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ON MONOTONIC SOLUTIONS OF SOME INTEGRAL EQUATIONS

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Abstract. The aim of this paper is to obtain monotonic solutions of an integral equation of Urysohn-Stieltjes type in $C[0,1]$. Existence will be established with the aid of the measure of noncompactness.

1. Introduction

Integral equations arise naturally in applications of real world problems [5, 6, 7, 8]. The theory of integral equations has been well developed with the help of various tools from functional analysis, topology and fixed-point theory. The classical theory of integral equations can be generalized if one uses the Stieltjes integral with kernels dependent on one or two variables. The aim of this paper is to investigate the existence of monotonic solutions of so-called nonlinear integral equation of Urysohn-Stieltjes type. Equations of such kind contain, among others, the integral equation of Chandrasekhar which arises in radioactive transfer, neutron transport and the kinetic theory of gases [5, 6, 7, 8].

2. Definitions, notations and facts

Assume E is a real Banach space with norm $\|\cdot\|$ and zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r and by B_r the ball $B(\theta,r)$. If X is a nonempty subset of E we denote by $\overline{X}$ and $\operatorname{Conv} X$ the closure and the convex closed closure of X, respectively. Finally, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of relatively compact sets.

Definition 1 (see [2])**.** A mapping $\mu : \mathfrak{M}_E \longrightarrow [0,\infty)$ is said to be a *measure of noncompactness* in the space E if it satisfies the following conditions:

(1) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

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(3) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(5) If $\{X_n\}_n$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

For properties of measures of noncompactness and results related to them we refer the reader to [2].

In section 3 we will need the following fixed point principle (cf. [2]). This result was formulated and proved first by Darbo (cf. [9]) in the case of the Kuratowski measure of noncompactness (cf. [12]).

Theorem 1 ([2])**.** *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and μ a measure of noncompactness in E. Let $F : Q \longrightarrow Q$ be a continuous operator such that $\mu(FX) \leq K\mu(X)$ for any nonempty subset of Q, where $K \in [0,1)$ is a constant. Then F has a fixed point in the set Q.*

Remark 1. *Under the assumptions of Theorem 1 the set* Fix F *of fixed points of F belonging to Q is a member of* $\ker\mu$. *In fact, as* $\mu\big(F(\text{Fix } F)\big) = \mu(\text{Fix } F) \leq K\mu(\text{Fix } F)$ *and* $0 \leq K < 1$, *we deduce that* $\mu(\text{Fix } F) = 0$.

Consider the space $C[0,1]$ of all real functions defined and continuous on the interval $[0,1]$ and equipped with the maximum norm

$$\|x\| = \sup\{|x(t)| : t \in [0,1]\}\,.$$

For convenience, we write $I = [0,1]$ and $C(I) = C[0,1]$. Next, we recall the definition of a measure of noncompactness in $C(I)$ which be used in section 3. This measure was introduced and studied in [3]. Fix a nonempty and bounded subset X of $C(I)$. For $\varepsilon > 0$ and $x \in X$ denote by $w(x,\varepsilon)$ the modulus of continuity of x defined by

$$w(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t-s| \leq \varepsilon\}\,.$$

Furthermore put

$$w(X,\varepsilon) = \sup\ \{w(x,\varepsilon) : x \in X\}$$

and

$$w_0(X) = \lim_{\varepsilon\to 0} w(X,\varepsilon)\,.$$

Next, let us define the following quantities:

$$i(x) = \sup\ \{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I,\ t \leq s\}$$

and

$$i(X) = \sup\{i(x) : x \in X\}\,.$$

Observe that $i(X) = 0$ if and only if all functions belonging to X are nondecreasing on I.

Finally, let
$$\mu(X) = w_0(X) + i(X)\,.$$
It can be see [3] that the function μ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $ker\mu$ consists of all sets X belonging to $\mathfrak{M}_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

We now collect some auxiliary facts related to functions of bounded variation and the Stieltjes integral. Let x be a given real function defined on the interval I. The symbol $\overset{1}{\underset{0}{V}}x$ will denote the variation of x on the interval I, defined by

$$\overset{1}{\underset{0}{V}}x = \sup_P \left\{ \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \ :\ \begin{array}{l} P = \{0 = t_0 < t_1 < \cdots < t_n = 1\} \\ \text{is a partition of } I \end{array} \right\}.$$

If $\overset{1}{\underset{0}{V}}x$ is finite then we say that x is of bounded variation on I. Recall

$$\begin{array}{ll} \text{a) } \overset{1}{\underset{0}{V}}x = \overset{1}{\underset{0}{V}}(-x) & \text{b) } \overset{1}{\underset{0}{V}}(x+y) \le \overset{1}{\underset{0}{V}}x + \overset{1}{\underset{0}{V}}y \\ \text{c) } \overset{1}{\underset{0}{V}}(x-y) \le \overset{1}{\underset{0}{V}}x + \overset{1}{\underset{0}{V}}y & \text{d) } \left|\overset{1}{\underset{0}{V}}x - \overset{1}{\underset{0}{V}}y\right| \le \overset{1}{\underset{0}{V}}(x-y)\,. \end{array}$$

For other properties of functions of bounded variation we refer the reader to the monographs [10] and [13]. Let $g(t,s) : I \times I \longrightarrow \mathbb{R}$ be a function, then the symbol $\overset{q}{\underset{t=p}{V}} g(t,s)$ indicates the variation of the function $t \longrightarrow g(t,s)$ on the interval $[p,q] \subset I$, where s is arbitrarily fixed in I. Similarly we define the quantity $\overset{q}{\underset{s=p}{V}} g(t,s)$. Now, let us assume that $x, \varphi : I \longrightarrow \mathbb{R}$ are bounded functions. Then, under some extra conditions ([10], [13]), we can define the Stieltjes integral $\int_0^1 x(t)\,d\varphi(t)$ of the function x with respect to the function φ. In this case, we say that x is Stieltjes integrable on the interval I with respect to φ. Recall the following results related to Stieltjes integrability (cf. [10], [13]). If x is continuous and φ is of bounded variation on the interval I, then x is Stieltjes integrable with respect to φ on I. Moreover, under the assumption that x and φ are of bounded variation on I, the Stieltjes integral $\int_0^1 x(t)\,d\varphi(t)$ exists if and only if the functions x and φ have no common points of discontinuity. Finally we recall a few properties of the Stieltjes integral which will be used later. These properties are contained in the following lemmas (cf. [10], [13]).

Lemma 1. *If x is Stieltjes integrable on the interval I with respect to a function φ of bounded variation then*
$$\Big|\int_0^1 x(t)\,d\varphi(t)\Big| \le \Big(\sup_{0\le t\le 1} |x(t)|\Big)\overset{1}{\underset{0}{V}}\varphi\,.$$
Moreover, the following inequality holds
$$\Big|\int_0^1 x(t)\,d\varphi(t)\Big| \le \int_0^1 |x(t)|\,d\Big(\overset{t}{\underset{0}{V}}\varphi\Big)\,.$$

Corollary 1. *If x is Stieltjes integrable with respect to a nondecreasing function φ then*

$$\Big|\int_0^1 x(t)\,d\varphi(t)\Big| \le \Big(\sup_{0\le t\le 1}|x(t)|\Big)\big(\varphi(1)-\varphi(0)\big)\,.$$

Lemma 2. *Let x_1, x_2 be Stieltjes integrable functions on the interval I with respect to a nondecreasing function φ and such that $x_1(t)\le x_2(t)$ for $t\in I$. Then*

$$\int_0^1 x_1(t)\,d\varphi(t) \le \int_0^1 x_2(t)\,d\varphi(t)\,.$$

Corollary 2. *Let x be Stieltjes integrable function on the interval I with respect to a nondecreasing function φ and such that $x(t)\ge 0$ for all $t\in I$. Then*

$$\int_0^1 x(t)\,d\varphi(t)\ge 0\,.$$

Lemma 3. *Let φ_1, φ_2 be nondecreasing functions on I such that $\varphi_2-\varphi_1$ is a nondecreasing function. If x is Stieltjes integrable on I and $x(t)\ge 0$ for $t\in I$ then*

$$\int_0^1 x(t)\,d\varphi_1(t) \le \int_0^1 x(t)\,d\varphi_2(t)\,.$$

We will also need later the Stieltjes integral of the form $\int_0^1 x(s)d_sg(t,s)$ where g is a function of two variables, $g: I\times I\longrightarrow\mathbb{R}$, and the symbol d_s indicates that the integration is taken with respect to s.

3. Main result

In this section we will study the integral equation of Urysohn-Stieltjes type

$$x(t)=a(t)+k\,x^2(t)+\int_0^1 u\big(t,s,x(s)\big)d_sg(t,s)\,,\quad t\in I\,; \tag{1}$$

here $k\ge 0$. The functions $a(t)$ and $u(t,s,x)$ are given while $x=x(t)$ is an unknown function.

We will examine this equation under the following assumptions:

(i) $a\in C(I)$ and it is a nonnegative and nondecreasing function on the interval I.

(ii) $g: I\times I\longrightarrow\mathbb{R}$ satisfies the following conditions:
  (a) The function $s\longrightarrow g(t,s)$ is a nondecreasing function on I for each $t\in I$.
  (b) For all t_1, $t_2\in I$ such that $t_1<t_2$ the function $s\longrightarrow g(t_2,s)-g(t_1,s)$ is nondecreasing on I.
  (c) The functions $t\longrightarrow g(t,0)$ and $t\longrightarrow g(t,1)$ are continuous on I.

(iii) $u: I\times I\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function such that $u: I\times I\times\mathbb{R}_+\longrightarrow\mathbb{R}_+$ and for arbitrarily fixed $s\in I$ and $x\in\mathbb{R}_+$ the function $t\longrightarrow u(t,s,x)$ is nondecreasing on I.

(iv) the function u verifies the following conditions:

(a) There exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow [0,\infty)$ with $|u(t,s,x)| \leq \psi(|x|)$ for each $x \in \mathbb{R}$ and $t, s \in I$.

(b) For any $M > 0$ there exists a continuous nondecreasing function $\phi_M : [0,\infty) \longrightarrow [0,\infty)$ with $\phi_M(0) = 0$, such that for each $s \in I$, $x \in \mathbb{R}$ with $|x| \leq M$ and for all $t_1, t_2 \in I$, $t_1 < t_2$ we have

$$|u(t_2,s,x) - u(t_1,s,x)| \leq \phi_M(t_2 - t_1)$$

(v) There exists $r_0 > 0$ with $\|a\| + k\,r_0^2 + T\psi(r_0) \leq r_0$ and $2r_0\,k < 1$; here $T = \sup\left\{ \bigvee_{s=0}^{1} g(t,s) \;:\; t \in [0,1] \right\}$ (see Proposition 2 below).

Proposition 1. *Assume that the function $g : I \times I \longrightarrow \mathbb{R}$ satisfies* (ii,b) *and* (ii,c). *Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for t_1, $t_2 \in I$, $t_1 < t_2$ with $t_2 - t_1 \leq \delta$ we have*

$$\bigvee_{s=0}^{1} [g(t_2,s) - g(t_1,s)] \leq \varepsilon$$

Proof. Take an arbitrary partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of the interval I. Now assumption (ii,b) yields

$$\begin{aligned} &\sum_{i=1}^{n} |[g(t_2,s_i) - g(t_1,s_i)] - [g(t_2,s_{i-1}) - g(t_1,s_{i-1})]| \\ &\qquad = \sum_{i=1}^{n} ([g(t_2,s_i) - g(t_1,s_i)] - [g(t_2,s_{i-1}) - g(t_1,s_{i-1})]) \\ &\qquad = [g(t_2,1) - g(t_1,1)] - [g(t_2,0) - g(t_1,0)]\,. \end{aligned}$$

Consequently

$$\bigvee_{s=0}^{1} [g(t_2,s) - g(t_1,s)] = [g(t_2,1) - g(t_1,1)] - [g(t_2,0) - g(t_1,0)]\,.$$

Finally from assumption (ii,c) we obtain the desired result. □

Proposition 2. *Assume that the function $g : I \times I \longrightarrow \mathbb{R}$ satisfies* (ii,b) *and* (ii,c) *and the function $s \longrightarrow g(t,s)$ is of bounded variation on I for each $t \in I$. Then the function $t \longrightarrow \bigvee_{s=0}^{1} g(t,s)$ is continuous on I.*

Proof. Now assumptions (ii,b) and (ii,c) and Proposition 1 imply that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $t_1, t_2 \in I$, $t_1 < t_2$ with $t_2 - t_1 \leq \delta$ we have

$$\bigvee_{s=0}^{1} [g(t_2,s) - g(t_1,s)] \leq \varepsilon\,.$$

Notice since $s \to g(t,s)$ is of bounded variation on I and $\bigvee_0^1 x = \bigvee_0^1 (-x)$ (see section 2), the inequality above is valid for $|\, t_1 - t_2 \,| \leq \delta$. Also, since $\left| \bigvee_0^1 x - \bigvee_0^1 y \right| \leq$

$\overset{1}{\underset{0}{V}}(x-y)$ (see section 2) we obtain

$$\Big|\overset{1}{\underset{s=0}{V}} g(t_2,s) - \overset{1}{\underset{s=0}{V}} g(t_1,s)\Big| \le \overset{1}{\underset{s=0}{V}} [g(t_2,s)-g(t_1,s)] \le \varepsilon\,.$$

Consequently the function $t \longrightarrow \overset{1}{\underset{s=0}{V}} g(t,s)$ is continuous on I. □

Remark 2. As every nondecreasing function is of bounded variation, in view of Proposition 2 and the compactness of the interval I there exists a constant $T > 0$ such that $\overset{1}{\underset{s=0}{V}} g(t,s) \le T$ for every $t \in I$ if g satisfies (ii).

Next, we formulate our main result

Theorem 2. *Under the assumptions (i) − (v) the integral equation (1) has at least one solution $x \in C(I)$ which is nondecreasing on I.*

Proof. Let r_0 be chosen as in (v). Let $M : [0,\infty) \longrightarrow [0,\infty)$ be

$$M(\varepsilon) = \sup\ \Big\{\overset{1}{\underset{s=0}{V}}\big[g(t_2,s)-g(t_1,s)\big]\ :\ t_1,t_2 \in I,\ t_1 \le t_2,\ t_2 - t_1 \le \varepsilon\Big\}\,.$$

Now Proposition 1 implies $M(\varepsilon) \to 0$ as $\varepsilon \to 0$. Consider the operator A defined on $C(I)$ by

$$(Ax)(t) = a(t) + k\,x^2(t) + \int_0^1 u\big(t,s,x(s)\big)\,d_s\,g(t,s)\,. \tag{2}$$

First we show that if $x \in C(I)$ then $Ax \in C(I)$. For this it is sufficient to show that if $x \in C(I)$ then $Bx \in C(I)$ where

$$(Bx)(t) = \int_0^1 u\big(t,s,x(s)\big)\,d_s\,g(t,s)\,.$$

Fix $\varepsilon > 0$ and take $t_1,\ t_2 \in I$ with $t_1 \le t_2$ and $t_2 - t_1 \le \varepsilon$. Let $x \in C(I)$ so there exists $M > 0$ with $\|x\| \le M$. Then our assumptions and Lemma 1 yield

$$\begin{aligned}
|(Bx)(t_2)-(Bx)(t_1)| &= \Big|\int_0^1 u\big(t_2,s,x(s)\big)d_s g(t_2,s) - \int_0^1 u\big(t_1,s,x(s)\big)d_s g(t_1,s)\Big| \\
&\le \Big|\int_0^1 u\big(t_2,s,x(s)\big)d_s g(t_2,s) - \int_0^1 u\big(t_1,s,x(s)\big)d_s g(t_2,s)\Big| \\
&\quad + \Big|\int_0^1 u\big(t_1,s,x(s)\big)d_s g(t_2,s) - \int_0^1 u\big(t_1,s,x(s)\big)d_s g(t_1,s)\Big| \\
&\le \int_0^1 \big|u\big(t_2,s,x(s)\big) - u\big(t_1,s,x(s)\big)\big|\,d_s\Big(\overset{s}{\underset{p=0}{V}} g(t_2,p)\Big) \\
&\quad + \int_0^1 \big|u\big(t_1,s,x(s)\big)\big|\,d_s\Big(\overset{s}{\underset{p=0}{V}}\big(g(t_2,p)-g(t_1,p)\big)\Big) \\
&\le \phi_M(t_2-t_1)\overset{1}{\underset{p=0}{V}} g(t_2,p) + \psi(\|x\|)\overset{1}{\underset{p=0}{V}}\big(g(t_2,p)-g(t_1,p)\big) \\
&\le \phi_M(\varepsilon)\cdot T + \psi(\|x\|)M(\varepsilon)\,.
\end{aligned}$$

Thus, we obtain the following estimate:

$$w(Bx,\varepsilon) \le \phi_M(\varepsilon)\cdot T + \psi(\|x\|)M(\varepsilon)\,.$$

Now $w(Bx,\varepsilon) \to 0$ as $\varepsilon \to 0$ so $Bx \in C(I)$.

Next, we show that A is a continuous operator. In order to prove this result it is sufficient to show the continuity of the operator B. Fix $\varepsilon > 0$ and let $x \in C(I)$. Next let $y \in C(I)$ with $\|x-y\| \le \varepsilon$. Then, for fixed $t \in I$ we have

$$\begin{aligned} |(Bx)(t)-(By)(t)| &= \Big|\int_0^1 u\big(t,s,x(s)\big)d_s g(t,s) - \int_0^1 u\big(t,s,y(s)\big)d_s g(t,s)\Big| \\ &\le \int_0^1 |u\big(t,s,x(s)\big) - u\big(t,s,y(s)\big)|\,d_s\Big(\bigvee_{p=0}^{s} g(t,p)\Big) \\ &\le \beta(\varepsilon)\bigvee_{p=0}^{1} g(t,p) \le \beta(\varepsilon)\cdot T \end{aligned}$$

where $\beta(\varepsilon)$ is given by

$$\beta(\varepsilon) = \sup\big\{|u(t,s,y)-u(t,s,y')|\; t,s\in I\,, y,y' \in [-\|x\|-\varepsilon, \|x\|+\varepsilon], |y-y'| \le \varepsilon\big\}\,.$$

From the uniform continuity of the function $u(t,s,x)$ on the set $I\times I\times[-\|x\|-\varepsilon, \|x\|+\varepsilon]$ we have that $\beta(\varepsilon)\to 0$ as $\varepsilon \to 0$. This fact and the last inequality prove that the operator B is continuous and consequently the operator A is continuous. Thus A transforms the space $C(I)$ into itself. Next assumption (iv) yields

$$\begin{aligned} |(Ax)(t)| &= \Big|a(t) + kx^2(t) + \int_0^1 u\big(t,s,x(s)\big)d_s g(t,s)\Big| \\ &\le \|a\| + k\|x\|^2 + \int_0^1 |u\big(t,s,x(s)\big)|\,d_s\Big(\bigvee_{p=0}^{s} g(t,p)\Big) \\ &\le \|a\| + k\|x\|^2 + \psi(\|x\|)\bigvee_{p=0}^{1} g(t,p) \\ &\le \|a\| + k\|x\|^2 + \psi(\|x\|)\cdot T\,. \end{aligned}$$

Thus if $\|x\| \le r_0$ we obtain from (v) that

$$\|Ax\| \le \|a\| + kr_0^2 + \psi(r_0)\cdot T \le r_0\,. \tag{3}$$

As a result A transforms the ball $B(0,r_0)$ into itself.

Consider the operator A on the subset $B^+_{r_0}$ of the ball B_{r_0} defined by

$$B^+_{r_0} = \{x \in B_{r_0} : x(t) \ge 0 \text{ for } t \in I\}\,.$$

Obviously, the set $B^+_{r_0}$ is nonempty, bounded, closed and convex. Let $x \in B^+_{r_0}$. Notice that in view of our assumptions (i) and (iii) if $x(t) \ge 0$ for $t \in I$ then $(Ax)(t) \ge 0$ for $t \in I$. Thus A transforms the set $B^+_{r_0}$ into itself. Moreover, A is continuous on $B^+_{r_0} \subset C(I)$. Let X be a nonempty subset of $B^+_{r_0}$. Fix $\varepsilon > 0$ and choose $x \in X$ and t_1, $t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we may assume that $t_1 \le t_2$. Notice

$$\begin{aligned}
|(Ax)(t_2)-(Ax)(t_1)| &= \Big|a(t_2)+kx^2(t_2)+\int_0^1 u\big(t_2,s,x(s)\big)d_sg(t_2,s)\\
&\quad -a(t_1)-kx^2(t_1)-\int_0^1 u\big(t_1,s,x(s)\big)d_sg(t_1,s)\Big|\\
&\le |a(t_2)-a(t_1)|+k|x(t_2)-x(t_1)|\,|x(t_2)+x(t_1)|\\
&\quad +\Big|\int_0^1 u\big(t_2,s,x(s)\big)d_sg(t_2,s)-\int_0^1 u\big(t_1,s,x(s)\big)d_sg(t_2,s)\Big|\\
&\quad +\Big|\int_0^1 u\big(t_1,s,x(s)\big)d_sg(t_2,s)-\int_0^1 u\big(t_1,s,x(s)\big)d_sg(t_1,s)\Big|\\
&\le w(a,\varepsilon)+2r_0\,kw(x,\varepsilon)\\
&\quad +\int_0^1 \big|u\big(t_2,s,x(s)\big)-u\big(t_1,s,x(s)\big)\big|d_s\Big(\bigvee_{p=0}^{s}g(t_2,p)\Big)\\
&\quad +\int_0^1 \big|u\big(t_1,s,x(s)\big)\big|d_s\Big(\bigvee_{p=0}^{s}\big(g(t_2,p)-g(t_1,p)\big)\Big)\\
&\le w(a,\varepsilon)+2r_0\,kw(x,\varepsilon)+\phi_{r_0}(t_2-t_1)\bigvee_{p=0}^{1}g(t_2,p)\\
&\quad +\psi(\|x\|)\Big(\bigvee_{p=0}^{1}\big(g(t_2,p)-g(t_1,p)\big)\Big)\\
&\le w(a,\varepsilon)+2r_0\,kw(x,\varepsilon)+T\cdot\phi_{r_0}(\varepsilon)+\psi(r_0)M(\varepsilon)\,.
\end{aligned}$$

Hence,

$$w(Ax,\varepsilon)\le w(a,\varepsilon)+2r_0\,kw(x,\varepsilon)+T\cdot\phi_{r_0}(\varepsilon)+\psi(r_0)M(\varepsilon)\,.$$

Thus

$$\sup_{x\in X} w(Ax,\varepsilon)\le w(a,\varepsilon)+2r_0\,k\cdot\sup_{x\in X} w(x,\varepsilon)+T\cdot\phi_{r_0}(\varepsilon)+\psi(r_0)M(\varepsilon)$$

so let $\varepsilon\to 0$ to obtain

$$w_0(AX)\le 2r_0\,k\cdot w_0(X)\,. \tag{4}$$

Let $x\in X$ and $t_1,t_2\in I$, $t_1\le t_2$. Then

$$\begin{aligned}
|(Ax)(t_2)-(Ax)(t_1)|&-[(Ax)(t_2)-(Ax)(t_1)]\\
&= \Big|a(t_2)+kx^2(t_2)+\int_0^1 u\big(t_2,s,x(s)\big)d_sg(t_2,s)\\
&\quad -a(t_1)-kx^2(t_1)-\int_0^1 u\big(t_1,s,x(s)\big)d_sg(t_1,s)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\quad - \Big[a(t_2) + kx^2(t_2) + \int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) \\
&\quad - a(t_1) - kx^2(t_1) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s)\Big] \\
&\le |a(t_2) - a(t_1)| - [a(t_2) - a(t_1)] \\
&\quad + k\big(|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)]\big)\, [x(t_2) + x(t_1)] \\
&\quad + \Big| \int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s)\Big| \\
&\quad - \Big[\int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s)\Big] \\
&\le 2r_0\, k \cdot i(x) + \Big| \int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s)\Big| \\
&\quad - \Big[\int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s)\Big]\,. \qquad (5)
\end{aligned}
$$

We next prove that $\int_0^1 u\big(t_2, s, x(s)\big)\, d_s\, g(t_2, s) - \int_0^1 u(t_1, s, x(s)) d_s g(t_1, s) \ge 0$. In fact notice

$$
\begin{aligned}
&\int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s) \\
&\qquad = \int_0^1 u\big(t_2, s, x(s) = \big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_2, s) \\
&\qquad\quad + \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s)\,. \qquad (6)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_2, s) \\
&\qquad = \int_0^1 \big[u\big(t_2, s, x(s)\big) - u\big(t_1, s, x(s)\big)\big] d_s g(t_2, s)
\end{aligned}
$$

so assumption (iii) and Corollary 2 yield

$$
\int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_2, s) \ge 0\,. \qquad (7)
$$

On the other hand,

$$
\begin{aligned}
&\int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s) \\
&\qquad = \int_0^1 u\big(t_1, s, x(s)\big) d_s \big(g(t_2, s) - g(t_1, s)\big)\,.
\end{aligned}
$$

Moreover, we have that $g(t_2, s) - g(t_1, s)$ is a nondecreasing function $\big($assumption (ii-b)$\big)$, $u(t_1, s, x) \geq 0$ $\big($assumption (iii)$\big)$ and $g(t_2, s), g(t_1, s)$ are nondecreasing functions $\big($assumption (ii-a)$\big)$. From these facts and Lemma 3 we deduce that

$$\int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s) \geq 0 \,. \tag{8}$$

Now (6), (7) and (8) imply

$$\int_0^1 u\big(t_2, s, x(s)\big) d_s g(t_2, s) - \int_0^1 u\big(t_1, s, x(s)\big) d_s g(t_1, s) \geq 0 \,.$$

This together with (5) yields

$$i(Ax) \leq 2r_0\, k \cdot i(x)$$

and so

$$i(AX) \leq 2r_0\, k \cdot i(X) \,. \tag{9}$$

Finally, combining (4) and (9) we get

$$\mu(AX) \leq 2r_0\, k \cdot \mu(X) \,.$$

Now, Theorem 1 guarantees that there exists $x \in B^+_{r_0}$ a solution of (1). Also, such a solution is nondecreasing in view of Remark 1 and the definition of the measure of noncompactness μ given in section 2. □

Remark 3. Suppose $k = 0$ and there exist $c, d \geq 0$ with $\psi(x) = c + dx$ with $dT < 1$ then it is easy to see that there exists $r_0 > 0$ with $\|a\| + (c + dr_0)T \leq r_0$.

Remark 4. The result in Theorem 2 holds for the integral equation

$$x(t) = a(t) + kx^n(t) + \int_0^1 u\big(t, s, x(s)\big)\, d_s\, g(t, s)$$

with $n \in \{1, 2, \dots\}$ provided assumption (v) is changed to: There exists $r_0 > 0$ with $\|a\| + k\, r_0^n + T\, \psi(r_0) \leq r_0$ and $n\, r_0^{n-1}\, k < 1$. Note the case $n = 1$ is easier since we can rewrite the equation as

$$x(t) = \frac{1}{1-k}\, a(t) + \frac{1}{1-k} \int_0^1 u\big(t, s, x(s)\big)\, d_s\, g(t, s) \,.$$

4. Remarks and Examples

First we discuss assumption (ii-a). Notice we cannot replace (ii-a) with the function $s \longrightarrow g(t, s)$ is of bounded variation on I. In fact, if we take $g(t, s) = e^{-s}$, it is easy to show that this function is decreasing and consequently is of bounded variation. Moreover, if we take $a(t) = 0$ and $u(t, s, x) = 1$, the integral equation (with $k = 0$) has the form

$$x(t) = \int_0^1 d_s g(t, s) = \int_0^1 -se^{-s} ds \leq 0.$$

Therefore, the operator A defined in the proof of Theorem 2 will not transform the set $B^+_{r_0}$ into itself.

Next, we give some examples of functions $g(t,s)$ which verify the assumptions of Theorem 2.

Example 1. Let us take a function $p : I \times I \longrightarrow \mathbb{R}_+$ which is bounded and integrable. Then, we consider the function $g(t,s)$ defined by

$$g(t,s) = \int_0^s \Big(\int_0^t p(z,v)\, dv \Big)\, dz\,.$$

Now, we show this function satisfies the assumptions of Theorem 2.

(ii-a) Fix $t \in [0,1]$ and take $s_1,\, s_2 \in I$ such that $s_1 < s_2$. Now $p \geq 0$ yields

$$\int_0^{s_2} \Big(\int_0^t p(z,v)\, dv \Big)\, dz \geq \int_0^{s_1} \Big(\int_0^t p(z,v)\, dv \Big)\, dz$$

and consequently, the function $s \longrightarrow g(t,s)$ is nondecreasing for each $t \in [0,1]$.

(ii-b) Take $t_1,\, t_2 \in I$ such that $t_1 < t_2$. Then the function $s \longrightarrow g(t_2,s) - g(t_1,s)$ is nondecreasing. In fact, if we take $s_1,\, s_2 \in I$ such that $s_1 < s_2$ we obtain

$$\begin{aligned}
&\big(g(t_2,s_2) - g(t_1,s_2)\big) - \big(g(t_2,s_1) - g(t_1,s_1)\big) \\
&\quad = \int_0^{s_2} \Big(\int_0^{t_2} p(z,v)\, dv \Big)\, dz - \int_0^{s_2} \Big(\int_0^{t_1} p(z,v)\, dv \Big)\, dz \\
&\qquad - \int_0^{s_1} \Big(\int_0^{t_2} p(z,v)\, dv \Big)\, dz + \int_0^{s_1} \Big(\int_0^{t_1} p(z,v)\, dv \Big)\, dz \\
&\quad = \int_0^{s_2} \Big(\int_{t_1}^{t_2} p(z,v)\, dv \Big)\, dz - \int_0^{s_1} \Big(\int_{t_1}^{t_2} p(z,v)\, dv \Big)\, dz \\
&\quad = \int_{s_1}^{s_2} \Big(\int_{t_1}^{t_2} p(z,v)\, dv \Big)\, dz \geq 0\,.
\end{aligned}$$

(ii-c) If $s = 0$ then

$$g(t,0) = \int_0^0 \Big(\int_0^t p(z,v)\, dv \Big) dz = 0\,,$$

so the function $t \longrightarrow g(t,0)$ is continuous. We next claim the function

$$t \longrightarrow g(t,1) = \int_0^1 \Big(\int_0^t p(z,v)\, dv \Big)\, dz$$

is continuous. Fix $\varepsilon > 0$ and $t_0 \in I$. Then there exists $\delta = \frac{\varepsilon}{\|p(z,v)\|} > 0$ (where $\|p(z,v)\| = \sup\{|p(z,v)| \ : \ p, v \in I\}$) such that if $t \in I$, $|t - t_0| \leq \delta$ and $t_0 \leq t$ then

$$\begin{aligned}
|g(t,1) - g(t_0,1)| &= \Big| \int_0^1 \Big(\int_0^t p(z,v)\, dv \Big)\, dz - \int_0^1 \Big(\int_0^{t_0} p(z,v)\, dv \Big)\, dz \Big| \\
&= \Big| \int_0^1 \Big(\int_{t_0}^t p(z,v)\, dv \Big)\, dz \Big| \leq \int_0^1 \Big(\int_{t_0}^t |p(z,v)|\, dv \Big)\, dz \\
&\leq \|p(z,v)\| (t - t_0) \leq \|p(z,v)\| \cdot \delta \leq \varepsilon\,.
\end{aligned}$$

Therefore, the function $t \longrightarrow g(t,1)$ is continuous.

These facts imply that the function $g(t,s) = \int_0^s \left(\int_0^t p(z,v)\,dv\right) dz$ satisfies assumption (ii). In this case, $d_s g(t,s) = \int_0^t p(s,v)\,ds$ and the integral equation has the following form

$$x(t) = a(t) + k\,x^2(t) + \int_0^1 u\big(t,s,x(s)\big)\Big(\int_0^t p(s,v)\,dv\Big)\,ds \tag{10}$$

which is an integral equation of Hammerstein type.

Note if we take suitable functions $a(t)$ and $u(t,s,x)$ (which satisfy the assumptions of Theorem 2) then the integral equation (10) has a nondecreasing solution on $C(I)$.

Example 2. Let us take the function $g : I \times I \longrightarrow \mathbb{R}$ defined by

$$g(t,s) = \begin{cases} t \cdot \ln\left(\frac{t+s}{t}\right) & \text{for} \quad t \in (0,1],\ s \in [0,1] \\ 0 & \text{for} \quad t = 0, s \in [0,1]\,. \end{cases}$$

We easily see that the function $s \longrightarrow g(t,s)$ in nondecreasing for each $t \in [0,1]$. In order to prove that $g(t,s)$ satisfies assumption (ii,b) and (ii,c), we fix $t_1, t_2 \in [0,1]$, $t_1 \leq t_2$ and we obtain

$$g(t_2,s) - g(t_1,s) = \begin{cases} t_2 \cdot \ln\left(\frac{t_2+s}{t_2}\right) & \text{for} \quad t_1 = 0 \\ t_2 \cdot \ln\left(\frac{t_2+s}{t_2}\right) - t_1 \cdot \ln\left(\frac{t_1+s}{t_1}\right) & \text{for} \quad t_1 > 0\,. \end{cases}$$

It is clear that the function $s \longrightarrow g(t_2,s) - g(t_1,s)$ is nondecreasing on the interval $[0,1]$. Moreover the functions $g(t,0)$ and $g(t,1)$ are continuous on $[0,1]$. As $d_s g(t,s) = \dfrac{t}{t+s}$ the integral equation (1) has the form

$$x(t) = a(t) + k\,x^2(t) + \int_0^1 u\big(t,s,x(s)\big)\frac{t}{t+s}\,ds$$

which is related to the Chandrasekhar equation [5, 6, 7, 8]. If we take suitable functions $a(t)$ and $u(t,s,x)$ then by Theorem 2 we know this integral equation has a nondecreasing solution on $C(I)$.

Example 3. Consider the following integral equation

$$x(t) = t^2 + 2t + 1 + \frac{1}{e}\int_0^1 s\big(t + \ln(1+|x(s)|)\big)\,d_s\big(e^{ts}\big)\,. \tag{11}$$

Here $a(t) = t^2 + 2t + 1$ which is continuous, nonnegative and bounded on the interval I. Thus, the function a satisfies assumption (i). Moreover, the function $g(t,s)$ is defined by $g(t,s) = e^{ts}$. Now, we will prove that this function satisfies assumption (ii).

(ii-a) Fix $t \in I$. The function $s \longrightarrow g(t,s) = e^{ts}$ is nondecreasing on I. In fact,

$$\frac{d}{ds}\left(e^{ts}\right) = te^{ts} \geq 0\,, \qquad t,s \in I\,.$$

(ii-b) For all t_1, $t_2 \in I$ such that $t_1 < t_2$ the function $s \longrightarrow e^{t_2 s} - e^{t_1 s}$ is nondecreasing. In fact,

$$\frac{d}{ds}\left(e^{t_2 s} - e^{t_1 s}\right) = t_2 e^{t_2 s} - t_1 e^{t_1 s} > 0\,.$$

(ii-c) The functions $t \longrightarrow g(t,0) = 1$ and $t \longrightarrow g(t,1) = e^t$ are continuous.

Consequently, the function $g(t,s) = e^{ts}$ satisfies assumption (ii). If $u(t,s,x)$ is given by

$$u(t,s,x) = \frac{1}{e} s\big(t + \ln(1+|x|)\big)\,,$$

then assumptions (iii) and (iv) of Theorem 2 are satisfied. Clearly $u(t,s,x)$ is a continuous function such that $u : I \times I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Moreover for fixed $s \in I$ and $x \in \mathbb{R}_+$ the function $t \longrightarrow u(t,s,x)$ is nondecreasing on I. In fact,

$$\frac{d}{dt}\Big(\frac{1}{e} s\big(t + \ln(1+|x|)\big)\Big) = \frac{1}{e} s \geq 0\,,$$

so the function u satisfies (iii).
Also we have the following estimate

$$|u(t,s,x)| = \Big|\frac{s}{e}\big(t + \ln(1+|x|)\big)\Big| \leq \frac{1}{e}\big(t + \ln(1+|x|)\big) \leq \frac{1}{e}(1+|x|) = \frac{1}{e} + \frac{|x|}{e}\,.$$

Therefore, $u(t,s,x)$ satisfies (iv-a) with $\psi(x) = c + dx$ and $c = d = \frac{1}{e}$.
Finally, for each $s \in I$, $x \in \mathbb{R}$ and for all $t_2, t_1 \in I$, $t_1 < t_2$, we have

$$\begin{aligned} |u(t_2,s,x) - u(t_1,s,x)| &= \Big|\frac{1}{e} s\big(t_2 + \ln(1+|x|)\big) - \frac{1}{e} s\big(t_1 + \ln(1+|x|)\big)\Big| \\ &= \frac{s}{e}(t_2 - t_1) \leq \frac{1}{e}(t_2 - t_1)\,. \end{aligned}$$

In view of the last inequality we can deduce that $u(t,s,x)$ satisfies assumption (iv-b). Now

$$T = \sup\Big\{\bigvee_{s=0}^{1} g(t,s) \ : \ t \in [0,1]\Big\} = \sup\{e^t - 1 : t \in [0,1]\} = e - 1\,,$$

so $d \cdot T = \frac{1}{e} \cdot (e-1) < 1$.

As a result the assumptions of Theorem 2 (see Remark 3) are satisfied. Thus, this integral equation has a nondecreasing solution in $C(I)$.

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