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ON THE NUMBER OF PERIODIC SOLUTIONS OF A GENERALIZED PENDULUM EQUATION

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Abstract. For a generalized pendulum equation we estimate the number of periodic solutions from below using lower and upper solutions and from above using a complex equation and Jensen's inequality.

Introduction

We are interested in an estimation from above of the number of periodic solutions of a generalized pendulum equation.

Our paper is motivated by a paper of Ortega [5], where a method of such estimation in case of classical pendulum equation is developed.

The estimation method of Ortega is based on use of a complex differential equation, Ljapunov-Schmidt reduction and Jensen's inequality for counting of zeros of an analytic complex function on a unit disc.

We use the same method in our case.

We deal with a generalized pendulum equation

$$x'' + cx' + g(x) = f(t) + s\,, \tag{1}$$

where $g \in C^\infty(R,R)$ is a 2π-periodic function such that $\int_{-\pi}^{\pi} g(x)\,dx = 0$, $f : R \to R$ is a 1–periodic continuous function such that $\int_0^1 f(t)\,dt = 0$ and $c, s \in R$.

We are interested in the number of 1-periodic solutions of the equation (1). The form of the equation (1) implies that if $x(t)$ is a 1-periodic solution then $x(t)+2k\pi$ is also a 1-periodic solution for each k integer. Estimating the number of solutions we regard such solutions as the same solution.

Assume $s \in R$ be such that there is a periodic solution of (1). Integrating the equation (1) we obtain that

$$-G \le s \le G\,,$$

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where $G = \max_{x\in R} |g(x)|$.

Let $S = \{s \in [-G, G]$, such that there is a periodic solution of (1)$\}$. We denote

$$s^- = \inf S, \qquad s^+ = \sup S\,. \tag{2}$$

Let $s^- < s_1 < s < s_2 < s^+$ and assume $s_1, s_2 \in S$. Then there exist 1-periodic solutions x_i of the equations $x'' + cx' + g(x) = f(t) + s_i$.

Moreover x_1 is a strict upper solution of (1) and x_2 a strict lower solution of (1). Periodicity of g implies that $x_1 + 2k\pi$ is also a strict upper solution. We choose k such that $x_2 < x_1 + 2k\pi$. The existence of a strict lower solution less then an upper one implies that there is a solution of the equation (1) (see [3], [6]). That means $(s^-, s^+) \subset S$ and using limitation in integral equation we obtain $S = [s^-, s^+]$.

Moreover for $c = 0$ it can be shown that $0 \in S$. The proof of this assertion we postpone at the end of the paper in Remark 2.

The estimation from below

It is possible to estimate the number of solutions from below by the method based on the existence of a lower and upper solutions. (Cf. [3], [6].) The following two lemmas can be found in [7].

Lemma 1 ([7]). *Let $|f(t, x, y)| < M$ and let α, β, $\alpha < \beta$ be strict lower and upper solutions of the boundary value problem*

$$x'' + cx' = f(t, x, x')\,,$$
$$x(0) = x(1), \qquad x'(0) = x'(1)\,.$$

Then there is a solution $x(t)$ such that $\alpha(t) < x(t) < \beta(t)$.

Lemma 2 ([7]). *Let $|f(t, x, y)| < M$ and let α, β, $\alpha \nleq \beta$ be strict lower and upper solutions of the boundary value problem*

$$x'' + cx' = f(t, x, x')\,,$$
$$x(0) = x(1), \qquad x'(0) = x'(1)\,.$$

Then there is a solution $x(t)$ and points $t_a, t_b \in (0, 1)$ such that $x(t_a) < \alpha(t_a)$, $x(t_b) > \beta(t_b)$.

Theorem 1. *For each $s \in (s_-, s_+)$ there are at least two periodic solutions of the equation* (1).

Proof. For each $s \in (s_-, s_+)$ there are $s_1, s_2 \in (s_-, s_+)$, $s_1 < s < s_2$. Let x_i be periodic solution of the equation $x'' + cx' + g(x) = f(t) + s_i$.

Then $x_1 + 2k\pi$ is a strict upper solution of (1) and x_2 a strict lower solution of (1). We choose k such that $x_2 < x_1 + 2k\pi$ and $x_2 \not< x_1 + 2(k-1)\pi$. Then Lemma 1 implies there is a solution $x(t)$ of (1) such that $x_2 < x(t) < x_1 + 2k\pi$.

As $x_2 \nleq x_1 + 2(k-1)\pi$, Lemma 2 implies there is a solution $y(t)$ such that there are $t_a, t_b \in [0, 1]$, $y(t_a) < x_2(t_a)$, $y(t_b) > x_1(t_b) + 2(k-1)\pi$.

Clearly $x(t) \neq y(t) + 2m\pi$. $\square$

The complex equation

Let

$$g(x)=\sum_{n=1}^{\infty}(a_n\sin nx+b_n\cos nx) \tag{3}$$

be the Fourier expansion of the periodic function $g(x)$.

We assume that there is a positive constant $d\in R$ and a sequence $\{c_n\}_{n=1}^{\infty}$, $c_n\ge 0$, $\sum_{n=1}^{\infty}c_n<\infty$, such that

$$|a_n|+|b_n|\le c_ne^{-nd}\,. \tag{4}$$

Then the function $g(x)$ can be extended to an analytic complex function

$$g(z)=\sum_{n=1}^{\infty}(a_n\sin nz+b_n\cos nz)$$

defined on the set $B_d=\{z,\ |\operatorname{Im} z|<d\}$.

Moreover (4) implies that $g(x)$ is a C^{∞} function.

Example. Let $r:(-1,1)\to R$ be an analytic function, $r(x)=\sum_{n=0}^{\infty}a_nx^n$ for $x\in(-1,1)$. Then the functions

$$g_1=\frac{1}{2}\big(r(qe^{ix})+r(qe^{-ix})\big)=\sum_{n=0}^{\infty}a_nq^n\cos nx\,,$$
$$g_2=\frac{1}{2i}\big(r(qe^{ix})-r(qe^{-ix})\big)=\sum_{n=0}^{\infty}a_nq^n\sin nx$$

satisfy for $|q|<1$ the condition (4).

For example for $r(x)=\dfrac{1}{1-x}$ we get

$$g_1(x)=\frac{1-q\cos x}{1-2q\cos x+q^2}\,,\quad g_2(x)=\frac{q\sin x}{1-2q\cos x+q^2}\,.$$

We deal with the complex equation

$$z''+cz'+g(z)=f(t)+s\,, \tag{5}$$

where $z:R\to C$.

Each real valued solution of (5) is a solution of (1). Therefore the number of periodic solutions of (1) is estimated from above by the number of periodic solutions of (5).

An abstract form of the equation (5) is an operator equation

$$Lz + Nz = f + s\,, \tag{6}$$

where $L : D(L) \subset Z \to Z$, $Lz = z'' + cz'$ is a linear operator, $Z = \{z(t),\ z : R \to C$ is a continuous 1-periodic function$\}$ is a complex Banach space with the norm $\|z\| = \max_{t\in R} |z(t)|$, $D(L)$ is a subspace of two times continuously differentiable functions, $N : Z \to Z$, $Nz = g(z)$ is a nonlinear operator, $f \in Z$ is real-valued (recall that $\int_0^1 f(t)\,dt = 0$), c and s are real numbers.

The kernel of L is $\ker L = C$. We denote by W the subspace $W = \{w \in Z,\ \int_0^1 w(t)\,dt = 0\}$.

Then $Z = C \oplus W$ and $z(t) = z_0 + w(t)$.

We use the Ljapunov-Schmidt reduction. The operator equation (6) is equivalent to the pair

$$QN(z) = s\,, \tag{7}$$

$$w = -K(I-Q)N(z_0 + w) + F(t) \tag{8}$$

where $K : W \to W$ is a right inverse operator to $L|_W$, $Q : Z \to C$ is a projection on C, $Qz = \int_0^1 z(t)\,dt$ and $F = K(I-Q)f$.

Lemma 3. *Let $\tilde{K} = -K(I-Q) : Z \to W$.*

For $c = 0$ the operator norm $\|\tilde{K}\| = \dfrac{1}{18\sqrt{3}}$.

For $c \neq 0$ the operator norm $\|\tilde{K}\| = \int_0^1 |\frac{1}{2c} - \frac{1}{c^2} + \frac{1}{c(e^c-1)}e^{ct} - \frac{t}{c}|\,dt$.

Proof. The case $c = 0$ is proved in [5]. Similarly we prove the case $c \neq 0$.

Let $p \in Z$ be such that $\tilde{K}p = w$, i.e. $w'' + cw' = -(I-Q)p$. Denote $\phi(t) \in W$,

$$\phi(t) = \frac{1}{2c} - \frac{1}{c^2} + \frac{1}{c(e^c - 1)}e^{ct} - \frac{t}{c}\,.$$

The function $\phi(t)$ is a solution of the boundary value problem

$$\begin{aligned} \phi'' - c\phi' &= 1\,, \\ \phi(0) &= \phi(1)\,, \\ \phi'(1) - c\phi(1) = -(\phi'(0) - c\phi(0)) &= \frac{1}{2}\,. \end{aligned}$$

Then

$$\int_0^1 p\phi\,dt = \int_0^1 (I-Q)p\phi\,dt = -\int_0^1 (w'' + cw')\phi\,dt = w(0)\,.$$

Due to periodicity we can assume that $\|w\| = |w(0)|$. Then

$$\|w\| \le \int_0^1 |\phi|\, dt\ \|p\| .$$

That means $\|\tilde{K}\| \le \int_0^1 |\phi|\, dt$.

Now we choose $p_n \in Z$, $\|p_n\| = 1$, $p_n \to \operatorname{sgn} \phi(t)$ a.e. $t \in (0,1)$. Computing the limit in the integral

$$\int_0^1 p_n \phi\, dt = w_n(0)$$

we obtain that $\|\tilde{K}\| \ge \int_0^1 |\phi|\, dt$. □

Now we prove that by the auxiliary equation (8) there is defined a contractive operator T_{z_0} for a suitable fixed z_0.

Let σ, η be positive real constants. We set

$$\begin{aligned} \Omega_\eta =& \{w \in W, |\operatorname{Im} w(t)| < \eta\} ,\\ B_\sigma =& \{z \in C, |\operatorname{Im} z_0| < \sigma\} . \end{aligned}$$

Lemma 4. *Assume that* (4) *is satisfied and*

$$\|\tilde{K}\| \sum_{n=1}^{\infty} n(|a_n| + |b_n|) < 1 . \tag{9}$$

Then there are σ, η positive, $\sigma + \eta < d$, such that for each $z_0 \in B_\sigma$ the operator $T_{z_0} : \bar{\Omega}_\eta \to \Omega_\eta$,

$$T_{z_0} w = -\tilde{K} g(z_0 + w) + F$$

is a contraction.

Proof. At first we prove that T_{z_0} is a contraction.

For each $z_0 \in B_\sigma$ and each w_1, $w_2 \in \bar{\Omega}_\eta$ there is

$$\begin{aligned} |T_{z_0} w_1 - T_{z_0} w_2| \le& \|\tilde{K}\|\, |g(z_0 + w_1) - g(z_0 + w_2)| \\ \le& \|\tilde{K}\| \sup_{t \in [0,1]} \|g'\big(z_0 + w_2 + t(w_1 - w_2)\big)\|\, \|w_1 - w_2)\|. \end{aligned}$$

Using the inequalities

$$|\cos z| \le \cosh |\operatorname{Im} z|, \qquad |\sin z| \le \cosh |\operatorname{Im} z|$$

we obtain that

$$|T_{z_0} w_1 - T_{z_0} w_2| \le \|\tilde{K}\| \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \cosh n(\sigma + \eta) \|w_1(t) - w_2(t)\| .$$

The condition (4) implies that for $\sigma+\eta < d$ the series $\sum_{n=1}^{\infty} n(|a_n|+|b_n|)\cosh n(\sigma+\eta)$ is convergent and depends continuously on $\sigma+\eta$. Now the assumption (9) implies that there are σ, η sufficiently small such that

$$\|\tilde{K}\| \sum_{n=1}^{\infty} n(|a_n|+|b_n|)\cosh n(\sigma+\eta) < 1\,. \tag{10}$$

That means T_{z_0} is a contraction.

Now we prove that $T_{z_0}(\bar{\Omega}_\eta) \subseteq \Omega_\eta$. We use the inequalities

$$|\mathrm{Im}\,\cos z| \le \sinh|\mathrm{Im}\,z|, \qquad |\mathrm{Im}\,\sin z| \le \sinh|\mathrm{Im}\,z|$$

to estimate

$$\begin{aligned} |\mathrm{Im}\,T_{z_0}(w)| =& |\mathrm{Im}\,(-\tilde{K}g(z_0+w)+F)| = |\tilde{K}(\mathrm{Im}\,g(z_0+w))| \\ \le& \|\tilde{K}\| \sum_{n=1}^{\infty}(|a_n|+|b_n|)\sinh n(\sigma+\eta) \\ \le& \|\tilde{K}\| \sum_{n=1}^{\infty}(|a_n|+|b_n|) n(\sigma+\eta)\cosh n(\sigma+\eta)\,. \end{aligned} \tag{11}$$

Arguing as above we obtain from the condition (4) and assumption (9) that there are σ, η sufficiently small such that

$$\|\tilde{K}\| \sum_{n=1}^{\infty}(|a_n|+|b_n|) n(\sigma+\eta)\cosh n(\sigma+\eta) < \eta$$

i.e.

$$T_{z_0}(\bar{\Omega}_\eta) \subseteq \Omega_\eta\,. \qquad \square$$

Remark 1. The same Ljapunov-Schmidt reduction is also possible for differential equation (1), where the operator equation (6) with operators L, N acting on real Banach space $X = \{x(t),\ x: R \to R$ is a continuous 1-periodic function$\}$ with the supreme norm is equivalent to the pair

$$QN(x) = s\,, \tag{12}$$

$$w = -K(I-Q)N(x_0+w) + F(t) \tag{13}$$

where K, Q and F have the same meaning as in the complex case.

Assuming (9), for each $x_0 \in R$ the operator T_{x_0} defined by the equation (13) is a contraction in the variable w. Then for each $x_0 \in R$ there is a unique solution $w(x_0)$ of (13) and this solution depends continuously on x_0. As g is 2π-periodic, for x_0 and $x_0+2\pi$ we get the same solution w. Using these facts, we can prove the equality $S = [s_-, s_+]$ stated before Lemma 1 in a different way.

Let us denote

$$h_r(x_0) = QN(x_0 + w_{x_0}) = \int_0^1 g(x_0 + w_{x_0})\,dt\,. \tag{14}$$

The function $h_r : R \to R$ is a continuous 2π-periodic function, so its range is a closed interval $S = [s_-, s_+]$. The set S is obviously nonempty and a solution of (1) exists if and only if $s \in S$.

It is not known if the case $S = \{s_0\}$ is possible (see [4]). If the case $S = \{s_0\}$ occurs, then for each $x_0 \in [0, 2\pi)$ there exists a solution x of (1) of the form

$$x(t) = x_0 + w(x_0)(t)\,.$$

As w depends continuously on x_0, the set $M = \{x_0 + w(x_0),\ x_0 \in [0, 2\pi)\}$ is a continuous and bijective image of the connected set $[0, 2\pi)$, so M is connected. If all functions from X that differ only by a multiple of 2π will be identified then M is a one dimensional continuum.

The function h_r depends continuously on f, so its range depends continuously on f, therefore the numbers s_-, s_+ are continuous functions of f.

The estimation from above

For σ, η from Lemma 4 we define the operator $z_0 \in B_\sigma \to w_{z_0} \in \Omega_\eta$, where w_{z_0} is the fixed point of T_{z_0}.

Now each solution $z(t) = z_0 + w_{z_0}$ of (5) satisfies the bifurcation equation (7) written in the form

$$h(z_0) = \int_0^1 g(z_0 + w_{z_0})\,dt - s = 0\,. \tag{15}$$

We estimate the number of roots of (15) by use of Jensen's inequality ([1], [2]) which estimates the number of zeros of a complex analytic function on a disk with radius ρ centered at origin using maximum value of modulus at unit disc.

In [5] the unit disc is transformed to a horizontal strip B_σ and the following result is proved.

Lemma 5 ([5])**.** *Let $h : B_\sigma \to C$ be an analytic 2π periodic function. Then for the number N_h of roots of h on each interval $[x, x + 2\pi)$ there is*

$$N_h \le \frac{-1}{\ln \tanh \frac{\pi^2}{4\sigma}} \ln \frac{M}{m}\,,$$

where

$$M = \sup_{z \in B_\sigma} |h(z)|, \qquad m = \max_{x \in R} |h(x)|\,.$$

Let $s \in [s_-, s_+]$ be fixed and $h(z)$ be given by (15). Then

$$\begin{aligned}|h(z_0)| &\le \int_0^1 |g(z_0 + w_{z_0})|\,dt + G\\ &\le \int_0^1 \sum_{n=1}^{\infty} |a_n \sin n(z_0 + w_{z_0}) + b_n \cos n(z_0 + w_{z_0})|\,dt + G\\ &\le G + \sum_{n=1}^{\infty} (|a_n| + |b_n|) \cosh n(\sigma + \eta)\,,\end{aligned}$$

where $G = \max_{x\in R} |g(x)|$.

The assumption (4) implies the convergence of the last sum.

As the range of the function $h_r(x_0)$ given by (14) is the interval $[s_-, s_+]$ we have

$$m = \max\{|s - s^-|, |s - s^+|\}\,.$$

Finally we estimate the number of periodic solutions of (1) using Lemma 5. To this aim we have to prove that the function h given by (15) is a differentiable function defined on B_σ. As h is a composition of maps $h_1 : z_0 \mapsto z_0 + w(z_0) =: w$, $h_2 : w \mapsto g(w) =: u$, $h_3 : u \mapsto \int_0^1 u\,dt$, it suffices to prove the differentiability of each h_i. This is obvious for h_3, the derivative of h_2 is the map

$$dh_2(w)\delta = g'(w)\delta\,.$$

The differentiability of $\tilde{h}_1 : z_0 \to w(z_0)$ is a consequence of the Implicit Function Theorem, namely w_{z_0} is the unique solution of the equation

$$H(z_0, w) := w + K(I - Q)g(z_0 + w) - F(t) = 0\,,$$

the operator H has continuous partial derivatives ∂H_{z_0} and $\partial H_w \delta = (I + K(I - Q)g'(z_0 + w))\delta$ and the derivative ∂H_w is a homeomorphism as $\|K(I - Q)g'\| < 1$.

So for the function h all assumptions of Lemma 5 are satisfied and we get the following

Theorem 2. *Let us consider the differential equation* (1) *with* $s \in [s^-, s^+]$, s^-, s^+ *given by* (2). *Let* $g(x) \in C^\infty(R, R)$ *be defined by* (3) *with coefficients* a_n, b_n *satisfying* (4) *and* (9).

Then there are σ, η *positive such that the number* $N(f, s)$ *of periodic solutions of the equation* (1) *is estimated from above by*

$$N(f, s) \le \frac{-1}{\ln \tanh \frac{\pi^2}{4\sigma}} \ln \frac{G + \sum_{n=1}^{\infty} (|a_n| + |b_n|) \cosh n(\sigma + \eta)}{m(s)}\,,$$

where $G = \max_{x\in R} |g(x)|$, $m(s) = \max\{|s - s^-|, |s - s^+|\}$.

The existence of positive constants σ, η is assured by Lemma 4, its proof shows that it is sufficient to choose them satisfying the inequalities

$$\|\tilde{K}\| \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \cosh n(\sigma + \eta) < 1 \tag{10}$$

and

$$\|\tilde{K}\| \sum_{n=1}^{\infty} (|a_n| + |b_n|) \sinh n(\sigma+\eta) \leq \eta \,. \tag{16}$$

An example

We consider the equation

$$x'' + \frac{\sin x}{1 - 2q\cos x + q^2} = f(t) + s \,, \tag{17}$$

where $q = e^{-\tilde{d}}$, $\tilde{d} > d > 0$.

So the function

$$g(x) = \sum_{n=1}^{\infty} e^{\tilde{d}} q^n \sin nx$$

and the condition (4) is satisfied with $c_n = e^{\tilde{d}} e^{-n(\tilde{d}-d)}$.

The inequalities (10), (16) give the following two conditions on σ, η.

$$\begin{aligned} \|\tilde{K}\| &\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \cosh n(\sigma+\eta) \\ &= \|\tilde{K}\| \sum_{n=1}^{\infty} n(e^{\tilde{d}} e^{-n\tilde{d}}) \cosh n(\sigma+\eta) \\ &= \frac{1}{18\sqrt{3}} e^{\tilde{d}} \frac{\cosh \tilde{d} \cosh(\sigma+\eta) - 1}{2(\cosh \tilde{d} - \cosh(\sigma+\eta))^2} < 1 \,. \end{aligned} \tag{18}$$

and

$$\begin{aligned} \|\tilde{K}\| &\sum_{n=1}^{\infty} (|a_n| + |b_n|) \sinh n(\sigma+\eta) \\ &\leq \|\tilde{K}\| \sum_{n=1}^{\infty} (e^{\tilde{d}} e^{-n\tilde{d}}) \sinh n(\sigma+\eta) \\ &= \frac{1}{18\sqrt{3}} e^{\tilde{d}} \frac{\sinh(\sigma+\eta)}{2(\cosh \tilde{d} - \cosh(\sigma+\eta))} < \eta \end{aligned} \tag{19}$$

The maximum of a real function $g(x)$ is given by a constant

$$G = \frac{1}{1-q^2} \tag{20}$$

and the estimation of $|g(z)|$ is

$$M_1 = \max |g(z)| \leq e^{\tilde{d}} \frac{\cosh(\sigma+\eta) - e^{-\tilde{d}}}{2(\cosh \tilde{d} - \cosh(\sigma+\eta))} \,. \tag{21}$$

If $\tilde{d} = 4$ then (18), (19) are satisfied for $\sigma = 2.49$ and $\eta = 0.51$, (20) implies $G = 1.0004$ and (21) gives $M_1 \le 15.92$.

Then Theorems 1, 2 imply there are at least two different periodic solutions of (17) and at most

$$N(f,s) \le 10.19 - 3.60 \ln m(s)$$

periodic solutions of (17) for each $s \in (s_-, s_+)$.

Remark 2. For the sake of completeness of the paper, let us recall the proof that for $c = 0$ and $s = 0$ there exists a 1-periodic solution of (1) belonging to $C^2[0,1]$.

The Sobolev space $W^{1,2}[0,1]$ is continuously embedded into $C[0,1]$, so the space

$$H = \{x \in W^{1,2}[0,1],\ x(0) = x(1)\}$$

is a Hilbert space with the norm $\|x\|_{1,2} = \|x\|_2 + \|x'\|_2$, where $\|\cdot\|_2$ is a L_2 norm.

According to the inequalities

$$\|x\|_2 \le |x(0)| + \|x'\|_2, \qquad |x(0)| \le \|x\| \le \|x\|_2 + \|x'\|_2$$

we get

$$\frac{1}{2}\|x\|_{1,2} \le |x(0)| + \|x'\|_2 \le 2\|x\|_{1,2}\,,$$

therefore

$$\|x\|_0 = |x(0)| + \|x'\|_2$$

is an equivalent norm on H.

Let us consider the functional $J : H \to R$,

$$J(x) = \int_0^1 \left[\frac{1}{2}(x'(t))^2 - \gamma(x(t)) + x(t)f(t)\right] dt\,,$$

where γ is a 2π-periodic function, $\gamma' = g$. Denote by M the closed convex subset $\{x \in H,\ -\pi \le x(0) \le \pi\}$ of H. We show that J is weakly coercive on M and weakly sequentially lower semicontinuous, accordingly J has a minimum on M. As $J(M) = J(H)$, this minimum is a global one.

The weak coercivity of J on M follows from the inequalities

$$|J(x)| \ge \frac{1}{2}\|x'\|_2^2 - B - A\|x\|_2 \ge \frac{1}{2}\left(\frac{1}{2}\|x\|_{1,2} - \pi\right)^2 - A\|x\|_{1,2} - B\,,$$

where $A = \|f\|$, $B = \|\gamma\|$.

Before the proof of the weak sequential lower semicontinuity of J let us remark that the weak convergence $x_n \rightharpoonup x$ in $W^{1,2}[0,1]$ implies

(i) the boundedness of $\{x_n\}_{n=1}^{\infty}$ in $W^{1,2}[0,1]$ and accordingly in $C[0,1]$,

(ii) the pointwise convergence $x_n(t) \to x(t)$ for each $t \in [0,1]$ and therefore the convergence $\int_0^1 |x_n(t) - x(t)|\,dt \to 0$,

(iii) the weak convergence $x_n - x_n(0) \rightharpoonup x - x(0)$ in $W^{1,2}[0,1]$.

Now the functional

$$J_2(x) = \int_0^1 x(t)f(t) - \gamma(x(t))\,dt$$

is weakly sequentially continuous as

$$|J_2(x_n) - J_2(x)| \le (\|f(t)\| + \|\gamma'\|) \int_0^1 |x_n(t) - x(t)|\,dt\,.$$

The first term of J can be expressed as

$$J_1(x) = \frac{1}{2}\int_0^1 (x'(t))^2\,dt = \|x - x(0)\|_0^2$$

and the norm is weakly sequentially lower semicontinuous.

Thus it exists an $x_0 \in H$ such that $J(x_0) = \min_{x\in H} J(x)$. For this x_0 and each $h \in H$ we have

$$0 = DJ(x,h) = \int_0^1 x_0'(t)h'(t) - g(x_0(t))h(t) + f(t)h(t)\,dt\,. \tag{22}$$

Integrating by parts we get

$$0 = \int_0^1 \left((x_0'(t) + \int_0^t g(x_0(\tau))\,d\tau - \int_0^t f(\tau)\,d\tau \right) h'(t)\,dt$$

for each $h \in C_0^\infty[0,1]$. As $h' \in \{C_0^\infty[0,1],\ \int_0^1 h'(t)\,dt = 0\}$, we get

$$x_0'(t) + \int_0^t g(x_0(\tau)) - f(\tau)\,d\tau = k_1$$

for a.e. $t \in [0,1]$, and

$$x_0(t) = k_2 + k_1 t - \int_0^t \left(\int_0^u g(x_0(\tau)) - f(\tau)\,d\tau \right) du$$

for each $t \in [0,1]$.

The function of the right hand side of the last equality belongs to $C^2[0,1]$, so $x_0 \in C^2[0,1]$, differentiating two times we get

$$x_0''(t) + g(x_0(t)) = f(t)\,. \tag{23}$$

As $x_0 \in H$ we have $x_0(0) = x_0(1)$. Let us now choose $h \in C^1[0,1]$ such that $h(0) = h(1) \ne 0$.

Integrating (22) by parts we get, with respect to (23),

$$0 = \int_0^1 (-x_0''(t) - g(x_0(t)) + f(t))h(t)\,dt + [x_0'(t)h(t)]_0^1 = [x_0'(t)h(t)]_0^1\,,$$

thus $x'(0) = x'(1)$.

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