will concentrate our attention to the transonic solutions since the study of the subsonic solutions is rather well understood and can be found for instance in [9], [14] and in [4]. Furthermore, for what about the proof of the existence of weak solutions to a hydrodynamic model for semiconductors and relaxation to drift-diffusion equation, it can be found in the papers [15], [16] of P. Marcati and R. Natalini.

Our goal is to study the stability of the transonic solutions with a single shock wave front Σ separating two states $U^+ = (\rho^+, u^+, E^+)^T$ and $U^- = (\rho^-, u^-, E^-)^T$, obeying the Rankine-Hugoniot jump conditions and the Lax entropy conditions. The assumption about the existence of only one shock is not too restrictive, since in [1] has been proved for steady states that every transonic solution has either exactly one shock or a jump at the boundary point β, which satisfies the condition

$$\lim_{x \to \beta^-} \rho(x) \leq J^2 / \,\overline{\rho}\,.$$

Furthermore in [17] has been proved that for an infinite current relaxation time τ only a countably many stationary transonic solutions are smooth for $x \in [0, \beta]$ and that all the others, which are infinite, have exactly one shock in $[0, \beta]$.

For the study of the stability we use the techniques developed by H. O. Kreiss [8], A. Majda [10], [11], [12], [13] and G. Métivier [18]. We will follow their works taking into account that our problem is not homogeneous.

We look for the algebraic conditions which guarantee the well posedness of the linearized equations in L^2. Simple computations show that the necessary and sufficient condition given in [11] to have a maximal L^2 estimate is satisfied. To obtain the maximal L^2 estimate we follow the analysis made by G. Métivier in [18]. When these conditions are satisfied, the linearized equations are also well posed in H^s, and, by using an iterative scheme, one can solve the nonlinear problem. We observe that these conditions are only sufficient for the existence of solutions of the nonlinear equations.

In the section 2, we begin giving the one-dimensional equations for a unipolar semiconductor and showing their properties. Then we pass in the subsection 2.2 to write down the jump conditions of Rankine-Hugoniot, the Lax entropy conditions and equivalent conditions introducing the flow velocity relative to the shock front u' and the Mach number M. We show also the properties of the electron current density J and find some explicit formulas that relate the two states U^+, U^- and the speed of propagation of the shock front σ. For these results we recover some computations found in [7]. Finally in the subsection 2.3 we study in some more details the stationary system for a semiconductor in a steady state.

In the section 3, we prove the stability of the transonic admissible shocks. To achieve this result the first canonical step is to introduce the linearized system which is obtained starting from the perturbed equations. There are two way to perturb the boundary. The former is to consider only a temporal perturbation of the boundary, as it is done in [11], the latter is to take a temporal-space perturbation of the boundary, as it is done in [18].

The second step in both cases is the derivation of the maximal L^2 estimates for the solutions of the resulting linearized boundary value problem, finally the stability follows from the "a priori" estimate as showed in [11] and in [18]. The main results are given in the Theorems 3.4, 3.6 and 3.7.

We shall give in details only the temporal-space perturbation case (as in [18]).

The principal tools that we will use to obtain the maximal L^2 estimate are the Kreiss' symmetrizers. In general one has pseudo-local symmetrizers, i.e. pseudo-differential operators which depend not only on x but also on the frequencies γ. For the construction of the symmetrizer we will follow that one found in [6].

The other important ingredient that we will use regards the paradifferential calculus of J. M. Bony [2], [3] and Y. Meyer [19]. In the appendix we summarized the results of this theory needed in our proofs.

2. One-dimensional equations for a unipolar semiconductor

The aim of this second section is to introduce the equations which govern a transonic state of current driven $n^+n\,n^+$ devices with the n-region of length $\alpha+\beta$, where $\alpha, \beta > 0$ are arbitrarily fixed constants.

We start providing in the first subsection the general equations for a smooth solution of the one-dimensional hydrodynamic model for a unipolar semiconductor in the isentropic case and observing that they represent a strictly hyperbolic and symmetrizable system.

Then in the second subsection we pass to analyze the shock equations. Beyond the internal equations, which are the canonical generalization of the general equations to the field of piecewise C^1 solutions, they include also the Rankine-Hugoniot jump conditions and the Lax entropy inequalities, which relate the values of an admissible shock on both sides of the shock front. Furthermore we write down simple explicit formulas which give u_L and σ as functions of the triple (u_R, ρ_L, ρ_R). Since the algebraic computations to obtain them are very simple and can be found in [7] we omit them.

Finally in the last subsection we analyze the stationary case specializing the results found before and also recalling a very interesting theorem given by U. M. Asher, P. A. Markowich, P. Pietra, C. Schmeiser in [1].

2.1. **One-dimensional equations for unipolar semiconductor devices.**

The one-dimensional isentropic model for a unipolar semiconductor device is given by the coupled Euler-Poisson system, where in the Poisson equation we denote by $d(x) \in C(-\alpha, \beta)$, $d(x) > 0$, the doping profile, namely the density of positively charged background ions. Therefore one has

$$
\begin{cases}
\rho_t + (\rho u)_x = 0 & -\alpha < x < \beta,\ t > 0\,, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x = \rho E - \frac{1}{\tau}\rho u & -\alpha < x < \beta, t > 0\,, \\
E_x = \rho - d & -\alpha < x < \beta,\ t > 0\,, \\
\rho(t,-\alpha) = \rho(t,\beta) = \overline{\rho}(t) & t > 0\,,
\end{cases} \tag{1}
$$

where $\rho = \rho(t,x) > 0$ is the electron density (namely there are no empty spaces), $u = u(t,x)$ is the electron velocity, $p(\rho) \in C^2(\overline{\mathbf{R}_+})$ is the pressure of the electron gas, which is such that $p''(\rho) > 0, p'(\rho) > 0$ for every $\rho > 0$ and $p(0) = 0$, $E = E(t,x)$ is the negative electric field generated by the Coulomb force of the particles, $\tau > 0$ is the current relaxation time and $\overline{\rho} > 1$ is the doping concentration in the n^+-regions. The first two equations of the system (1) describe respectively mass conservation and momentum balance, the third is the Poisson equation and determines the electric field.

Let us assume that $\rho, \rho u \in L^\infty_{loc}$, and that $E(t,-\alpha) = \overline{E}(t)$ for $t \in (0,+\infty)$; then the problem (1) is equivalent to

(2)

$$
\begin{cases}
\rho_t + (\rho u)_x = 0 & -\alpha<x<\beta,\ t>0\,, \\
(\rho u)_t+(\rho u^2+p(\rho))_x=\rho\Big(\overline{E}+\int_{-\alpha}^x(\rho(t,y) - d(y))dy - \frac{1}{\tau}u\Big) & -\alpha<x<\beta,\ t>0\,, \\
\rho(t,-\alpha) = \rho(t,\beta) = \overline{\rho}(t) & t>0\,, \\
E(t,-\alpha) = \overline{E}(t) & t>0\,.
\end{cases}
$$

Let us denote by

$$
V = \begin{pmatrix} \rho \\ u \end{pmatrix}\ ,\ f_0(V) = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}\ ,\ f_1(V) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \end{pmatrix}
$$

and

$$
F(V,x) = \left(0, \rho\left(\overline{E} + \int_{-\alpha}^x (\rho(t,y) - d(y))dy - \frac{1}{\tau}u\right)\right)^T,
$$

then the first two equations of system (2) can be written in the following conservative form

$$
f_0(V)_t + f_1(V)_x = F(V,x) \qquad -\alpha < x < \beta,\ t > 0\,, \tag{3}
$$

or in the nonconservative form

$$
A_0(V)\partial_t V + A_1(V)\partial_x V = F(V,x) \qquad -\alpha < x < \beta,\ t > 0\,, \tag{4}
$$

where $A_0(V) = \begin{pmatrix} 1 & 0 \\ u & \rho \end{pmatrix}$, $A_1(V) = \begin{pmatrix} u & \rho \\ u^2 + c^2 & 2\rho u \end{pmatrix}$ are the Jacobian matrices corresponding to f_0, f_1 respectively. Here $c = \sqrt{\frac{dp}{d\rho}(\rho)} > 0$ is the sound speed. We recall the following basic fact.

Proposition 2.1. *The system* (4) *is strictly hyperbolic and symmetrizable.*

2.2. **Shock equations.** We shall assume the flow is subsonic in the outer n^+-regions since it is the most relevant situation in practical situations. Since we shall concentrate only on solutions having a finite number of jumps, we shall restrict the class of our weak solutions to the piecewise smooth case. More precisely we consider weak solution in the sense of the following definition.

Definition 2.2. $V = (\rho, u)^T$ is a piecewise discontinuous admissible solution if there exists a finite number of smooth orientable lines Σ in the (x,t)-space, outside of which V is a C^1 solution and across which V has a jump discontinuity satisfying the conditions of Rankine-Hugoniot and the Lax entropy conditions.

Let $V = (\rho, u)^T$ be a piecewise discontinuous admissible solution with only one jump across the line Σ and assume that Σ has a parameterization of the form $(t, \varphi(t))$, where $\varphi : t \to \varphi(t)$ is a C^1 function such that $-\alpha < \varphi(t) < \beta$. The usual interpretation of $\sigma(t) = \frac{d\varphi}{dt}(t)$ is that it represents the speed of propagation of the shock front. We denote by V^- and by V^+ the restriction of V respectively to the left region and to the right region with respect to the line Σ, and by $V_{R,L}(t) = \lim_{\varepsilon\to 0^+} V^{\pm}(t \mp \sigma\varepsilon, x \pm \varepsilon) \in \mathbf{R}$, $(t,\, x = \varphi(t)) \in \Sigma$. The system resulting from (4) is

$$
(5) \qquad \begin{cases} A_0(V^-)V_t^- + A_1(V^-)V_x^- = F_-(V^-, x) & -\alpha < x < \varphi(t),\ t > 0\,, \\ A_0(V^+)V_t^+ + A_1(V^+)V_x^+ = F_+(V^-, V^+, x) & \varphi(t) < x < \beta,\ t > 0 \end{cases}
$$

and the Rankine-Hugoniot jump conditions writes

$$
(6) \qquad \sigma[f_0(V)] = [f_1(V)] \qquad x = \varphi(t),\ t > 0
$$

where the brackets denote the jump of any quantity g across the interface (namely $[g] = g_R - g_L$),

$$
F_-(V^-, x) = \Big(0, \rho^-\Big(\overline{E} + \int_{-\alpha}^{x} (\rho^-(t,y) - d(y))\, dy - \frac{1}{\tau}u^-\Big)\Big)^T,
$$

$$
\begin{aligned}
F_+(V^-, V^+, x) = \Big(0, \rho^+\Big(\overline{E} &+ \int_{-\alpha}^{\varphi(t)} (\rho^-(t,y) - d(y))\, dy \\
&+ \int_{\varphi(t)}^{x} (\rho^+(t,y) - d(y))\, dy - \frac{1}{\tau}u^+\Big)\Big)^T.
\end{aligned}
$$

Since the eigenvalue of $A_0^{-1}A_1(V)$ are $\lambda_1(V) = u - c$ and $\lambda_2(V) = u + c$, the Lax entropy conditions for the 1-shock are given by

$$
(7) \qquad \begin{cases} \sigma < u_L - c_L\,, \\ u_R - c_R < \sigma < u_R + c_R \end{cases}
$$

and for the 2-shock are given by

$$
(8) \qquad \begin{cases} u_L - c_L < \sigma < u_L + c_L\,, \\ u_R + c_R < \sigma\,. \end{cases}
$$

Remark. Note that the above conditions imply the shocks are noncharacteristic since the speed of propagation of the shock front σ is different from the characteristic speeds $\lambda_1(V)$, $\lambda_2(V)$ on both sides of the interface.

Let us introduce the flow velocity relative to the shock front $u' = u - \sigma$, then the Rankine-Hugoniot jump conditions can be written in the form

$$[f_1(\rho, u')] = 0\,. \tag{9}$$

Therefore the electron current density $J = \rho u'$ is in $C\big((0,\infty) \times (-\alpha, \beta)\big)$, furthermore algebraic computations show that on the discontinuity line Σ holds

$$J = -(p_R - p_L)/(u_R - u_L) \tag{10}$$

and that, denoted by $\nu = 1/\rho$ the specific electron volume, on Σ holds also

$$J = (u_R - u_L)/(\nu_R - \nu_L)\,, \tag{11}$$

$$J^2 = -(p_R - p_L)/(\nu_R - \nu_L)\,. \tag{12}$$

Remark 2.3. Physically relevant solutions must satisfy the Lax entropy conditions, hence in the case of steady shock analyzed by [1] this lead to restrict to 1-shock satisfying

$$\rho_L c_L < J < \rho_R c_R\,,$$

therefore in particular we get $J > 0$, as we shall see later in the subsection 2.3.

An immediate consequence of the foregoing remark is that $J > 0$ along the discontinuity line Σ, and therefore $u_R, u_L > \sigma$. Another consequence is that the condition (8) must be thrown away, and so we can only have a 1-shock. Finally from (7) and (9) it follows that

$$\begin{cases} \rho_R \neq \rho_L\,, \\ u_R \neq u_L\,. \end{cases} \tag{13}$$

Clearly an alternative form to write (7) using the Mach number $M = u'/c$ is

$$M_L > 1 > M_R > 0\,. \tag{14}$$

An elementary result is given in the following proposition.

Proposition 2.4. *On the discontinuity line Σ one has*

$$u_L = u_R + \operatorname{sign}\{\rho_R - \rho_L\} \sqrt{\frac{(\rho_R - \rho_L)(p_R - p_L)}{\rho_R \rho_L}}\,, \tag{15}$$

$$\sigma = u_R - \sqrt{\frac{\rho_L(p_R - p_L)}{\rho_R(\rho_R - \rho_L)}}\,. \tag{16}$$

2.3. **The stationary case.** If the semiconductor is in a steady state then all its parameters are independent of time and the equations become

$$
\begin{cases}
(\rho u)_x = 0 & -\alpha < x < \beta\,,\\
(\rho u^2 + p(\rho))_x = \rho E - \frac{1}{\tau}\rho u & -\alpha < x < \beta\,,\\
E_x = \rho - d & -\alpha < x < \beta\,,\\
\rho(-\alpha) = \rho(\beta) = \overline{\rho}\,.
\end{cases}
\tag{17}
$$

Clearly the electron current density J is a constant greater than zero and so also the velocity u is positive. Furthermore the above system can be written in the form

$$
\begin{cases}
\left(\frac{1}{\rho}J^2 + p(\rho)\right)_x = \rho E - \frac{1}{\tau}J & -\alpha < x < \beta\,,\\
E_x = \rho - d & -\alpha < x < \beta\,,\\
\rho(-\alpha) = \rho(\beta) = \overline{\rho}\,.
\end{cases}
\tag{18}
$$

If $U = (\rho, u = J/\rho, E)^T$ is a stationary transonic admissible shock, then it is not limitative to assume that $\Sigma = \{\, x = 0 \,\} \subset \mathbf{R}^2_+$ is his discontinuity line.
The corresponding Rankine-Hugoniot jump conditions are

$$
\left[\frac{1}{\rho}J^2 + p(\rho)\right] = 0\,,
\tag{19}
$$

and the stationary Lax entropy inequalities for the 1-shock are

$$
\begin{cases}
u_L > c_L > 0\,,\\
|u_R| < c_R\,,
\end{cases}
\tag{20}
$$

which can be equivalently written in the form

$$
c_L\rho_L < J < c_R\rho_R\,.
\tag{21}
$$

For the proposition 2.4 with $\sigma = 0$

$$
u_R = \sqrt{\frac{\rho_L(p_R - p_L)}{\rho_R(\rho_R - \rho_L)}}\,, \quad u_L = \sqrt{\frac{\rho_R(p_R - p_L)}{\rho_L(\rho_R - \rho_L)}}\,.
\tag{22}
$$

Further in [1] it can be found the proof of the following important theorem.

Theorem 2.5. *If $\overline{\rho} > J > 1$ and the pressure is $p(\rho) = \rho$, namely the gas is polytrophic, then there exists a set S of piecewise discontinuous admissible solutions $(\rho, E, \alpha + \beta)$ for the isentropic stationary boundary value problem*

$$
\begin{cases}
\rho_x\left(1 - \frac{1}{\rho^2}J^2\right) = \rho E - \frac{1}{\tau}J & -\alpha < x < \beta\,,\\
E_x = \rho - 1 & -\alpha < x < \beta\,,\\
\rho(-\alpha) = \rho(\beta) = \overline{\rho}\,,
\end{cases}
\tag{23}
$$

which contains a solution (ρ, E) *for every* $\alpha + \beta > 0$. *Furthermore if* $S^{\alpha,\beta}$ *is the set of rescaled-translate solutions* $(\rho^{\alpha,\beta}, E^{\alpha,\beta}, \alpha, \beta)$ *defined by*

$$(\rho^{\alpha,\beta}(y), E^{\alpha,\beta}(y)) = (\rho((\alpha+\beta)\, y - \alpha), E((\alpha+\beta)\, y - \alpha))\ , \quad 0 \le y \le 1\ , \tag{24}$$

then it forms a continuum subset of $L^q(0,1) \times C[0,1) \times \mathbf{R}_+ \times \mathbf{R}_+$ *for every* $1 \le q < \infty$. *Finally every transonic solution* $\rho^{\alpha,\beta}$ *of the set* $S^{\alpha,\beta}$ *has either exactly one shock in* $[0,1]$ *or a jump at the boundary satisfying*

$$\lim_{x \to 1^-} \rho^{\alpha,\beta}(x) \le \frac{1}{\overline{\rho}} J^2\ . \tag{25}$$

3. Stability of the stationary solutions

To study the stability of an admissible transonic shock V solution of the problem (5) the first difficulty we meet is due to the fact that the discontinuity line $\Sigma = \{x = \varphi(t)\}$ is unknown; for this reason we must add the jump conditions (6) to the system (5), obtaining

$$\begin{cases} A_0(V^-)V_t^- + A_1(V^-)V_x^- = F_-(V^-, x) & -\alpha < x < \varphi(t)\ ,\ t > 0\ , \\ A_0(V^+)V_t^+ + A_1(V^+)V_x^+ = F_+(V^-, V^+, x) & \varphi(t) < x < \beta\ ,\ t > 0\ , \\ \sigma[f_0(V)] = [f_1(V)] & \text{on } \ x = \varphi(t), t > 0\ . \end{cases} \tag{26}$$

The second step is to introduce the linearized system which is obtained starting from the perturbed equations for $(V_\varepsilon, \psi_\varepsilon) = (V + \varepsilon W', \varphi - \varepsilon\phi')$. This will be done in the subsection 3.1.

The third step is to prove a maximal L^2 estimate for the solutions of the linearized boundary value problem. The principal tool that we will use to obtain the maximal L^2 estimate is the Kreiss' symmetrizer. This will be the object of the subsection 3.2. Finally, since the stability follows from the apriori estimate, the aim of the section is achieved.

We will study only the stability of stationary admissible transonic shocks solution of the boundary value problem (18). Let us remark that it is always satisfied the necessary and sufficient condition given in [11] (see Proposition 3.1 on page 26) to have a maximal L^2 estimate under our hypotheses.

We have to warn the reader that many computations will be not reproduced here to avoid overloading the section, but we shall often refer to the appendix sections or to previous works where some details can be found.

3.1. **Linearization of the equations around the stationary solution.**

In this subsection we introduce the linearized system resulting form the perturbation of the system (26).

Let us consider $(V^+(x), V^-(x), \varphi \equiv 0)$ the stationary admissible transonic shock studied in the subsection 2.3 and let us suppose that $V^- \in W^{1,1}(-\alpha, 0)$, $V^+ \in W^{1,1}(0,\beta)$. If furthermore $d \in W^{1,1}(-\alpha,\beta)$, then we can extend V and d to all $\mathbf{R}$ in such way that the support of V and d is a subset of $(-\alpha-\beta, \alpha+\beta)$ and there exists a constant $C > 0$ such that $\|V\|_{W^{1,1}(\mathbf{R})} \le C\|V\|_{W^{1,1}(-\alpha,\beta)}$

and $\|d\|_{W^{1,1}(\mathbf{R})} \leq C\|d\|_{W^{1,1}(-\alpha,\beta)}$. Now, to have that the stationary admissible transonic shock is stable, we have to require that for any perturbation $\left(W'^{+}, W'^{-}, \phi'\right) \in C_0^\infty(\{(t,x) \in \mathbf{R}_+ \times (-\alpha,\beta)\})$ it results that $V_\varepsilon^\pm = V^\pm + \varepsilon W'^{\pm}$ and $\varepsilon\phi'$ satisfy, at least for $\varepsilon \in \mathbf{R}$ sufficiently close to zero and at least until to a time $T > 0$ independent from ε, the system

(27)
$$\begin{cases} A_0(V_\varepsilon^-)\partial_t V_\varepsilon^- + A_1(V_\varepsilon^-)\partial_x V_\varepsilon^- = F_-(V_\varepsilon^-, x) & \text{in } -\alpha - \varepsilon\phi'(t,x) < x < -\varepsilon\phi'(t,x),\ t>0\,, \\ A_0(V_\varepsilon^+)\partial_t V_\varepsilon^+ + A_1(V_\varepsilon^+)\partial_x V_\varepsilon^+ = F_+(V_\varepsilon^-, V_\varepsilon^+, x) & \text{in } -\varepsilon\phi'(t,x) < x < \beta - \varepsilon\phi'(t,x),\ t>0\,, \\ \varepsilon\partial_t\phi'[f_0(V_\varepsilon)] + (1+\varepsilon\partial_x\phi')[f_1(V_\varepsilon)] = 0 & \text{on } x = -\varepsilon\phi'(t,x),\ t > 0\,, \end{cases}$$

where

$$F_-(V_\varepsilon^-, x) = \Big(0, \rho_\varepsilon^-(t,x)\Big(\overline{E} + \int_{\phi_\varepsilon(t,\,.\,)^{-1}(-\alpha)}^{x} (\rho_\varepsilon^-(t,y) \ - d(y))dy - \frac{1}{\tau}u_\varepsilon^-(t,x)\Big)\Big)^T,$$

$$\begin{aligned} F_+(V_\varepsilon^- V_\varepsilon^+, x) &= \Big(0, \rho_\varepsilon^+(t,x)\Big(\overline{E} + \int_{\phi_\varepsilon(t,\,.\,)^{-1}(-\alpha)}^{\phi_\varepsilon(t,\,.\,)^{-1}(0)} (\rho_\varepsilon^-(t,y) - d(y))dy \\ &\qquad + \int_{\phi_\varepsilon(t,\,.\,)^{-1}(0)}^{x} (\rho_\varepsilon^+(t,y) - d(y))dy - \frac{1}{\tau}u_\varepsilon^+(t,x)\Big)\Big)^T. \end{aligned}$$

Here we have introduced the function $\phi_\varepsilon(t,x) = x + \varepsilon\phi'(t,x)$.
In the Appendix A we show the computations to obtain the linearized problem, which is given in the following proposition.

Proposition 3.1. *The corresponding linearized equations are*

(28)
$$\begin{cases} A_0(\tilde V^-)\partial_{\tilde t}(\tilde W'^{-} + \phi'\partial_{\tilde x}\tilde V^-) + A_1(\tilde V^-)\partial_{\tilde x}(\tilde W'^{-} + \phi'\partial_{\tilde x}\tilde V^-) = & \\ \quad = \tilde C(\tilde V^-, \tilde x)\tilde W'^{-} + \phi' D(\tilde V^-) + \tilde\rho^- \int_{-\alpha}^{\tilde x}(0, \tilde\rho'^{-}(\tilde t, y) - \phi'_{\tilde x}(\tilde t, y)(\tilde\rho^- - d)(y))^T dy & \textit{in} \quad -\alpha < \tilde x < 0, \tilde t > 0\,, \\ A_0(\tilde V^+)\partial_{\tilde t}(\tilde W'^{+} + \phi'\partial_{\tilde x}\tilde V^+) + A_1(\tilde V^+)\partial_{\tilde x}(\tilde W'^{+} + \phi'\partial_{\tilde x}\tilde V^+) = & \\ \quad = \tilde C(\tilde V^+, \tilde x)\tilde W'^{+} + \phi' D(\tilde V^+) + \tilde\rho^+ \int_{-\alpha}^{\tilde x}(0, \tilde\rho'(\tilde t, y) - \phi'_{\tilde x}(\tilde t, y)(\tilde\rho - d)(y))^T dy & \textit{in} \quad 0 < \tilde x < \beta, \tilde t > 0\,, \\ \partial_{\tilde t}\phi'[f_0(\tilde V)] + [A_1(\tilde V)\tilde W'] = 0 & \textit{on} \quad \tilde x = 0\,, \end{cases}$$

where

$$\tilde V(\tilde t, \tilde x) = V(\tilde t, \phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\,, \quad \tilde W'(\tilde t, \tilde x) = W'(\tilde t, \phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\,,$$

$$\tilde C(\tilde V, \tilde x) := \begin{pmatrix} 0 & 0 \\ \overline{E} + \int_{-\alpha}^{\tilde x}(\tilde\rho - d)(y)\,dy - \frac{1}{\tau}\tilde u & -\frac{1}{\tau}\tilde\rho \end{pmatrix} - \partial_{\tilde x}A_1(\tilde V)$$

and

$$D(\tilde V) := A_1(\tilde V)\partial_{\tilde x}^2\tilde V\,.$$

Let us introduce two constants $\alpha' \in (0,\alpha)$ and $\beta' \in (0,\beta)$ such that $\tilde V$ is supersonic in $\{ -\alpha' \leq \tilde x < 0 \}$ and subsonic in $\{ 0 < \tilde x \leq \beta' \}$, then to study the stability of the shock it is sufficient to consider the linear equations on the strip $\{ -\alpha' < \tilde x < \beta', \tilde t > 0 \}$. Furthermore it is convenient to change the above transmission problem to a boundary value problem in a half-space. This can be done through the changing coordinates $(\tilde t, \tilde x) \mapsto (t,x) = (\tilde t, -\tilde x/\alpha')$ in the superficial region $\{ \tilde x < 0 \}$, and $(\tilde t, \tilde x) \mapsto (t,x) = (\tilde t, \tilde x/\beta')$ in the superficial region $\{ \tilde x > 0 \}$. In fact, taking

$$V = \begin{pmatrix} V^+/\beta' \\ -V^-/\alpha' \end{pmatrix}, \qquad W' = \begin{pmatrix} W'^+ \\ W'^- \end{pmatrix},$$

$$\mathcal{A}_0(V) = \begin{pmatrix} A_0(V^+) & 0 \\ 0 & A_0(V^-) \end{pmatrix}, \qquad \mathcal{A}_1(V) = \begin{pmatrix} \frac{1}{\beta'} A_1(V^+) & 0 \\ 0 & -\frac{1}{\alpha'} A_1(V^-) \end{pmatrix},$$

$$\mathcal{C}(V,x) = \begin{pmatrix} C_+(V,\, x) & 0 \\ 0 & C_-(V^-,\, x) \end{pmatrix},$$

where

$$C_+(V,x) = \begin{pmatrix} 0 & 0 \\ \overline{E} + \alpha' \int_0^{\alpha/\alpha'} (\rho^- - d)(y) dy + \beta' \int_0^x (\rho^+ - d)(y) dy - \frac{1}{\tau} u^+ & -\frac{1}{\tau}\rho^+ \end{pmatrix} - \frac{1}{\beta'}\, \partial_x A_1(V^+),$$

$$C_-(V^-,x) = \begin{pmatrix} 0 & 0 \\ \overline{E} + \alpha' \int_x^{\alpha/\alpha'} (\rho^- - d)(y)\, dy - \frac{1}{\tau} u^- & -\frac{1}{\tau}\rho^- \end{pmatrix} + \frac{1}{\alpha'}\, \partial_x A_1(V^-),$$

$$\mathcal{D}(V) = \begin{pmatrix} D_+(V^+) \\ D_-(V^-) \end{pmatrix},$$

where

$$D_+(V^+) = \frac{1}{\beta'^2} A_1(V^+) \partial_x^2 V^+, \qquad D_-(V^-) = \frac{1}{\alpha'^2} A_1(V^-) \partial_x^2 V^-,$$

$$\begin{aligned}\mathcal{L}(V,W',x) = \Big(0, \rho^+ \Big(\alpha' \int_0^{\alpha/\alpha'} \Big(\rho'^-(t,y) + \frac{1}{\alpha'} \phi'_x(t,y)(\rho^- - d)(y)\Big)\, dy \\ + \beta' \int_0^x \Big(\rho'^+(t,y) - \frac{1}{\beta'} \phi'_x(t,y)(\rho^+ - d)(y)\Big)\, dy\Big), 0, \\ \alpha' \rho^- \int_x^{\alpha/\alpha'} \Big(\rho'^-(t,y) + \frac{1}{\alpha'} \phi'_x(t,y)(\rho^- - d)(y)\Big)\, dy\Big)^T,\end{aligned}$$

$$b(V) = f_0(V_R) - f_0(V_L), \qquad M(V)W' = A_1(V_R)W'_R - A_1(V_L)W'_L,$$

the problem can be written in the form

$$(29)\qquad \begin{cases} \mathcal{A}_0(V)(W' + \phi'\partial_x V)_t + \mathcal{A}_1(V)(W' + \phi'\partial_x V)_x \\ \quad = \mathcal{C}(V,x)W' + \mathcal{D}(V)\phi' + \mathcal{L}(V,W',x) & \text{in } 0 < x < 1,\ t > 0\,, \\ b(V)\partial_t\phi' + M(V)W' = 0 & \text{on } x = 0,\ t > 0\,. \end{cases}$$

Let us suppose that W', ϕ' grow at most like $e^{\gamma_0 t}$ as $t \to +\infty$, where $\gamma_0 > 1$. For a fixed $\gamma \geq \gamma_0$, we introduce $\phi = e^{-\gamma t}\phi'$ and $w = e^{-\gamma t}\mathcal{S}(V)^{-1}(W' + \phi'\partial_x V)$, where $\mathcal{S}(V) = \begin{pmatrix} \rho^+ & \rho^+ & 0 & 0 \\ -c^+ & c^+ & 0 & 0 \\ 0 & 0 & \rho^- & \rho^- \\ 0 & 0 & -c^- & c^- \end{pmatrix}$. Then the system for these new variables is

$$(30)\qquad \begin{cases} \partial_x w + \Lambda(V)\partial_t w + \gamma\Lambda(V)w = F(V,\partial_x V; w,\phi) & \text{in } 0 < x < 1,\ t > 0\,, \\ b(V)\partial_t\phi + M\mathcal{S}(V)w + \gamma\, b(V)\,\phi = G(V,\partial_x V;\phi) & \text{on } x = 0,\ t > 0\,, \end{cases}$$

where

$$\begin{aligned} \Lambda(V) &= \mathcal{S}(V)^{-1}\mathcal{A}_1(V)^{-1}\mathcal{A}_0(V)\mathcal{S}(V)\,, \\ F(V,\partial_x V; w,\phi) &= \mathcal{S}(V)^{-1}\mathcal{A}_1(V)^{-1}\Big(\phi\Big(\mathcal{D}(V) - \mathcal{C}(V,x)\partial_x V\Big) \\ &\qquad + \Big(\mathcal{C}(V,x)\mathcal{S}(V) - \mathcal{A}_1(V)\partial_x\mathcal{S}(V)\Big)w + \mathcal{L}(V,\mathcal{S}(V)w - \phi\partial_x V, x)\Big)\,, \\ G(V,\partial_x V;\phi) &= \phi\, M(V)\partial_x V\,. \end{aligned}$$

By now we extend W' and ϕ' to all the space $\mathbf{R} \times (0,1)$ taking for all $t < 0$ $W'(t,\,.\,), \phi'(t,\,.\,) \equiv 0$.

3.2. **The symmetrizer.** In this subsection we give the Kreiss' symmetrizer for our system.

Definition 3.2. A symmetrizer is a matrix valued function

$$\mathcal{R} : (V,\tau,\gamma) \in (\mathcal{K} = V(\{0 \leq x \leq 1\})) \times \mathbf{R} \times [1,\infty] \longrightarrow \mathcal{R}(V,\tau,\gamma) \in \text{Aut } (4)\,,$$

which is smooth, homogeneous of degree zero in (τ,γ) and such that

(a) there exists a constant $C > 0$ such that for all $(V,\tau,\gamma) \in \mathcal{K} \times \mathbf{R} \times [1,\infty]$

$$(31)\qquad \left(\mathcal{R} + (\Pi M\mathcal{S})^T(\Pi M\mathcal{S})\right)(V,\tau,\gamma) \geq C\,\mathbf{I}d_4\,,$$

where $\Pi(V)$ is the projector on $b(V)^\perp$;

(b) there exist a finite set of smooth matrix valued functions

$$\begin{aligned} &H_j : (V^+,\tau,\gamma) \in \mathcal{K} \times \mathbf{R} \times [1,\infty) \longrightarrow H_j(V,\tau,\gamma) \in \text{Aut } (2)\,, \\ &K_j : (V^-,\tau,\gamma) \in \mathcal{K} \times \mathbf{R} \times [1,\infty) \longrightarrow K_j(V,\tau,\gamma) \in \text{Aut } (2)\,, \\ &Z_j : (V,\tau,\gamma) \in \mathcal{K} \times \mathbf{R} \times [1,\infty) \longrightarrow Z_j(V,\tau,\gamma) \in \mathbf{C}^{4\times 4}\,, \end{aligned}$$

$j = 1, \dots, l$, homogeneous of degree zero in (τ, γ), and a constant $C > 0$ such that for all $(V, \tau, \gamma) \in \mathcal{K} \times \mathbf{R} \times [1, \infty]$

$$H_j(V,\tau,\gamma)\,,\ K_j(V,\tau,\gamma)\,,\ \sum_{j=1}^{l} Z_j^* Z_j(V,\tau,\gamma) \geq C\,\mathbf{I}d \tag{32}$$

and furthermore

$$\begin{aligned}\operatorname{Im}\left((\tau - i\gamma)\mathcal{R}\Lambda\right)(V) \\ = \gamma \sum_{j=1}^{l} Z_j^*(V,\tau,\gamma) \begin{pmatrix} H_j(V,\tau,\gamma) & 0 \\ 0 & K_j(V,\tau,\gamma)\end{pmatrix} Z_j(V,\tau,\gamma)\,.\end{aligned} \tag{33}$$

Here we have denoted with $\operatorname{Aut}(k)$, $k \in \mathbf{N}$, the space of self adjoint matrices with dimension k and with $\mathbf{C}^{4\times 4}$ the space of the complex matrices with dimension 4×4. We are lucky because holds the following theorem.

Theorem 3.3. *There exists a constant $C \in (0,1)$ such that the matrices*

$$H(V) = \begin{pmatrix} \frac{\beta'}{c^+ - u^+} & 0 \\ 0 & \frac{(c^+ + u^+)^3 \rho^{+2} \beta'}{1 + 2(c^- + u^-)^4 \rho^{-2} + (c^+ + u^+)^4 \rho^{+2}} \end{pmatrix}, \quad K(V^-) = \begin{pmatrix} \frac{\alpha'}{u^- - c^-} & 0 \\ 0 & \frac{\alpha'}{u^- + c^-}\end{pmatrix},$$

$$\mathcal{R}(V) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(c^+ + u^+)^4 \rho^{+2}}{1 + 2(c^- + u^-)^4 \rho^{-2} + (c^+ + u^+)^4 \rho^{+2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Z = \mathbf{I}d_4\,,$$

satisfy the conditions of Definition 3.2.

3.3. **The stability estimates.** Denoted by V the stationary solution analyzed in the subsection 2.3, let us assume that

(**H1**) $\qquad V$ is a Lipshitz function on $(0,1)$.

Remark. Let us remark again that it is always satisfied the necessary and sufficient Lopatinski type condition given in [11] to have a maximal L^2 estimate for the problem (26). In fact tedious but elementary computations show that this condition is equivalent to require $J \neq -\rho_L c_R$, which is trivially true.

Theorem 3.4. *There exist $\gamma_0 > 1$, $C > 0$, such that for all $\gamma \geq \gamma_0$ and for any solution $(w, \phi) \in \left(H^1(\mathbf{R} \times (0,1))\right)^2$ to the problem (30), satisfying $w(\,.\,,1) \equiv 0$, the following estimate holds*

$$\begin{aligned}\sqrt{\gamma}\,\|w\|_{L^2(\mathbf{R}\times(0,1))} + \|w(\,.\,,0)\|_{L^2(\mathbf{R})} + \|\varphi\|_{1,\gamma} \\ \leq \frac{C}{\sqrt{\gamma}}\,\|F(V, \partial_x V; w, \phi)\|_{L^2(\mathbf{R}\times(0,1))}\,.\end{aligned} \tag{34}$$

Here we have denoted by φ the trace of ϕ on $\{x = 0\}$ and introduced the norm

$$\|\varphi\|_{1,\gamma} = \|\varphi_t\|_{L^2(\mathbf{R})} + \gamma \|\varphi\|_{L^2(\mathbf{R})}\,. \tag{35}$$

We also introduce the weighted spaces $L^2_\gamma = e^{\gamma t} L^2$, $H^1_\gamma = e^{\gamma t} H^1$ and equip them with the norms $\|\varphi'\|_{L^2_\gamma} = \|e^{-\gamma t}\varphi'\|_{L^2}$, $\|\varphi'\|_{H^1_\gamma} = \gamma\|\varphi'\|_{L^2_\gamma} + \|\partial_t \varphi'\|_{L^2_\gamma}$. Clearly

Lemma 3.5. $\|e^{-\gamma t}\,.\,\|_{1,\gamma} \sim \|\,.\,\|_{H^1_\gamma(\mathbf{R})}$ *on* $H^1_\gamma(\mathbf{R})$.

And now in the following theorem we give the first maximal L^2 estimate for V.

Theorem 3.6 (The first maximal L^2 estimate)**.** *Let us assume that*

$$\textbf{(H2)} \qquad V \text{ is a Lipschitz function such that } \partial_x^2 V \in L^2(0,1).$$

Then there exist $\gamma_0 > 1$*,* $C > 0$*, such that for all* $\gamma \geq \gamma_0$ *and for any solution* $(W', \phi') \in \big(H^1_\gamma(\mathbf{R}\times(0,1))\big)^2$ *to the problem* (29)*, satisfying the conditions* $\phi'(\,.\,,1), W'(\,.\,,1) \equiv 0$*, the following estimate holds*

$$\begin{aligned}(36)\qquad &\sqrt{\gamma}\,\|W' + \phi'\partial_x V\|_{L^2_\gamma(\mathbf{R}\times(0,1))} + \|W'(\,.\,,0)\|_{L^2_\gamma(\mathbf{R})} + \|\varphi'\|_{H^1_\gamma(\mathbf{R})} \\ &\leq \frac{C}{\sqrt{\gamma}} \left\| \mathcal{C}(V,x)W' + \mathcal{D}(V)\phi' - \mathcal{L}(V,W',x) \right\|_{L^2_\gamma(\mathbf{R}\times(0,1))}.\end{aligned}$$

Proof. Let us consider $(W', \phi') \in \big(H^1_\gamma(\mathbf{R}\times(0,1))\big)^2$ the solution of the problem (29). If we pose $w = e^{-\gamma t}\mathcal{S}(V)^{-1}(W' + \phi'\partial_x V)$ and $\phi = e^{-\gamma t}\phi'$, then the couple $(w, \phi) \in \big(H^1(\mathbf{R}\times(0,1))\big)^2$ is a solution of the problem (30) and satisfies the estimate (34). So (W', ϕ') satisfies the estimate (36) for the preceding lemma and because for γ sufficiently big hold the following two estimates

$$\begin{aligned}\|F(V,\partial_x V; w,\phi)\|_{L^2(\mathbf{R}\times(0,1))} \leq C\Big(&\|\mathcal{C}(V,x)W' + \mathcal{D}(V)\phi' - \mathcal{L}(V,W',x)\|_{L^2_\gamma(\mathbf{R}\times(0,1))} \\ &+ \|W' + \phi'\partial_x V\|_{L^2_\gamma(\mathbf{R}\times(0,1))}\Big),\end{aligned}$$

$$\begin{aligned}&\sqrt{\gamma}\,\|w\|_{L^2(\mathbf{R}\times(0,1))} + \|w(\,.\,,0)\|_{L^2(\mathbf{R})} + \|\varphi\|_{1,\gamma} \\ &\geq \sqrt{\gamma}\,\|\mathcal{S}(V)\|^{-1}_{L^\infty(0,1)}\,\|W' + \phi'\partial_x V\|_{L^2_\gamma(\mathbf{R}\times(0,1))} \\ &\quad + \|\mathcal{S}(V(0))\|^{-1}\|W'(\,.\,,0) + \varphi'\partial_x V(0)\|_{L^2_\gamma(\mathbf{R})} + \|\varphi\|_{1,\gamma} \\ &\geq \frac{1}{k}\left(\sqrt{\gamma}\,\|W' + \phi'\partial_x V\|_{L^2_\gamma(\mathbf{R}\times(0,1))} + \|W'(\,.\,,0)\|_{L^2_\gamma(\mathbf{R})} + \|\varphi\|_{1,\gamma}\right).\end{aligned}$$

□

We observe that if one takes into consideration a simple temporal perturbation of the boundary (namely taking ϕ' independent from the space variable x) then, recovering the computations made in [11], it results that the corresponding linearized problem is similar to the one we have found before, and furthermore one has the following result.

Theorem 3.7 (The second maximal L^2 estimate)**.** *Let us assume that* **(H2)** *holds and that* $\rho_x(1)=0$*, then there exist* $\gamma_0 > 1$*,* $C > 0$*, such that for all* $\gamma \geq \gamma_0$ *and for* *any solution* $(W',\phi')\in H^1_\gamma(\mathbf{R}\times(0,1))\times H^1_\gamma(\mathbf{R})$ *to the problem* $(29)'$*, such that* $W'(\,.\,,1)\equiv 0$*, the following estimate holds*

$$\sqrt{\gamma}\,\|W'\|_{L^2_\gamma(\mathbf{R}\times(0,1))} + \|W'(\,.\,,0)\|_{L^2_\gamma(\mathbf{R})} + \|\phi'\|_{H^1_\gamma(\mathbf{R})} \leq \frac{C}{\sqrt{\gamma}}\left\|\mathcal{C}(V,x)W' + \mathcal{D}(V)\varphi' + \mathcal{L}(V,W',x)\right\|_{L^2_\gamma(\mathbf{R}\times(0,1))}. \tag{37}$$

It remains now to prove of the Theorem 3.4.
By using the paradifferential theory developed in the appendix B, we can give a paradifferential approximation of the boundary value problem (30). In order to apply this theory we have to remark the following fact. We know that on the boundary $\{x=0\}$ the only variable is $t\in\mathbf{R}$ and therefore the paradifferential calculus directly applies. Instead if we have a symbol σ and a function w, defined on $\mathbf{R}\times(0,1)$, we will continue to use the symbol $P^\gamma_\sigma w$ to denote the tangential paraproduct of the functions taking x as a constant, i.e.

$$(P^\gamma_\sigma w)(\,.\,,x) = P^\gamma_{\sigma(\,.\,,x)}w(\,.\,,x) \quad \text{for all} \quad x\in(0,1)\,.$$

So we can consider the symbol $J(x,\tau,\gamma)=(\tau-i\gamma)\Lambda(V(x))\in\Gamma^1_1$ and the paradifferential operator $J^\gamma = iT^\gamma_J$. For the Theorem B.7 we have that

Proposition 3.8. *If V is such that $u^- - c^-$, $c^+ - u^+\geq k$ for a constant $k>0$, then there exists a constant $C=C\left(\|\Lambda(V)\|_{L^\infty(0,1)}\right)>0$, such that for all $\gamma\geq 1$ and $w\in H^1(t\in\mathbf{R})\cap L^2(x\in(0,1))$, one has*

$$\|\Lambda(V)\,\partial_t w + \gamma\,\Lambda(V)\,w - J^\gamma w\|_{L^2(\mathbf{R}\times(0,1))}\leq C\,\|w\|_{L^2(\mathbf{R}\times(0,1))}\,. \tag{38}$$

Let us also introduce the boundary symbols $b(\tau,\gamma)=(\tau-i\gamma)\,b(V)\in\Gamma^1_1$ and $M=M(V)$, $\mathcal{S}=\mathcal{S}(V(0))\in\Gamma^0_1$. Another consequence of the Theorem B.7 is the following result.

Proposition 3.9. *There exists a constant $C=C(\,\|b\|,\|M\mathcal{S}\|\,)>0$, such that for all $(\varphi,w(\,.\,,0))\in H^1(\mathbf{R})\times L^2(\mathbf{R})$ and $\gamma\geq 1$, one has*

$$\|b(V)\,\partial_t\varphi + M\mathcal{S}w(\,.\,,0) + \gamma b(V)\,\varphi - iT^\gamma_b\varphi - T^\gamma_{M\mathcal{S}}w(\,.\,,0)\|_{L^2(\mathbf{R})} \leq \frac{C}{\gamma}\left(\|\varphi\|_{1,\gamma} + \|w(\,.\,,0)\|_{L^2(\mathbf{R})}\right). \tag{39}$$

Therefore the next theorem implies the foregoing theorem.

Theorem 3.10. *There exist $\gamma_0>1$, $C>0$ such that for all $\gamma\geq\gamma_0$ and for any solution $(w,\phi)\in H^1(\mathbf{R}\times(0,1))\times H^1(\mathbf{R}\times(0,1))$ to the problem* (30)*, satisfying $w(\,.\,,1)\equiv 0$, it follows*

$$\sqrt{\gamma}\,\|w\|_{L^2(\mathbf{R}\times(0,1))} + \|w(\,.\,,0)\|_{L^2(\mathbf{R})} + \|\varphi\|_{1,\gamma} \leq C\Big(\frac{1}{\sqrt{\gamma}}\,\|(\partial_x + J^\gamma)w\|_{L^2(\mathbf{R}\times(0,1))} + \|iT^\gamma_b\varphi + T^\gamma_{M\mathcal{S}}w(\,.\,,0)\|_{L^2(\mathbf{R})}\Big). \tag{40}$$

Now we only need to show that this theorem is true.
Let us consider the self-adjoint paradifferential operator $R^\gamma = \mathrm{Re}\,T^\gamma_{\mathcal{R}}$, where the symbol $\mathcal{R}=\mathcal{R}(V)\in\Gamma^0_1$ is our symmetrizer. We want to show that R^γ satisfies the hypotheses of the following elementary lemma.

Lemma 3.11. *Assume that there exist C, $k > 0$, such that for all $\gamma \geq 1$ and $w \in H^1(\mathbf{R} \times (0,1))$, with $w(\,.\,,1) \equiv 0$, the following estimates hold*

$$\|R^\gamma w\|_{L^2} \leq C\,\|w\|_{L^2}\,, \tag{41}$$

$$\|[\partial_x, R^\gamma]w\|_{L^2} \leq C\,\|w\|_{L^2}\,, \tag{42}$$

$$\langle (R^\gamma J^\gamma + (J^\gamma)^* R^\gamma)w, w\rangle_{L^2} \leq -k\,\gamma\,\|w\|^2_{L^2}\,, \tag{43}$$

$$\langle R^\gamma w(\,.\,,0), w(\,.\,,0)\rangle_{L^2} + C\|T^\gamma_{\Pi M \mathcal{S}} w(\,.\,,0)\|^2_{L^2} \geq k\|w(\,.\,,0)\|^2_{L^2}\,, \tag{44}$$

then there are $\gamma_0 \geq 1$, $C_1 > 0$, which depend only on C and k, such that for all $\gamma \geq \gamma_0$ and $w \in H^1(\mathbf{R} \times (0,1))$ with $w(\,.\,,1) \equiv 0$

$$\sqrt{\gamma}\|w\|_{L^2} + \|w(\,.\,,0)\|_{L^2} \leq C_1\Big(\frac{1}{\sqrt{\gamma}}\|(\partial_x + J^\gamma)w\|_{L^2} + \|T^\gamma_{\Pi M \mathcal{S}} w(\,.\,,0)\|_{L^2}\Big)\,. \tag{45}$$

The first two estimates are immediate, so we start directly from the estimate (43).
Let us consider the symbol $P(x,\gamma) = -2\gamma \mathcal{R}\Lambda(V(x)) \in \Gamma^1_1$. Then by the Theorems B.8, B.9 we get the next lemma.

Lemma 3.12. *Let V be such that $u^- - c^-$, $c^+ - u^+ \geq k$ for some constant $k > 0$, then there exists a constant $C = C\left(\|\Lambda(V)\|_{L^\infty(0,1)}, \|\mathcal{R}(V)\|_{L^\infty(0,1)}\right) > 0$, such that for all $w \in L^2(\mathbf{R} \times (0,1))$, it follows*

$$\|(R^\gamma J^\gamma + (J^\gamma)^* R^\gamma + T^\gamma_P)\, w\|_{L^2} \leq C\|w\|_{L^2}\,.$$

Let us denote $F(x,\gamma) \in \Gamma^1_1$ the block diagonal matrix valued symbol with blocks $2\,\gamma H(V(x))$ and $2\,\gamma K(V^-(x))$.

Lemma 3.13. *Let V be such that $u^- - c^-$, $c^+ - u^+ \geq \tilde{k}$ for some constant $\tilde{k} > 0$, then there exist C, $k > 0$, such that for every $w \in L^2(\mathbf{R} \times (0,1))$, it follows*

(a) $\operatorname{Re}\langle T^\gamma_P w, w\rangle_{L^2} - 2\operatorname{Re}\langle T^\gamma_F T^\gamma_Z w, T^\gamma_Z w\rangle_{L^2} \leq C\|w\|^2_{L^2}$,
(b) $\operatorname{Re}\langle T^\gamma_F T^\gamma_Z w, T^\gamma_Z w\rangle_{L^2} \geq k\,\gamma\|T^\gamma_Z w\|^2_{L^2}$,
(c) $\|w\|^2_{L^2} \leq C\Big(\,\|T^\gamma_Z w\|^2_{L^2} + \frac{1}{\gamma}\,\|w\|^2_{L^2}\Big)$.

Proof.

(a) Let us take γ sufficiently large, then our result follows from the properties of the symmetrizer and the Theorems B.8, B.9.
(b) Since $H(V(x))$, $K(V^-(x)) \in \Gamma^0_1$ satisfy the hypotheses of the Theorem B.10 for $m = 0$, the second estimate holds.
(c) Since $Z^*Z \in \Gamma^0_1$ satisfies the conditions of the Theorem B.10, for $m = 0$, by using the Theorems B.8, B.9, we get the last estimate. □

Hence by the estimates of the previous lemma, we have

$$\operatorname{Re}\langle T^\gamma_P w, w\rangle_{L^2} \geq 2\operatorname{Re}\langle T^\gamma_F T^\gamma_Z w, T^\gamma_Z w\rangle_{L^2} - C\|w\|^2_{L^2} \geq \frac{\gamma\, k}{2}\|w\|^2_{L^2}\,.$$

Thus for γ sufficiently large

$$\langle (R^\gamma J^\gamma + (J^\gamma)^* R^\gamma) w, w\rangle_{L^2} = -\mathrm{Re}\,\langle T_P^\gamma w, w\rangle_{L^2} + \mathrm{Re}\,\langle (R^\gamma J^\gamma + (J^\gamma)^* R^\gamma + T_P^\gamma)\, w, w\rangle_{L^2} \leq -\frac{k\,\gamma}{3}\|w\|_{L^2}^2 ,$$

so the estimate (43) holds. Finally the estimate (44) is easily obtained by using the next lemma.

Lemma 3.14. *There exist $C,\ k > 0$, such that for all $w(\,.\,,0) \in L^2(\mathbf{R})$, one has*

(a) $\langle R^\gamma w(\,.\,,0), w(\,.\,,0)\rangle_{L^2} + C\,\mathrm{Re}\,\langle T^\gamma_{(\Pi M\mathcal{S})^T \Pi M\mathcal{S}} w(\,.\,,0), w(\,.\,,0)\rangle_{L^2} \geq k\|w(\,.\,,0)\|_{L^2}^2$,

(b) $\mathrm{Re}\,\langle T^\gamma_{(\Pi M\mathcal{S})^T\,\Pi M\mathcal{S}} w(\,.\,,0), w(\,.\,,0)\rangle_{L^2} \leq \|T^\gamma_{\Pi M\mathcal{S}} w(\,.\,,0)\|_{L^2}^2 + \frac{1}{\gamma} C\|w\|_{L^2}^2$.

Proof. The first estimate follows from the Theorem B.10 and from the properties of the symmetrizer. The second estimate is a consequence of the Theorems B.8, B.9. $\square$

Since $\Pi \in \Gamma_1^0$, $b \in \Gamma_1^1$, $M \in \Gamma_1^0$ and $\mathcal{S} \in \Gamma_1^0$, by the Theorem B.8, we get

$$\|T_\Pi^\gamma T_b^\gamma \varphi\|_{L^2} = \|(T_\Pi^\gamma T_b^\gamma - T_{\Pi b}^\gamma)\varphi\|_{L^2} \leq C\,\|\varphi\|_{L^2} \leq \frac{C}{\gamma}\|\varphi\|_{1,\gamma} ,$$

$$\|(T_\Pi^\gamma T_{M\mathcal{S}}^\gamma - T_{\Pi M\mathcal{S}}^\gamma) w(\,.\,,0)\|_{L^2} \leq C\|w(\,.\,,0)\|_{-1,\gamma} \leq \frac{C}{\gamma}\|w(\,.\,,0)\|_{L^2} .$$

Because of these two estimates, it follows

$$\|T_{\Pi M\mathcal{S}}^\gamma w(\,.\,,0)\|_{L^2} \leq C\|iT_b^\gamma\varphi + T_{M\mathcal{S}}^\gamma w(\,.\,,0)\|_{L^2} + \frac{C}{\gamma}\left(\|\varphi\|_{1,\gamma} + \|w(\,.\,,0)\|_{L^2}\right) . \tag{46}$$

In order to complete the proof of the Theorem 3.11 we need this last lemma.

Lemma 3.15. *There exists $\gamma_0 \geq 1$, such that for all $\gamma \geq \gamma_0$ and $\varphi, w(\,.\,,0) \in L^2(\mathbf{R})$, one has*

$$\|\varphi\|_{1,\gamma} \leq C\left(\|iT_b^\gamma\varphi + T_{M\mathcal{S}}^\gamma w(\,.\,,0)\|_{L^2} + \|w(\,.\,,0)\|_{L^2}\right) .$$

Proof. Since the symbol $p(\tau,\gamma) = b^*\, b(\tau,\gamma) = (\tau^2+\gamma^2)\,(\rho_R - \rho_L)^2 \in \Gamma_1^2$ satisfies the hypotheses of the Theorem B.10, with $m = 2$, by using the Theorem B.9, one can see there exists $\gamma_0 \geq 1$, such that for all $\gamma \geq \gamma_0$, it follows

$$(\rho_R - \rho_L)^2\|\varphi\|_{1,\gamma}^2 \leq 2\,\mathrm{Re}\ \langle T_p^\gamma\varphi,\varphi\rangle_{L^2} \leq \frac{C}{\gamma}\|\varphi\|_{1,\gamma}^2 + 2\langle\varphi, T_p^\gamma\varphi\rangle_{L^2} \Rightarrow \frac{C}{2}\|\varphi\|_{1,\gamma}^2 \leq \|T_b^\gamma\varphi\|_{L^2}^2 \Rightarrow \sqrt{\frac{C}{2}}\,\|\varphi\|_{1,\gamma} \leq \|iT_b^\gamma\varphi + T_{M\mathcal{S}}^\gamma w(\,.\,,0)\|_{L^2} + \|w(\,.\,,0)\|_{L^2} .$$

$\square$

Now by the Lemmas 3.12, 3.16 and the estimate (46) the proof of the Theorem 3.11 is achieved.

Appendix A. Computations for the linearization

The linearized system in (W', ϕ') will be obtained from the perturbed system (27), that we rewrite in the following way

$$\begin{cases} A_0(V_\varepsilon^-)\partial_t V_\varepsilon^- + A_1(V_\varepsilon^-)\partial_x V_\varepsilon^- = F_-(V_\varepsilon^-, x) & \text{in } -\alpha < \phi_\varepsilon(t,x) < 0,\, t > 0\,, \\ A_0(V_\varepsilon^+)\partial_t V_\varepsilon^+ + A_1(V_\varepsilon^+)\partial_x V_\varepsilon^+ = F_+(V_\varepsilon^-, V_\varepsilon^+, x) & \text{in } 0 < \phi_\varepsilon(t,x) < \beta,\, t > 0\,, \\ \partial_t \phi_\varepsilon [f_0(V_\varepsilon)] + \partial_x \phi_\varepsilon [f_1(V_\varepsilon)] = 0 & \text{on } \phi_\varepsilon(t,x) = 0,\, t > 0\,, \end{cases}$$

where $\phi_\varepsilon(t,x) = x + \varepsilon\phi'(t,x)$. The dependence of the domain on ε is a problem which can be solved turning to a fixed domain. For this purpose it is sufficient to make the following change of coordinates

$$(t,x) \mapsto (\tilde t, \tilde x) = (t, \phi_\varepsilon(t,x))\,.$$

We observe that this change of coordinates makes sense for ε small enough, since in this case $\partial_x \phi_\varepsilon(\tilde t, .) > 0$, for every $\tilde t > 0$. In the new variables the surface $\{-\alpha < \phi_\varepsilon(t,x) < 0\}$ becomes the fixed domain $\{-\alpha < \tilde x < 0\}$, the surface $\{0 < \phi_\varepsilon(t,x) < -\beta\}$ becomes the fixed domain $\{0 < \tilde x < -\beta\}$ and the line $\{\phi_\varepsilon(t,x) = 0\}$ becomes $\{\tilde x = 0\}$. Since moreover

$$\partial_t = \partial_{\tilde t} + \partial_t \phi_\varepsilon \, \partial_{\tilde x} \quad \text{ and } \quad \partial_x = \partial_x \phi_\varepsilon \, \partial_{\tilde x}\,,$$

the previous system becomes

$$\begin{cases} A_0(\tilde V_\varepsilon^-)\partial_{\tilde t} \tilde V_\varepsilon^- + \tilde A_1(\tilde V_\varepsilon^-, (\nabla\phi_\varepsilon)(\tilde t, \phi_\varepsilon(\tilde t, \,.\,)^{-1}(\tilde x))\, \partial_{\tilde x} \tilde V_\varepsilon^- & \\ \quad = F_-(\tilde V_\varepsilon^-, \tilde x) & \text{in } -\alpha < \tilde x < 0,\, \tilde t > 0\,, \\ A_0(\tilde V_\varepsilon^+)\partial_{\tilde t} \tilde V_\varepsilon^+ + \tilde A_1(\tilde V_\varepsilon^+, (\nabla\phi_\varepsilon)(\tilde t, \phi_\varepsilon(\tilde t, \,.\,)^{-1}(\tilde x))\, \partial_{\tilde x} \tilde V_\varepsilon^+ & \\ \quad = F_+(\tilde V_\varepsilon^-, \tilde V_\varepsilon^+, \tilde x) & \text{in } 0 < \tilde x < \beta,\, \tilde t > 0\,, \\ (\partial_t\phi_\varepsilon)(\tilde t, \phi_\varepsilon(\tilde t, \,.\,)^{-1}(\tilde x))\,[f_0(\tilde V_\varepsilon)] & \\ \quad + (\partial_x\phi_\varepsilon)(\tilde t, \phi_\varepsilon(\tilde t, \,.\,)^{-1}(\tilde x))\,[f_1(\tilde V_\varepsilon)] = 0 & \text{on } \tilde x = 0,\, t > 0\,, \end{cases}$$

where $\tilde V_\varepsilon^\pm(\tilde t, \tilde x) := V_\varepsilon^\pm(\tilde t, \phi_\varepsilon(\tilde t, \,.\,)^{-1}(\tilde x))$,

$$\begin{aligned} &F_-(\tilde V_\varepsilon^-, \tilde x) \\ &= \Big(0, \tilde\rho_\varepsilon^-(\tilde t, \tilde x)\Big(\overline{E} + \int_{-\alpha}^{\tilde x} (\tilde\rho_\varepsilon^-(\tilde t, y) - \tilde d(y))\, \frac{\partial}{\partial y}\Big(\phi_\varepsilon(\tilde t, \,.\,)^{-1}(y)\Big)\, dy - \frac{1}{\tau}\tilde u_\varepsilon^-(\tilde t, \tilde x)\Big)\Big)^T, \end{aligned}$$

$$\begin{aligned} F_+(\tilde V_\varepsilon^-, \tilde V_\varepsilon^+, \tilde x) = \Big(0, \tilde\rho_\varepsilon^+(\tilde t, \tilde x)\Big(\overline{E} &+ \int_{-\alpha}^{0} (\tilde\rho_\varepsilon^-(\tilde t, y) - \tilde d(y))\, \frac{\partial}{\partial y}\Big(\phi_\varepsilon(\tilde t, .)^{-1}(y)\Big)\, dy \\ &+ \int_0^{\tilde x} (\tilde\rho_\varepsilon^+(\tilde t, y) - \tilde d(y))\, \frac{\partial}{\partial y}\Big(\phi_\varepsilon(\tilde t, .)^{-1}(y)\Big)\, dy - \frac{1}{\tau}\tilde u_\varepsilon^+(\tilde t, \tilde x)\Big)\Big)^T, \end{aligned}$$

and

$$\begin{aligned} \tilde A_1(V_\varepsilon^\pm(t,x), \nabla\phi_\varepsilon(t,x)) : &= (\partial_x\phi_\varepsilon)(t,x)\, A_1(V_\varepsilon^\pm(t,x)) \\ &\quad + (\partial_t\phi_\varepsilon)(t,x)\, A_0(V_\varepsilon^\pm(t,x))\,. \end{aligned}$$

Elementary computations show that

$$\frac{d}{d\varepsilon}\Big(\,(\nabla\phi_\varepsilon)\,(\tilde t,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\Big)\Big|_{\varepsilon=0}=(\nabla\phi')\,(\,\tilde t,\tilde x),(\nabla\phi_\varepsilon)\,(\tilde t,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\Big|_{\varepsilon=0}=\begin{pmatrix}0\\1\end{pmatrix},$$
$$\frac{d}{d\varepsilon}\Big(\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x)\Big)\Big|_{\varepsilon=0}=-\phi'(\,\tilde t,\tilde x),\Big(\frac{\partial}{\partial y}\left(\phi_\varepsilon(t,\,.\,)^{-1}(y)\right)\Big)\Big|_{\varepsilon=0}=1\,.$$

Therefore by using the foregoing equalities easily one finds that

$$\frac{d}{d\varepsilon}\Big(A_0(\tilde V_\varepsilon)\partial_{\tilde t}\tilde V_\varepsilon+\tilde A_1(\tilde V_\varepsilon,(\nabla\phi_\varepsilon)(\tilde t,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\,\partial_{\tilde x}\tilde V_\varepsilon\Big)\Big|_{\varepsilon=0}=A_0(\tilde V)\partial_{\tilde t}(\tilde W'+\phi'\partial_{\tilde x}\tilde V)$$
$$+A_1(\tilde V)\partial_{\tilde x}(\tilde W'+\phi'\partial_{\tilde x}\tilde V)-\phi' A_1(\tilde V)\partial^2_{\tilde x}\tilde V+\partial_{\tilde x}A_1(\tilde V)\tilde W'\,,$$

$$\frac{d}{d\varepsilon}\Big(F_-(\tilde V_\varepsilon^-,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\Big)\Big|_{\varepsilon=0}=\tilde\rho^{'-}\Big(\overline{E}+\int_{-\alpha}^{\tilde x}(\tilde\rho^--d)(y)\,dy-\frac{1}{\tau}\tilde u^-\Big)$$
$$+\tilde\rho^-\Big(\int_{-\alpha}^{\tilde x}\big(\tilde\rho^{'-}(\tilde t,y)-\phi'_{\tilde x}(\tilde t,y)(\tilde\rho^--d)(y)\big)\,dy-\frac{1}{\tau}\tilde u^{'-}\Big)\,,$$
$$\frac{d}{d\varepsilon}\big(F_+(\tilde V_\varepsilon^-,\tilde V_\varepsilon^+,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\big)\Big|_{\varepsilon=0}=\tilde\rho^{'+}\Big(\overline{E}+\int_{-\alpha}^{0}\big(\tilde\rho^-(y)-d(y)\big)\,dy$$
$$+\int_0^{\tilde x}\big(\tilde\rho^+(y)-d(y)\big)\,dy-\frac{1}{\tau}\tilde u^+\Big)+\tilde\rho^+\Big(\int_{-\alpha}^0\Big(\tilde\rho^{'-}(\tilde t,y)-\phi'_{\tilde x}(\tilde t,y)(\tilde\rho^--d)(y)\Big)\,dy$$
$$+\int_0^{\tilde x}\big(\tilde\rho^{'+}(\tilde t,y)-\phi'_{\tilde x}(\tilde t,y)(\tilde\rho^+-d)(y)\big)\,d-\frac{1}{\tau}\tilde u^{'+}\Big)\,,$$
$$\frac{d}{d\varepsilon}\big(\partial_t\;\phi_\varepsilon(\tilde t,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\,[f_0(V_\varepsilon)]+\partial_x\phi_\varepsilon(\tilde t,\phi_\varepsilon(\tilde t,\,.\,)^{-1}(\tilde x))\,[f_1(V_\varepsilon)]\big)\Big|_{\varepsilon=0}=0$$
$$\Leftrightarrow\partial_{\tilde t}\phi'(\tilde t,\tilde x)\,[f_0(\tilde V(\tilde t,\tilde x))]+[A_1(\tilde V(\tilde t,\tilde x))\tilde W'(\tilde t,\tilde x)]=0\,,$$

and the linear equations are

$$\begin{cases}A_0(\tilde V^-)\partial_{\tilde t}(\tilde W^{'-}+\phi'\partial_{\tilde x}\tilde V^-)+A_1(\tilde V^-)\partial_{\tilde x}(\tilde W^{'-}+\phi'\partial_{\tilde x}\tilde V^-)\\ \quad=\tilde C(\tilde V^-,\tilde x)\tilde W^{'-}+\phi' D(\tilde V^-)+\tilde\rho^-\int_{-\alpha}^{\tilde x}(0,\tilde\rho^{'-}(\tilde t,y)\\ \quad-\phi'_{\tilde x}(\tilde t,y)(\tilde\rho^--d)(y))^T dy & \text{in } -\alpha<\tilde x<0,\,\tilde t>0\\ A_0(\tilde V^+)\partial_{\tilde t}(\tilde W^{'+}+\phi'\partial_{\tilde x}\tilde V^+)+A_1(\tilde V^+)\partial_{\tilde x}(\tilde W^{'+}+\phi'\partial_{\tilde x}\tilde V^+)\\ \quad=\tilde C(\tilde V^+,\tilde x)\tilde W^{'+}+\phi' D(\tilde V^+)+\tilde\rho^+\int_{-\alpha}^{\tilde x}(0,\tilde\rho'(\tilde t,y)\\ \quad-\phi'_{\tilde x}(\tilde t,y)(\tilde\rho-d)(y))^T dy & \text{in } 0<\tilde x<\beta,\,\tilde t>0\\ \partial_{\tilde t}\phi'[f_0(\tilde V)]+[A_1(\tilde V)\tilde W']=0 & \text{on } \tilde x=0\end{cases}$$

where

$$\tilde C(\tilde V,\tilde x):=\begin{pmatrix}0 & 0\\ \overline{E}+\int_{-\alpha}^{\tilde x}(\tilde\rho-d)(y)\,dy-\frac{1}{\tau}\tilde u & -\frac{1}{\tau}\tilde\rho\end{pmatrix}-\partial_{\tilde x}A_1(\tilde V)$$

and

$$D(\tilde{V}) := A_1(\tilde{V})\partial_{\tilde{x}}^2\tilde{V}\,.$$

Appendix B. The paradifferential calculus

The purpose of this appendix is to recall the results of the paradifferential theory that are necessary for our proofs. We will not give the proofs because they can be found in [18] and because they are extentions of the results given in [2] and [5] to the framework of parameter depending operators.

B.1. **Introduction.**

Definition B.1. For $s \in \mathbf{R}$, we denote with $H^s(\mathbf{R})$ the Sobolev space of temperate distributions $v \in S'(\mathbf{R})$ such that $(1+\xi^2)^{s/2}\mathcal{F}_t v \in L^2(\mathbf{R})$. We equip this space with the γ-family of norms

$$\| v \|_{s,\gamma}^2 = \int_{\mathbf{R}} (\gamma^2+\xi^2)^s \, |\mathcal{F}_t v|^2(\xi)\, d\xi\,.$$

Clearly this norm is equivalent to the norm $\| \,.\, \|_{1,\gamma}$ given in the section 3 when $s=1$. Let us consider a fixed $\gamma \geq 1$. Recalling that the spectrum of a function is the support of its Fourier transform, we introduce the spaces Γ_k^m and the spaces Σ_k^m of the symbols for our paradifferential operators, $m \in \mathbf{R}$, $k = 0{,}1$.

Definition B.2. Let $m \in \mathbf{R}$.

$\Gamma_{\mathbf{0}}^{\mathbf{m}} = \{\, a : (t,\tau,\gamma) \in \mathbf{R}\times\mathbf{R}\times[1,\infty) \to a(t,\tau,\gamma) \in \mathbf{C} \,|\, a \in L^\infty_{\mathrm{loc}}(\mathbf{R}\times\mathbf{R}\times[1,\infty)), a(t,\,.\,,\gamma) \in C^\infty(\mathbf{R})$ for all $(t,\gamma) \in \mathbf{R}\times[1,\infty)$, and for all $\beta \in \mathbf{N}$ there exists $C_\beta > 0$ such that $|\partial_\tau^\beta a|(t,\tau,\gamma) \leq C_\beta(\gamma+|\tau|)^{m-\beta}$ for all $(t,\tau,\gamma) \in \mathbf{R}\times\mathbf{R}\times[1,\infty)\,\}$,

$\Gamma_{\mathbf{1}}^{\mathbf{m}} = \{\, a \in \Gamma_0^m \,|\, \partial_t a \in \Gamma_0^m \,\}$,

$\Sigma_{\mathbf{0}}^{\mathbf{m}} = \{\, \sigma \in \Gamma_0^m \,|$ there exists $\varepsilon \in (0,1]$ such that $spec\,\sigma(\,.\,,\tau,\gamma) \subseteq \{\, \eta \in \mathbf{R} : |\eta| \leq \varepsilon(\gamma^2+\tau^2)^{1/2} \,\}$ for all $(\tau,\gamma) \in \mathbf{R}\times[1,\infty)\,\}$,

$\Sigma_{\mathbf{1}}^{\mathbf{m}} = \Gamma_1^m \cap \Sigma_0^m$.

Now, if a symbol $\sigma \in \Sigma_0^m$, $m \in \mathbf{R}$, the corresponding paradifferential operator is

$$P_\sigma^\gamma v(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\tau t}\sigma(t,\tau,\gamma)(\mathcal{F}_t v)(\tau)\, d\tau\,, \quad v \in S'(\mathbf{R})\,. \tag{47}$$

To build a symbol starting from a Γ_k^m-function we need an admissible cut-off.

Definition B.3. A function $\psi(\eta,\tau,\gamma) \in C^\infty(\mathbf{R}\times\mathbf{R}\times[1,\infty))$ is an admissible cut-off if

- $0 \leq \psi(\eta,\tau,\gamma) \leq 1$ for all $(\eta,\tau,\gamma) \in \mathbf{R}\times\mathbf{R}\times[1,\infty)$;
- there are two constants ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 \leq 1$ and

$$\psi(\eta,\tau,\gamma) = 1 \quad \text{for} \quad |\eta| \leq \varepsilon_1(\gamma^2+\tau^2)^{1/2}\,,$$

$$\psi(\eta,\tau,\gamma) = 0 \quad \text{for} \quad |\eta| \geq \varepsilon_2(\gamma^2+\tau^2)^{1/2}\,;$$

- for all $(\alpha,\beta)\in\mathbf{N}\times\mathbf{N}$ there is $C_{\alpha,\beta}>0$ such that

$$|\partial_\tau^\alpha\partial_\eta^\beta\psi|(\eta,\tau,\gamma)\le\frac{C_{\alpha,\beta}}{(\gamma+|\tau|)^{\alpha+\beta}}$$

The way to associate to any function of Γ_k^m, $k=0,\,1$, a symbol of Σ_k^m is given by the following proposition.

Proposition B.4. *Let $m\in\mathbf{R}$. If $a\in\Gamma_k^m$, $k=0,\,1$, then*

$$\sigma_a(t,\tau,\gamma)=(2\pi)^{-1/2}(\mathcal{F}_\eta^{-1}\psi*_t a)(t,\tau,\gamma)\in\Sigma_k^m\,.$$

So we can associate to $a\in\Gamma_0^m$, $m\in\mathbf{R}$, the operator $T_a^\gamma : S'(\mathbf{R})\to S'(\mathbf{R})$ defined by $T_a^\gamma=P_{\sigma_a}^\gamma$ for all $\gamma\ge1$.

Definition B.5. A family of operators $\{P^\gamma\}_{\gamma\ge1}$ is m-regularizing, $m\in\mathbf{R}$, if for all $s\in\mathbf{R}$ it results that

$$P^\gamma:(H^s(\mathbf{R}),\,\|\,.\,\|_{s,\gamma})\longrightarrow(H^{s+m}(\mathbf{R}),\,\|\,.\,\|_{s+m,\gamma})$$

is continuous uniformly respect to γ, i.e. $P^\gamma(H^s(\mathbf{R}))\subseteq H^{s+m,\gamma}(\mathbf{R})$ and there exists a constant $C>0$ such that $\|P^\gamma v\|_{s+m,\gamma}\le C\,\|v\|_{s,\gamma}$ for all $v\in H^s(\mathbf{R})$ and $\gamma\ge1$.

Proposition B.6. *If $\sigma\in\Sigma_0^m$, $m\in\mathbf{R}$, then $\{P_\sigma^\gamma\}_{\gamma\ge1}$ is $-m$-regularizing.*

B.2. **The main theorems of paradifferential calculus.** In this last subsection we give the four theorems which have been used to obtain the maximal L^2 estimates.

Theorem B.7. *If $a(t)\in L^\infty(\mathbf{R})$ then $\{T_a^\gamma\}_{\gamma\ge1}$ is 0-regularizing.*
If further $a\in W^{1,\infty}(\mathbf{R})$ then there is a constant $C>0$ such that for all $\gamma\ge1$

$$\|av-T_a^\gamma v\|_{L^2(\mathbf{R})}\le\frac{C}{\gamma}\|v\|_{L^2(\mathbf{R})}\,\|a\|_{W^{1,\infty}(\mathbf{R})}\quad\text{for all}\quad v\in L^2(\mathbf{R})\,,$$

$$\|a\,\partial_t v-T_a^\gamma\partial_t v\|_{L^2(\mathbf{R})}\le C\|v\|_{L^2(\mathbf{R})}\,\|a\|_{W^{1,\infty}(\mathbf{R})}\quad\text{for all}\quad v\in H^1(\mathbf{R})\,.$$

Theorem B.8. *If $a\in\Gamma_1^m$, $m\in\mathbf{R}$, $b\in\Gamma_1^{m'}$, $m'\in\mathbf{N}$, then $T_a^\gamma\circ T_b^\gamma-T_{ab}^\gamma$ is $(1-m-m')$-regularizing.*

Theorem B.9. *If $a\in\Gamma_1^m$, then $(T_a^\gamma)^*-T_{\overline{a}}^\gamma$ is $1-m$-regularizing.*

Theorem B.10. *If $a\in\Gamma_1^m$ and there is a constant $C>0$ such that*

$$\mathrm{Re}\;a(t,\tau,\gamma)\ge C(\gamma^2+\tau^2)^{m/2}\quad\text{for all}\quad(t,\tau,\gamma)\in\mathbf{R}\times\mathbf{R}\times[1,\infty)\,,$$

then there exists a constant $\gamma_0\ge1$ such that

$$\frac{C}{2}\|v\|_{m/2,\gamma}^2\le\mathrm{Re}\;\langle T_a^\gamma v,v\rangle_{L^2(\mathbf{R})}\quad\text{for all}\quad v\in H^m(\mathbf{R})\quad\text{and}\quad\gamma\ge\gamma_0\,.$$

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