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ON THE STABILITY OF THE SOLUTIONS OF CERTAIN FIFTH ORDER NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. Our aim in this paper is to present sufficient conditions under which all solutions of (1.1) tend to zero as $t \to \infty$.

1. Introduction

The equation studied here is of the form

$$x^{(5)} + f(t, \dot{x}, \ddot{x}, \dddot{x})x^{(4)} + \phi(t, \ddot{x}, \dddot{x}) + \psi(t, \ddot{x}) + g(t, \dot{x}) + e(t)h(x) = 0, \tag{1.1}$$

where f, ϕ, ψ, g, e and h are continuous functions which depend only on the displayed arguments, $\phi(t, 0, 0) = \psi(t, 0) = g(t, 0) = h(0) = 0$. The dots indicate differentiation with respect to t and all solutions considered are assumed real.

Chukwu [3] discussed the stability of the solutions of the differential equation

$$x^{(5)} + ax^{(4)} + f_2(\dddot{x}) + c\ddot{x} + f_4(\dot{x}) + f_5(x) = 0\,.$$

In [1], sufficient conditions for the uniform global asymptotic stability of the zero solution of the differential equation

$$x^{(5)} + f_1(\dddot{x})x^{(4)} + f_2(\dddot{x}) + f_3(\ddot{x}) + f_4(\dot{x}) + f_5(x) = 0$$

were investigated.

Tiryaki & Tunc [6] and Tunc [7] studied the stability of the solutions of the differential equations

$$x^{(5)} + \phi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + b\dddot{x} + h(\dot{x}, \ddot{x}) + g(x, \dot{x}) + f(x) = 0\,,$$

$$x^{(5)} + \phi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + \psi(\ddot{x}, \dddot{x}) + h(\ddot{x}) + g(\dot{x}) + f(x) = 0\,.$$

We shall present here sufficient conditions, which ensure that all solutions of (1.1) tend to zero as $t \to \infty$. Many results have been obtained on asymptotic properties of non-autonomous equations of third order in Swich [5], Hara [4] and Yamamoto [8].

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2. Assumptions and theorems

We shall state the assumptions on the functions f, ϕ, ψ, g, e and h appeared in the equation (1.1).

Assumptions:

(1) $h(x)$ is a continuously differentiable function in $\Re^1$, and $e(t)$ is a continuously differentiable function in $\Re^+ = [0, \infty)$.

(2) The function $g(t, y)$ is continuous in $\Re^+ \times \Re^1$, and for the function $g(t, y)$ there exist non-negative functions $d(t)$, $g_0(y)$ and $g_1(y)$ which satisfy the inequalities
$$d(t)g_0(y) \leq g(t, y) \leq d(t)g_1(y)$$
for all $(t, y) \in \Re^+ \times \Re^1$. The function $d(t)$ is continuously differentiable in $\Re^+$. Let
$$\widetilde{g}(y) \equiv \frac{1}{2}\{g_0(y) + g_1(y)\},$$
$\widetilde{g}(y)$ and $\widetilde{g}'(y)$ are continuous in $\Re^1$.

(3) The function $\psi(t, z)$ is continuous in $\Re^+ \times \Re^1$. For the function $\psi(t, z)$ there exist non-negative functions $c(t)$, $\psi_0(z)$ and $\psi_1(z)$ which satisfy the inequalities
$$c(t)\psi_0(z) \leq \psi(t, z) \leq c(t)\psi_1(z)$$
for all $(t, z) \in \Re^+ \times \Re^1$. The function $c(t)$ is continuously differentiable in $\Re^+$. Let
$$\widetilde{\psi}(z) \equiv \frac{1}{2}\{\psi_0(z) + \psi_1(z)\},$$
$\widetilde{\psi}(z)$ is continuous in $\Re^1$.

(4) The function $\phi(t, z, w)$ is continuous in $\Re^+ \times \Re^2$. For the function $\phi(t, z, w)$ there exist non-negative functions $b(t)$, $\phi_0(z, w)$ and $\phi_1(z, w)$ which satisfy the inequalities
$$b(t)\phi_0(z, w) \leq \phi(t, z, w) \leq b(t)\phi_1(z, w)$$
for all $(t, z, w) \in \Re^+ \times \Re^2$. The function $b(t)$ is continuously differentiable in $\Re^+$. Let
$$\widetilde{\phi}(z, w) \equiv \frac{1}{2}\{\phi_0(z, w) + \phi_1(z, w)\},$$
$\widetilde{\phi}(z, w)$ and $\partial\widetilde{\phi}(z, w)/\partial z$ are continuous in $\Re^2$.

(5) The function $f(t, y, z, w)$ is continuous in $\Re^+ \times \Re^3$, and for the function $f(t, y, z, w)$ there exist functions $a(t)$, $f_0(y, z, w)$ and $f_1(y, z, w)$ which satisfy the inequality
$$a(t)f_0(y, z, w) \leq f(t, y, z, w) \leq a(t)f_1(y, z, w)$$
for all $(t, y, z, w) \in \Re^+ \times \Re^3$. Further the function $a(t)$ is continuously differentiable in $\Re^+$, and let
$$\widetilde{f}(y, z, w) \equiv \frac{1}{2}\{f_0(y, z, w) + f_1(y, z, w)\},$$
$\widetilde{f}(y, z, w)$ is continuous in $\Re^3$.

Theorem 1. *Further to the basic assumptions* (1)–(5), *suppose the following* ($\epsilon, \epsilon_1, \dots, \epsilon_5$ *are small positive constants*):

(i) $A \geq a(t) \geq a_0 \geq 1$, $B \geq b(t) \geq b_0 \geq 1$, $C \geq c(t) \geq c_0 \geq 1$, $D \geq d(t) \geq d_0 \geq 1$, $E \geq e(t) \geq e_0 \geq 1$, *for* $t \in \Re^+$.

(ii) $\alpha_1, \dots, \alpha_5$ *are some constants satisfying*

$$\alpha_1 > 0\,,\ \alpha_1\alpha_2 - \alpha_3 > 0\,,\ (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0\,,$$

$$\delta_0 := (\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0\,,\ \alpha_5 > 0\,; \tag{2.1}$$

$$\Delta_1 := \frac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \{\alpha_1\, d(t)\widetilde{g}'(y) - \alpha_5\} > 2\epsilon\alpha_2\,, \tag{2.2}$$

for all y *and all* $t \in \Re^+$;

$$\Delta_2 := \frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5)\gamma d(t)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{\epsilon}{\alpha_1} > 0\,, \tag{2.3}$$

for all y *and all* $t \in \Re^+$, *where*

$$\gamma := \begin{cases} \widetilde{g}(y)/y, & y \neq 0 \\ \widetilde{g}'(0), & y = 0. \end{cases} \tag{2.4}$$

(iii) $\epsilon_0 \leq \widetilde{f}(y,z,w) - \alpha_1 \leq \epsilon_1$ *for all* z *and* w.

(iv) $\widetilde{\phi}(0,0) = 0$, $0 \leq \widetilde{\phi}(z,w)/w - \alpha_2 \leq \epsilon_2 \quad (w \neq 0)$, $\frac{\partial}{\partial z}\widetilde{\phi}(z,w) \leq 0$.

(v) $\widetilde{\psi}(0) = 0$, $0 \leq \widetilde{\psi}(z)/z - \alpha_3 \leq \epsilon_3 \quad (z \neq 0)$.

(vi) $\widetilde{g}(0) = 0$, $\widetilde{g}(y)/y \geq \frac{E\alpha_4}{d_0} \quad (y \neq 0)$, $|\alpha_4 - \widetilde{g}'(y)| \leq \epsilon_4$ *for all* y *and*

$$\widetilde{g}'(y) - \widetilde{g}(y)/y \leq \alpha_5\delta_0/D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3) \quad (y \neq 0)\,.$$

(vii) $h(0) = 0$, $h(x)$ *sgn* $x > 0 (x \neq 0)$, $H(x) \equiv \int_0^x h(\xi)d\xi \to \infty$ *as* $|x| \to \infty$ *and*

$$0 \leq \alpha_5 - h'(x) \leq \epsilon_5 \quad \text{for all} \quad x\,.$$

(viii) $\int_0^\infty \beta_0(t)dt < \infty$, $e'(t) \to 0$ *as* $t \to \infty$, *where*

$$\beta_0(t) := b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)|\,,$$
$$b'_+(t) := \max\{b'(t), 0\} \quad \text{and} \quad c'_+(t) := \max\{c'(t), 0\}\,.$$

(ix)
$$|A(f_1 - f_0) + B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)| \leq \Delta(y^2 + z^2 + w^2 + u^2)^{1/2}\,,$$

where Δ *is a non-negative constant.*

Then every solution of (1.1) *satisfies*

$$x(t) \to 0, \dot{x}(t) \to 0\,, \quad \ddot{x}(t) \to 0\,, \quad \dddot{x}(t) \to 0\,, \quad x^{(4)}(t) \to 0 \quad as \quad t \to \infty\,.$$

Next, considering the equation

$$x^{(5)} + a(t)f(\dot{x}, \ddot{x}, \dddot{x})x^{(4)} + b(t)\phi(\dot{x}, \ddot{x}) + c(t)\psi(\ddot{x}) + d(t)g(\dot{x}) + e(t)h(x) = 0\,, \tag{2.5}$$

we can take the function $g(y)$ in place of $g_0(y)$ and $g_1(y)$; the function $\phi(y, z)$ in place of $\phi_0(y, z)$ and $\phi_1(y, z)$; the function $\psi(z)$ in place of $\psi_0(z)$ and $\psi_1(z)$, and the function $f(y, z, w)$ in place of $f_0(y, z, w)$ and $f_1(y, z, w)$ in the Assumptions $(2) - (5)$. Thus in this case the functions $\widetilde{g}(y), \widetilde{\phi}(y, z), \widetilde{\psi}(z), \widetilde{f}(y, z, w)$ coincide with $g(x, y), \phi(y, z), \psi(z), f(y, z, w)$ respectively. Thus from Theorem 1, we have

Theorem 2. *Suppose that the functions $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $e(t)$ are continuously differentiable in $\Re^+$, and the functions $h(x)$, $g(x, y)$, $\phi(y, z)$, $\psi(z)$, $f(y, z, w)$, $g'(y)$, $h'(x)$, $\frac{\partial}{\partial z}\phi(y, z)$ and that these functions satisfy the following conditions:*

(i) $A \ge a(t) \ge a_0 \ge 1$, $B \ge b(t) \ge b_0 \ge 1$, $C \ge c(t) \ge c_0 \ge 1$, $D \ge d(t) \ge d_0 \ge 1$, $E \ge e(t) \ge e_0 \ge 1$ *for* $t \in \Re^+$.

(ii) $\alpha_1, \dots, \alpha_5$ *are some constants satisfying*

$$\alpha_1 > 0\,, \quad \alpha_1\alpha_2 - \alpha_3 > 0\,, \quad (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0\,,$$
$$\delta_0 := (\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0\,, \quad \alpha_5 > 0\,;$$
$$\Delta_1 := \frac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \{\alpha_1 d(t)\ g'(y) - \alpha_5\} > 2\epsilon\alpha_2\,,$$

for all y and all $t \in \Re^+$;

$$\Delta_2 := \frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5)\gamma d(t)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{\epsilon}{\alpha_1} > 0\,,$$

for all y and all $t \in \Re^+$, where

$$\gamma := \begin{cases} g(y)/y\,, & y \ne 0 \\ g'(0)\,, & y = 0\,. \end{cases}$$

(iii) $\epsilon_0 \le f(y, z, w) - \alpha_1 \le \epsilon_1$, *for all* z *and* w.

(iv) $\phi(0, 0) = 0$, $0 \le \phi(z, w)/w - \alpha_2 \le \epsilon_2$ $(w \ne 0)$, $\frac{\partial}{\partial z}\phi(z, w) \le 0$.

(v) $\psi(0) = 0$, $0 \le \psi(z)/z - \alpha_3 \le \epsilon_3$ $(z \ne 0)$.

(vi) $g(0) = 0$, $g(y)/y \ge \frac{E\alpha_4}{d_0}$ $(y \ne 0)$, $|\alpha_4 - g'(y)| \le \epsilon_4$ *for all* y *and*

$$g'(y) - g(y)/y \le \alpha_5\delta_0/D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3) \qquad (y \ne 0)\,.$$

(vii) $h(0) = 0$, $h(x)\ \operatorname{sgn} x > 0 \quad (x \neq 0)$, $H(x) \equiv \int_0^x h(\xi)\, d\xi \to \infty$ *as* $|x| \to \infty$ *and*

$$0 \leq \alpha_5 - h'(x) \leq \epsilon_5 \quad \textit{for all} \quad x\,.$$

(viii) $\int_0^\infty \beta_0(t)\, dt < \infty$, $e'(t) \to 0$ *as* $t \to \infty$, *where*

$$\beta_0(t) : = b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)|\,,$$
$$b'_+(t) : = \max\{b'(t), 0\} \quad \textit{and} \quad c'_+(t) := \max\{c'(t), 0\}\,.$$

Then every solution of (2.5) *satisfies*

$$x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), x^{(4)}(t) \to 0 \quad as \quad t \to \infty\,.$$

3. The Lyapunov function $V_0(t, x, y, z, w, u)$

We consider, in place of (1.1), the equivalent system

$$\begin{aligned} &\dot{x} = y\,, \quad \dot{y} = z\,, \quad \dot{z} = w\,, \quad \dot{w} = u\,, \\ &\dot{u} = -f(t, y, z, w)u - \phi(t, z, w) - \psi(t, z) - g(t, y) - e(t)h(x)\,. \end{aligned} \tag{3.1}$$

The proof of the theorem is based on some fundamental properties of a continuously differentiable function $V_0 = V_0(t, x, y, z, w, u)$ defined by

$$\begin{aligned} 2V_0 = {} & u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz + 2\delta yu + 2b(t)\int_0^w \widetilde{\phi}(z, \omega)\, d\omega \\ & + \Big\{\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\Big\} w^2 + 2\Big\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\Big\} wz \\ & + 2\alpha_1\delta wy + 2d(t)w\widetilde{g}(y) + 2e(t)wh(x) + 2\alpha_1 c(t)\int_0^z \widetilde{\psi}(\zeta)\, d\zeta \\ & + \Big\{\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta\Big\} z^2 + 2\delta\alpha_2 yz + 2\alpha_1 d(t)z\widetilde{g}(y) - 2\alpha_5 yz \\ & + 2\alpha_1 e(t)zh(x) + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\, d(t)\int_0^y \widetilde{g}(\eta)\, d\eta + (\delta\alpha_3 - \alpha_1\alpha_5)y^2 \\ & + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} e(t)yh(x) + 2\delta e(t)\int_0^x h(\xi)\, d\xi\,, \end{aligned} \tag{3.2}$$

where

$$\delta := \alpha_5(\alpha_1\alpha_2 - \alpha_3)/(\alpha_1\alpha_4 - \alpha_5) + \epsilon\,. \tag{3.3}$$

The properties of the function $V_0 = V_0(t, x, y, z, w, u)$ are summarized in Lemma 1 and Lemma 2.

Lemma 1. *Subject to the hypotheses* (i)–(vii) *of the theorem, there are positive constants D_7 and D_8 such that*

$$D_7\{H(x)+y^2+z^2+w^2+u^2\} \le V_0 \le D_8\{H(x)+y^2+z^2+w^2+u^2\}\,. \tag{3.4}$$

Proof. We observe that $2V_0$ in (3.2) can be rearranged as

$$\begin{aligned}2V_0 = &\Big\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\Big\}^2\\ &+\frac{\alpha_4(\alpha_1\alpha_4-\alpha_5)}{(\alpha_1\alpha_2-\alpha_3)\gamma d(t)}\Big\{\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}e(t)h(x)\\ &+\frac{\alpha_1\alpha_2-\alpha_3}{\alpha_1\alpha_4-\alpha_5}\gamma d(t)y+\frac{\alpha_1}{\alpha_4}\gamma d(t)z\\ &+\frac{1}{\alpha_4}\gamma d(t)w\Big\}^2+\frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\Big(z+\frac{\alpha_5}{\alpha_4}y\Big)^2+\Delta_2(w+\alpha_1 z)^2\\ &+2\epsilon\Big(\frac{\alpha_4\alpha_3-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\Big)yz+\sum_{i=1}^{4}S_i\,,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}S_1 &:= 2\delta e(t)\int_0^x h(\xi)\,d\xi-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)\gamma d(t)}e^2(t)h^2(x)\,,\\ S_2 &:= \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{\alpha_1\alpha_4-\alpha_5}\Big\{2\int_0^y \widetilde{g}(\eta)\,d\eta-y\widetilde{g}(y)\Big\}\\ &\quad+\Big\{\delta\alpha_3-\alpha_1\alpha_5-\frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)^2}-\delta^2\Big\}y^2\,,\\ S_3 &:= \frac{\epsilon}{\alpha_1}w^2+2b(t)\int_0^w \widetilde{\phi}(z,\omega)\,d\omega-\alpha_2 w^2\,,\\ S_4 &:= 2\alpha_1 c(t)\int_0^z \widetilde{\psi}(\zeta)\,d\zeta-\alpha_1\alpha_3 z^2\,.\end{aligned}$$

It can be seen from the estimates arising in the course of the proof of [2; Lemma 1] that

$$2\alpha_5\int_0^x h(\xi)\,d\xi-h^2(x)\ge 0\,, \tag{3.6}$$

$$S_1 \ge 2\epsilon e_0\int_0^x h(\xi)\,d\xi\,.$$

Since

$$y\widetilde{g}(y)\equiv\int_0^y \widetilde{g}(\eta)\,d\eta+\int_0^y \eta\widetilde{g}'(\eta)\,d\eta\,,$$

we have

$$
\begin{aligned}
S_2 &= \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{\alpha_1\alpha_4-\alpha_5}\Big\{2\int_0^y \widetilde{g}(\eta)\,d\eta - y\widetilde{g}(y)\Big\} \\
&\quad + \Big[\frac{\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - \epsilon\Big\{\epsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\Big\}\Big]y^2 \\
&= \int_0^y \Big[\frac{2\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{\alpha_1\alpha_4-\alpha_5}\Big\{\widetilde{g}'(\eta) - \frac{\widetilde{g}(\eta)}{\eta}\Big\} \\
&\quad - 2\epsilon\Big\{\epsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\Big\}\Big]\eta\,d\eta \\
&\geq \int_0^y \Big[\frac{\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - 2\epsilon\Big\{\epsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\Big\}\Big]\eta\,d\eta\,,
\end{aligned}
$$

by (vi) and (i)

$$
\geq \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2\,,
$$

provided that

$$
\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)} \geq \epsilon\Big\{\epsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\Big\}\,, \tag{3.7}
$$

which we now assume. From (i), (iv) and (v) we find

$$
\begin{aligned}
S_3 &= \frac{\epsilon}{\alpha_1}w^2 + 2b(t)\int_0^w \widetilde{\phi}(z,\omega)d\omega - \alpha_2 w^2 \\
&\geq \frac{\epsilon}{\alpha_1}w^2 + 2\int_0^w \Big\{\frac{\widetilde{\phi}(z,\omega)}{\omega} - \alpha_2\Big\}\omega\,d\omega \geq \frac{\epsilon}{\alpha_1}w^2\,, \\
S_4 &= 2\alpha_1 c(t)\int_0^z \widetilde{\psi}(\zeta)d\zeta - \alpha_1\alpha_3 z^2 \geq 2\alpha_1\int_0^z \Big\{\frac{\widetilde{\psi}(\zeta)}{\zeta} - \alpha_3\Big\}\zeta\,d\zeta \geq 0\,.
\end{aligned}
$$

On gathering all of these estimates into (3.5) we deduce

$$
\begin{aligned}
2V_0 &\geq \Big\{u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z + \delta y\Big\}^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\Big(z + \frac{\alpha_5}{\alpha_4}y\Big)^2 \\
&\quad + \Delta_2(w+\alpha_1 z)^2 + 2\epsilon e_0\int_0^x h(\xi)\,d\xi + \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2 + \frac{\epsilon}{\alpha_1}w^2 \\
&\quad + 2\epsilon\Big(\frac{\alpha_4\alpha_3-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\Big)yz\,,
\end{aligned}
$$

by (ii) and (vi). It is clear that there exist sufficiently small positive constants $D_1, \ldots, D_5$ such that

$$
2V_0 \geq D_1 H(x) + 2D_2 y^2 + 2D_3 z^2 + D_4 w^2 + D_5 u^2 + +2\epsilon\Big(\frac{\alpha_4\alpha_3-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\Big)yz\,.
$$

Let

$$S_5 := D_2 y^2 + 2\epsilon\Big(\frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\Big) yz + D_3 z^2.$$

By using the inequality $|yz| \leq \frac{1}{2}(y^2 + z^2)$, we obtain

$$S_5 \geq D_2 y^2 + D_3 z^2 - \epsilon\Big(\frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\Big)(y^2 + z^2) \geq D_6(y^2 + z^2)\,,$$

for some $D_6 > 0, D_6 = \frac{1}{2}\min\{D_2, D_3\}$, if

$$\epsilon \leq (\alpha_1\alpha_4 - \alpha_5)/\big(2(\alpha_4\alpha_3 - \alpha_2\alpha_5)\big)\min\{D_2, D_3\}\,, \tag{3.8}$$

which we also assume. Then

$$2V_0 \geq D_1 H(x) + (D_2 + D_6)y^2 + (D_3 + D_6)z^2 + D_4 w^2 + D_5 u^2\,.$$

Consequently there exists a positive constant D_7 such that

$$V_0 \geq D_7\{H(x) + y^2 + z^2 + w^2 + u^2\}\,,$$

provided ϵ is so small that (3.7) and (3.8) hold. From (i), (iv), (v),(vi) and (3.6) we can verify that there exists a positive constant D_8 satisfying

$$V_0 \leq D_8\{H(x) + y^2 + z^2 + w^2 + u^2\}\,.$$

Thus (3.4) follows. □

Lemma 2. *Assume that all conditions of the theorem hold. Then there exist positive constants D_i $(i = 11, 12)$ such that*

$$\dot V_0 \leq -D_{12}(y^2 + z^2 + w^2 + u^2) + D_{11}\beta_0 V_0\,. \tag{3.9}$$

Proof. From (3.2) and (3.1) it follows that (for $y, z, w \neq 0$)

$$
\begin{aligned}
\frac{d}{dt}V_0 \le\; & -u^2\{a(t)\widetilde{f}(y,z,w)-\alpha_1\} \\
& - w^2\Big[\alpha_1\frac{b(t)\widetilde{\phi}(z,w)}{w} - \Big\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\Big\}\Big] \\
& - z^2\Big[\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)c(t)}{\alpha_1\alpha_4-\alpha_5}\frac{\widetilde{\psi}(z)}{z} - \{\delta\alpha_2 + \alpha_1 d(t)\widetilde{g}'(y) - \alpha_5\}\Big] \\
& - y^2\Big\{\delta d(t)\frac{\widetilde{g}(y)}{y} - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)h'(x)\Big\} \\
& + wb(t)\int_0^w \frac{\partial}{\partial z}\widetilde{\phi}(z,\omega)\,d\omega - \alpha_1 wua(t)\{\widetilde{f}(y,z,w)-\alpha_1\} - uzc(t)\Big\{\frac{\widetilde{\psi}(z)}{z} - \alpha_3\Big\} \\
& - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uza(t)\{\widetilde{f}(y,z,w)-\alpha_1\} \\
& - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}wzb(t)\Big\{\frac{\widetilde{\phi}(z,w)}{w} - \alpha_2\Big\} \\
& - wzd(t)\{\alpha_4 - \widetilde{g}'(y)\} - \delta yua(t)\{\widetilde{f}(y,z,w)-\alpha_1\} - ywe(t)\{\alpha_5 - h'(x)\} \\
& - \delta ywb(t)\Big\{\frac{\widetilde{\phi}(z,w)}{w} - \alpha_2\Big\} - \alpha_1 yze(t)\{\alpha_5 - h'(x)\} - \delta yzc(t)\Big\{\frac{\widetilde{\psi}(z)}{z} - \alpha_3\Big\} \\
& + \Big\{\alpha_1^2 uw + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uz + \alpha_1\delta yu\Big\}\{1-a(t)\} \\
& + \Big\{\frac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}wz + \delta\alpha_2 yw\Big\}\{1-b(t)\} + (\alpha_3 uz + \delta\alpha_3 yz)\{1-c(t)\} \\
& - \alpha_4 wz\{1-d(t)\} - (\alpha_5 yw + \alpha_1\alpha_5 yz)\{1-e(t)\} \\
& + \frac{1}{2}\{a(t)(f_1-f_0) + b(t)(\phi_1-\phi_0) + c(t)(\psi_1-\psi_0) + d(t)(g_1-g_0)\} \\
& \Big\{u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z + \delta y\Big\} + \frac{\partial V_0}{\partial t}\,.
\end{aligned}
\tag{3.10}
$$

By (i) and (iii), $a(t)\widetilde{f}(y,z,w) - \alpha_1 \ge \epsilon_0$. From (i), (iv) and (3.3) we have (for $w \neq 0$)

$$
\begin{aligned}
& \alpha_1\frac{b(t)\widetilde{\phi}(z,w)}{w} - \Big\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\Big\} \\
& \qquad \ge \alpha_1\Big\{\frac{\widetilde{\phi}(z,w)}{w} - \alpha_2\Big\} + \Big\{\alpha_1\alpha_2 - \alpha_3 + \delta - \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\Big\} \ge \epsilon\,.
\end{aligned}
$$

By using (i), (v), (3.3) and (2.2) we obtain (for $z \neq 0$)

$$
\begin{aligned}
& \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)c(t)}{\alpha_1\alpha_4-\alpha_5}\frac{\widetilde{\psi}(z)}{z} - \{\delta\alpha_2 + \alpha_1 d(t)\widetilde{g}'(y) - \alpha_5\} \\
& \qquad \ge \frac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \{\alpha_1 d(t)\widetilde{g}'(y) - \alpha_5\} - \epsilon\alpha_2 \ge \epsilon\alpha_2\,.
\end{aligned}
$$

From (i), (vi) and (vii) we find (for $y \neq 0$)

$$\begin{aligned}\delta d(t)\frac{\widetilde{g}(y)}{y} &- \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)h'(x)\\ &\geq \epsilon\alpha_4 E + \frac{\alpha_4 E(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\{\alpha_5 - h'(x)\} \geq \epsilon\alpha_4 E\,.\end{aligned}$$

Therefore, the first four terms involving u^2, w^2, z^2 and y^2 in (3.10) are majorizable by

$$-(\epsilon_0 u^2 + \epsilon w^2 + \epsilon\alpha_2 z^2 + \epsilon\alpha_4 E y^2)\,.$$

Let $R(t,x,y,z,w,u)$ denote the sum of the remaining terms in (3.10). By using hypotheses (i), (iii)–(vii) and the inequalities

$$\begin{aligned}&|uw| \leq \frac{1}{2}(u^2+w^2)\,, \quad |uz| \leq \frac{1}{2}(u^2+z^2)\,, \quad |uy| \leq \frac{1}{2}(u^2+y^2)\,,\\ &|wz| \leq \frac{1}{2}(w^2+z^2)\,, \quad |wy| \leq \frac{1}{2}(w^2+y^2)\,, \quad |yz| \leq \frac{1}{2}(y^2+z^2)\,;\end{aligned}$$

it follows that

$$\begin{aligned}|R(t,x,y,z,w,u)| &\leq D_9(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)(y^2+z^2+w^2+u^2)\\ &\quad + \frac{1}{2}\{a(t)(f_1-f_0)+b(t)(\phi_1-\phi_0)+c(t)(\psi_1-\psi_0)+d(t)(g_1-g_0)\}\\ &\qquad \Big\{u+\alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\Big\} + \frac{\partial V_0}{\partial t}\,,\end{aligned}$$

for some $D_9 > 0$. Thus, after substituting in (3.10), one obtains

$$\begin{aligned}\dot{V}_0 &\leq -(\epsilon_0 u^2+\epsilon w^2+\epsilon\alpha_2 z^2+\epsilon\alpha_4 E y^2)+D_9(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)(y^2+z^2+w^2+u^2)\\ &\quad + \Big|\frac{1}{2}\{a(t)(f_1-f_0)+b(t)(\phi_1-\phi_0)+c(t)(\psi_1-\psi_0)+d(t)(g_1-g_0)\}\\ &\qquad \Big\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\Big\}\Big| + \frac{\partial V_0}{\partial t}\\ &\leq -\frac{1}{2}\min\{\epsilon_0,\epsilon,\epsilon\alpha_2,\epsilon\alpha_4 E\}(y^2+z^2+w^2+u^2)\\ &\quad + \Big|\frac{1}{2}\{a(t)(f_1-f_0)+b(t)(\phi_1-\phi_0)+c(t)(\psi_1-\psi_0)+d(t)(g_1-g_0)\}\\ &\qquad \Big\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\Big\}\Big| + \frac{\partial V_0}{\partial t}\,,\end{aligned} \tag{3.11}$$

provided that

$$D_9(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5) \leq \frac{1}{2}\min\{\epsilon_0,\epsilon,\epsilon\alpha_2,\epsilon\alpha_4 E\}\,. \tag{3.12}$$

Now we assume that D_9 and $\epsilon_1, \dots, \epsilon_5$ are so small that (3.12) holds. The case $y, z, w = 0$ is trivially dealt with. From (3.2) we find

$$\begin{aligned}\frac{\partial V_0}{\partial t} &= b'(t)\int_0^w \widetilde{\phi}(z,\omega)\,d\omega + \alpha_1 c'(t)\int_0^z \widetilde{\psi}(\zeta)\,d\zeta \\ &\quad + d'(t)\Big\{w\widetilde{g}(y) + \alpha_1 z\widetilde{g}(y) + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^y \widetilde{g}(\eta)\,d\eta\Big\} \\ &\quad + e'(t)\Big\{wh(x) + \alpha_1 zh(x) + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}yh(x) + 2\delta\int_0^x h(\xi)\,d\xi\Big\}.\end{aligned}$$

From (iv), (v), (vi), (3.6) and (3.4) we can find a positive constant D_{10} which satisfies

$$\frac{\partial V_0}{\partial t} \le D_{10}\{b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)|\}\{H(x) + y^2 + z^2 + w^2\}$$

$$\le D_{11}\beta_0 V_0\,, \tag{3.13}$$

where $D_{11} = \frac{D_{10}}{D_7}$. Let

$$D_{12} = \frac{1}{4}\min\{\epsilon_0, \epsilon, \epsilon\alpha_2, \epsilon\alpha_4 E\}, \quad \text{and} \quad D_{13} = \max\Big\{1, \alpha_1, \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}, \delta\Big\}$$

then from (3.11), (3.13) and (ix) we obtain the estimate

$$\dot{V}_0 \le -2D_{12}(y^2+z^2+w^2+u^2) + 2D_{13}\Delta(y^2+z^2+w^2+u^2) + D_{11}\beta_0 V_0\,.$$

Let Δ be fixed, in what follows, to satisfy $\Delta = \frac{D_{12}}{2D_{13}}$. With this limitation on Δ we find

$$\dot{V}_0 \le -D_{12}(y^2+z^2+w^2+u^2) + D_{11}\beta_0 V_0\,. \tag{3.14}$$

Now (3.9) is verified and the lemma is proved.

4. Completion of the proof of Theorem 1

Define the function $V(t,x,y,z,w,u)$ as follows

$$V(t,x,y,z,w,u) = e^{-\int_0^t D_{11}\beta_0(\tau)d\tau}V_0(t,x,y,z,w,u)\,. \tag{4.1}$$

Then one can verify that there exist two functions U_1 and U_2 satisfying

$$U_1(\|\bar{x}\|) \le V(t,x,y,z,w,u) \le U_2(\|\bar{x}\|)\,, \tag{4.2}$$

for all $\bar{x} = (x,y,z,w,u) \in \Re^5$ and $t \in \Re^+$; where U_1 is a continuous increasing positive definite function, $U_1(r) \to \infty$ as $r \to \infty$ and U_2 is a continuous increasing function.

Along any solution (x,y,z,w,u) of (3.1) we have

$$\begin{aligned}\dot{V} &= e^{-\int_0^t D_{11}\beta_0(\tau)d\tau}\{\dot{V}_0 - \beta(t)V_0\} \\ &\le -D_{12}e^{-\int_0^t D_{11}\beta_0(\tau)d\tau}(y^2+z^2+w^2+u^2)\,.\end{aligned}$$

Thus we can find a positive constant D_{14} such that

$$\dot{V} \le -D_{14}(y^2+z^2+w^2+u^2)\,. \tag{4.3}$$

From the inequalities (4.2) and (4.3), we obtain the uniform boundedness of all solutions (x,y,z,w,u) of (3.1) [9; Theorem 10.2].

Auxiliary lemma

Consider a system of differential equations

$$\dot{\bar{x}} = F(t, \bar{x})\,, \tag{4.4}$$

where $F(t, \bar{x})$ is continuous on $\Re^+ \times \Re^n$, $F(t, \bar{0}) = \bar{0}$.

The following lemma is well-known [9].

Lemma 3. *Suppose that there exists a non-negative continuously differentiabla scalar function $V(t, \bar{x})$ on $\Re^+ \times \Re^n$ such that $\dot{V}_{(4.4)} \leq -U(\|\bar{x}\|)$, where $U(\|\bar{x}\|)$ is positive definite with respect to a closed set Ω of $\Re^n$. Moreover, suppose that $F(t, \bar{x})$ of system (4.4) is bounded for all t when $\bar{x}$ belongs to an arbitrary compact set in $\Re^n$ and that $F(t, \bar{x})$ satisfies the following two conditions with respect to Ω:*

(1) $F(t, \bar{x})$ tends to a function $H(\bar{x})$ for $\bar{x} \in \Omega$ as $t \to \infty$, and on any compact set in Ω this convergence is uniform.

(2) corresponding to each $\epsilon > 0$ and each $\bar{y} \in \Omega$, there exist a $\delta, \delta = \delta(\epsilon, \bar{y})$ and $T, T = T(\epsilon, \bar{y})$ such that if $t \geq T$ and $\|\bar{x} - \bar{y}\| < \delta$, we have $\|F(t, \bar{x}) - F(t, \bar{y})\| < \epsilon$.

Then every bounded solution of (4.4) approaches the largest semi-invariant set of the system $\bar{x} = H(\bar{x})$ contained in Ω as $t \to \infty$.

From the system (3.1) we set

$$F(t, \bar{x}) = \begin{bmatrix} y \\ z \\ w \\ u \\ -f(t, y, z, w)u - \phi(t, z, w) - \psi(t,z) - g(t, y) - e(t)h(x) \end{bmatrix}.$$

It is clear that F satisfies the conditions of Lemma 3. Let $U(\|\bar{x}\|) = D_{14}(y^2 + z^2 + w^2 + u^2)$, then

$$\dot{V}(t, x, y, z, w, u) \leq -U(\|\bar{x}\|) \tag{4.5}$$

and $U(\|\bar{x}\|)$ is positive definite with respect to the closed set $\Omega := \{(x, y, z, w, u) \mid x \in \Re,\ y = 0,\ z = 0,\ w = 0,\ u = 0\}$. It follows that in Ω

$$F(t, \bar{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -e(t)h(x) \end{bmatrix}. \tag{4.6}$$

According to condition (viii) of the theorem and the boundedness of e, we have $e(t) \to e_\infty$ as $t \to \infty$, where $1 \le e_0 \le e_\infty \le E$. If we set

$$H(\bar{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -e_\infty h(x) \end{bmatrix}, \tag{4.7}$$

then the conditions on $H(\bar{x})$ of Lemma 3 are satisfied. Since all solutions of (3.1) are bounded, it follows from Lemma 3 that every solution of (3.1) approaches the largest semi-invariant set of the system $\dot{\bar{x}} = H(\bar{x})$ contained in Ω as $t \to \infty$. From (4.7); $\dot{\bar{x}} = H(\bar{x})$ is the system

$$\dot{x} = 0,\ \dot{y} = 0,\ \dot{z} = 0,\ \dot{w} = 0 \quad \text{and} \quad \dot{u} = -e_\infty h(x)\,,$$

which has the solutions

$$x = k_1,\ y = k_2,\ z = k_3,\ w = k_4, \quad \text{and} \quad u = k_5 - e_\infty h(k_1)(t - t_0)\,.$$

In order to remain in Ω, the above solutions must satisfy

$$k_2 = 0,\ k_3 = 0,\ k_4 = 0 \quad \text{and} \quad k_5 - e_\infty h(k_1)(t - t_0) = 0 \quad \text{for all} t \ge t_0\,,$$

which implies $k_5 = 0$, $h(k_1) = 0$, and thus $k_1 = k_5 = 0$.

Therefore the only solution of $\dot{\bar{x}} = H(\bar{x})$ remaining in Ω is $\bar{x} = \bar{0}$, that is, the largest semi-invariant set of $\dot{\bar{x}} = H(\bar{x})$ contained in Ω is the point $(0, 0, 0, 0, 0)$. Consequently we obtain

$$x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), x^{(4)}(t) \to 0 \quad \text{as} \quad t \to \infty\,.$$

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