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COUNTABLY THICK MODULES

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Abstract. The purpose of this paper is to further the study of countably thick modules via weak injectivity. Among others, for some classes $\mathcal{M}$ of modules in $\sigma[M]$ we study when direct sums of modules from $\mathcal{M}$ satisfies a property $\mathbb{P}$ in $\sigma[M]$. In particular, we get characterization of locally countably thick modules, a generalization of locally *q.f.d.* modules.

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see, [29]). Given a module X_R the injective hull of X in Mod-R (resp., in $\sigma[M]$) is denoted by $E(X)$ (resp., $\widehat{X}$). The M-injective hull $\widehat{X}$ is the trace of M in $E(X)$, i.e. $\widehat{X} = \sum\{f(M), f \in \operatorname{Hom}(M, E(X))\}$ (see [29, 3.17.9]).

The purpose of this paper is to further the study of the concepts of weak injectivity (tightness, and weak tightness) in $\sigma[M]$ studied in [4], [9], [21], [24], [23], [25], [27], [30], [31]. For a locally *q.f.d.* module M, there exists a module $K \in \sigma[M]$ such that $K \oplus N$ is weakly injective in $\sigma[M]$, for any $N \in \sigma[M]$. For some classes $\mathcal{M}$ of modules in $\sigma[M]$ we study when direct sums of modules from $\mathcal{M}$ are weakly tight in $\sigma[M]$. In particular, we get necessary and sufficient conditions for $\sum$-weak tightness of the injective hull of a simple module. As a consequence, we get characterizations of *q.f.d.* rings by means of weakly injective (tight) modules given by A. Al-Huzali, S. K. Jain and S. R. López-Permouth [2].

Given two modules Q and $N \in \sigma[M]$, we call Q *weakly N-injective* in $\sigma[M]$ if for every homomorphism $\varphi : N \to \widehat{Q}$, there exists a homomorphism $\widehat{\varphi} : N \to Q$ and a monomorphism $\sigma : Q \to \widehat{Q}$ such that $\varphi = \sigma\widehat{\varphi}$. Equivalently, if there exists a submodule X of $\widehat{Q}$ such that $\varphi(N) \subseteq X \simeq Q$. A module $Q \in \sigma[M]$ is called *weakly injective* in $\sigma[M]$ if Q is weakly N-injective for all finitely generated modules N in

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$\sigma[M]$. A module X is *N-tight* in $\sigma[M]$ if every quotient of N which is embeddable in the M-injective hull of X is embeddable in X. A module is *tight* (*R-tight*) in $\sigma[M]$ if it is tight relative to all finitely generated (cyclic) submodules of its M-injective hull, and Q is *weakly tight* (*weakly R-tight*) in $\sigma[M]$ if every finitely generated (cyclic) submodule N of $\widehat{Q}$ is embeddable in a direct sum of copies of Q. It is clear that every weakly injective module in $\sigma[M]$ is tight in $\sigma[M]$, and every tight module in $\sigma[M]$ is weakly tight in $\sigma[M]$, but weak tightness does not imply tightness, (see [4], [31]). A module M_R is called *locally q.f.d.* [3], [7], [18] in case every finitely generated (or cyclic) module $N \in \sigma[M]$ has finite uniform dimension. A module Q is called *weakly* (N)*-injective* (resp., *weakly* (N)*-tight, tight*) [17], [14], [15], [16] if it is weakly (N)-injective (resp., weakly (N)-tight, tight) in $\sigma[R_R] = \text{Mod-}R$. An essential (large) submodule X of an R-module Y will be denoted by $X \subseteq' Y$.

2. Preliminaries

The class of weak injectivity (tightness, weak tightness) in $\sigma[M]$ is closed under finite direct sums, and essential extensions.

First, we list below some of known results on weak injectivity, tightness, and weak tightness in $\sigma[M]$ that will be needed through this paper (cf. [4], [24], [26]).

Lemma 2.1 ([24, Proposition 3.6, Corollary 3.5])**.** *Given modules $N, Q \in \sigma[M]$.*

(i) *If Q is self-injective and N-tight in $\sigma[M]$, then Q is N-injective in $\sigma[M]$.*
(ii) *If Q is a uniform module, then Q is N-tight in $\sigma[M]$ iff Q is weakly N--injective in $\sigma[M]$.*

Lemma 2.2 ([24, Proposition 3.3])**.** *For a module M_R, we have the following:*

(i) *A finite direct sum of weakly injective (tight, weakly tight) modules in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$.*
(ii) *An essential extension of a weakly injective (tight, weakly tight) module in $\sigma[M]$ is weakly injective (tight, weakly tight) in $\sigma[M]$.*

Lemma 2.3. *A uniform module $X \in \sigma[M]$ is weakly tight in $\sigma[M]$ iff X is weakly injective in $\sigma[M]$.*

Proof. Let X be uniform and weakly tight in $\sigma[M]$, and let N be a finitely generated submodule of $\widehat{X}$. Then N is embeddable in $X^{(\alpha)}$ via a monomorphism, say, ϕ. Let $\pi_i : X^{(\alpha)} \to X$ be the i-th projection map. Then $\cap_{i\in\alpha} \ker(\pi_i\phi) \subseteq \ker\phi = 0$. Since X is uniform, then $\ker(\pi_i\phi) = 0$, and thus N embeds in X, proving that X is tight in $\sigma[M]$. By Lemma 2.1(ii), X is weakly injective in $\sigma[M]$. □

Example 2.4. (i) [17, Example 2.11], [19]. Let R be the ring of endomorphisms of an infinite dimensional vector space V over a field F. Then $M = \text{Soc}(R_R) \oplus R$ is tight but not weakly injective.
(ii) [4]. Let $R = Z$ and $X = (Q/Z) \oplus (Z/pZ)$, where p is a prime number. Then X is weakly tight in $\sigma[M]$ but not tight.

(iii) [17, Example 4.4(d)] Let F be a field. Then $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is weakly injective but the summand $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ as an R-module is not weakly injective.

As a direct consequence of Theorem 2.8 in [17], we get the following corollary.

Corollary 2.5. *Let M be a locally q.f.d. module. Then every tight module in $\sigma[M]$ is weakly injective in $\sigma[M]$.*

Lemma 2.6. *Let M be a locally q.f.d. module. Then there exists a module $K \in \sigma[M]$ such that $Q = K \oplus N$ is a weakly injective module in $\sigma[M]$, for every module $N \in \sigma[M]$.*

Proof. Let $\mathcal{F}$ be the family of all indecomposable $\sigma[M]$-injectives up to isomorphism, and let $K = \bigoplus \sum_{F \in \mathcal{F}} F^{(\alpha)}$ where α is an infinite cardinal number greater than both the cardinality of M and the cardinality of the ring R. Let $Q = K \oplus N$. Then Q is weakly injective in $\sigma[M]$, for every module $N \in \sigma[M]$, since every finitely generated module over a locally q.f.d. module is embeddable in a finite direct sum of indecomposable injectives and thus embeddable in Q. Thus Q is tight in $\sigma[M]$ and thus, Q is weakly injective in $\sigma[M]$. □

In [19], it is shown that any semisimple module is a direct summand of a weakly injective module, recently in [26], it is shown that in fact any module is a direct summand of a weakly injective module.

Lemma 2.7 ([26])**.** *For any module X in $\sigma[M]$, $X \oplus \widehat{X^{(\alpha)}}$, where α is an infinite cardinal number, is weakly injective in $\sigma[M]$.*

Lemma 2.7 generalizes 2.12, 2.13, 2.14, in [17], 2.1, 2.2., and 2.3 in [19].

We call a module M_R *weakly semisimple* (*weakly R-semisimple*) if every module $N \in \sigma[M]$ is weakly injective (weakly R-injective) in $\sigma[M]$. As a direct applications of the above results, we get the following characterizations of semisimple and weakly (R)-semisimple modules in terms of weak injectivity, tightness, and weak tightness. The proofs are straightforward, for the sake of convenience of the reader we provide proofs to some of these implications. The texts [22], [8] are good general references for module theoretic notions of continuous and discrete modules (see also [17]).

Theorem 2.8. *For a module M_R, the following are equivalent:*

(a) *M is semisimple;*
(b) *every weakly injective module in $\sigma[M]$ is* (*quasi*)*-discrete;*
(c) *every weakly injective module in $\sigma[M]$ is* (*quasi*)*-continuous;*
(d) *every* (*direct summand of* a) *weakly injective module in $\sigma[M]$ is* (*injective*) *projective in $\sigma[M]$;*
(e) *every direct summand of a weakly injective module in $\sigma[M]$ is quasi-injective in $\sigma[M]$.*

Proof. (d) $\Longrightarrow$ (a). Let $X \in \sigma[M]$. By Lemma 2.7, $X \oplus \widehat{X^{(\alpha)}}$, where α is an infinite cardinal number, is weakly injective in $\sigma[M]$. Thus X is projective, proving that M is semisimple. The other implications are similar and thus are left to the reader. □

Theorem 2.9. *For a module M_R. The following are equivalent:*

(a) *M is weakly semisimple (resp., weakly R-semisimple);*
(b) *every direct summand of a weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) module in $\sigma[M]$ is weakly injective (or tight, weakly tight) (resp., weakly R-injective) (or R-tight, weakly R-tight) in $\sigma[M]$.*

Proof. (b) $\Longrightarrow$ (a). Let $N \in \sigma[M]$. By Lemma 2.7, there exists a module $Q \in \sigma[M]$ such that $Q \oplus N$ is weakly injective and thus N is weakly injective, proving that M is weakly semisimple. The other cases are similar and thus are left to the reader. □

In case $M = R$ in the above two theorems, we get characterizations of semisimple, weakly semisimple, and weakly R-semisimple rings.

3. Weak-injectivity and countably thick modules

Let M_R be a fixed module and $\mathcal{K}$ a class of simple modules in $\sigma[M]$. We denote

$$\operatorname{Soc}_{\mathcal{K}}(X) = \sum \{A \subseteq X \mid A \simeq P \text{ for some } P \in \mathcal{K}\}.$$

Recall in [4], [5], [6] that $X \in \sigma[M]$ is said to be *countably thick relative to* $\mathcal{K}$ if $\operatorname{Soc}_{\mathcal{K}}(X/A)$ is finitely generated for all $A \subseteq X$. In particular, if $\mathcal{K}$ is the class of all simple modules in $\sigma[M]$ then $X \in \sigma[M]$ is countably thick relative to $\mathcal{K}$ if and only if all factor modules of X have finite uniform dimension, that is X is *locally q.f.d.* (see [4, Lemma 1], [5], [6]).

For a module X_R and a module theoretic property $\mathbb{P}$, X is said to be $\sum -\mathbb{P}$ in case every direct sum of copies of X has the property $\mathbb{P}$. Also we call X *locally* $\mathbb{P}$ in case every finitely generated submodule of X has the property $\mathbb{P}$ (see [1], [3], [18]).

Lemma 3.1 ([4, Corollary 5]). *For a module M_R and any class $\mathcal{K}$ of simple modules in $\sigma[M]$, the following are equivalent.*

(a) *M is locally countably thick relative to $\mathcal{K}$;*
(b) *every cyclic submodule of M is countably thick relative to $\mathcal{K}$;*
(c) *every finitely generated (cyclic) module in $\sigma[M]$ is countably thick relative to $\mathcal{K}$;*
(d) *every module in $\sigma[M]$ is locally countably thick relative to $\mathcal{K}$.*

Theorem 3.2. *For a module M_R, the following holds.*

(a) *if every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$ is weakly injective in $\sigma[M]$, then every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injective modules in $\sigma[M]$ is weakly injective in $\sigma[M]$;*
(b) *if every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules in $\sigma[M]$ is tight in $\sigma[M]$, then every direct sum $\bigoplus_\Lambda M_\lambda$ of tight modules in $\sigma[M]$ is tight in $\sigma[M]$;*

(c) *if every direct sum $\bigoplus_\Lambda E_\lambda$ of injective modules in $\sigma[M]$ is weakly tight in $\sigma[M]$, then every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly tight modules in $\sigma[M]$ is weakly tight in $\sigma[M]$.*

Proof. (a) Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of weakly injective modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X}$. By our hypothesis, the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly injective in $\sigma[M]$ and $X = \bigoplus_\Lambda M_\lambda \subseteq' \bigoplus_\Lambda \widehat{M_\lambda} \subseteq' \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. Thus, there exists a submodule $Y \subseteq \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$ such that $N \subseteq Y \cong \bigoplus_\Lambda \widehat{M}_\lambda$. Write $Y = \bigoplus_\Lambda \widehat{Y}_\lambda$, where $Y_\lambda \cong M_\lambda, \lambda \in \Lambda$. Since N is finitely generated, there exists a finite subset Γ of Λ such that $N \subseteq \bigoplus_\Gamma \widehat{Y_\lambda} = \widehat{\bigoplus_\Gamma Y_\lambda}$. Since $Y_\lambda, \lambda \in \Gamma$ are weakly injective in $\sigma[M]$, the finite direct sum $\bigoplus_\Gamma \widehat{Y_\lambda}$ is weakly injective in $\sigma[M]$ (cf. Lemma 2.2, (i)). Therefore, there exists $X_1 \cong \bigoplus_\Gamma Y_\lambda \cong \bigoplus_\Gamma M_\lambda$ such that $N \subseteq X_1 \subseteq \widehat{\bigoplus_\Gamma Y_\lambda}$. Thus $N \subseteq X_1 \oplus \bigoplus_{\lambda \notin \Gamma} Y_\lambda \simeq X$, proving that X is weakly injective.

(b) Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X} = \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. By our hypothesis, the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is tight in $\sigma[M]$. Thus, N embeds in $\bigoplus_\Lambda \widehat{M}_\lambda$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subseteq \bigoplus_\Gamma \widehat{M}_\lambda$ for some finite $\Gamma \subseteq \Lambda$. Since $\bigoplus_\Gamma M_\lambda$ is tight, $N \simeq \varphi(N)$ embeds in the finite direct sum $\bigoplus_\Gamma M_\lambda$, proving that X is tight.

(c) Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of weakly tight modules in $\sigma[M]$. Let N be a finitely generated submodule of $\widehat{X} = \widehat{\bigoplus_\Lambda \widehat{M_\lambda}}$. By the hypothesis, the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly tight in $\sigma[M]$. Thus, N embeds in $(\bigoplus_\Lambda \widehat{M}_\lambda)^{(\aleph_0)}$ via a monomorphism, say, φ. Since $\varphi(N)$ is finitely generated, $N \subseteq \bigoplus_\Gamma \widehat{M}_\lambda$ for some finite $\Gamma \subseteq \Lambda$. Since $\bigoplus_\Gamma M_\lambda$ is weakly tight, $N \simeq \varphi(N)$ embeds in a direct sum of copies of $\bigoplus_\Gamma M_\lambda$, proving that X is weakly tight. □

Notice that in Theorem 3.2, we can restrict to modules X which are essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$ for a given class $\mathcal{K}$ of simple modules in $\sigma[M]$. The next theorem provides several characterizations of countably thick (consequently, *locally q.f.d.*) modules which extends the main result in [26]. Consequently, we get the main result in [2] as a corollary to the main results of this section.

Theorem 3.3. *For a module M_R, and any class $\mathcal{K}$ of simples in $\sigma[M]$, the following conditions are equivalent:*

(a) *every cyclic submodule of M is countably thick relative to $\mathcal{K}$;*
(b) *M is locally countably thick relative to $\mathcal{K}$;*
(c) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$, is tight in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda E_\lambda$ of tight modules in $\sigma[M]$, where each E_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$, is tight in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda E_\lambda$ of weakly tight modules in $\sigma[M]$, where each E_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$, is weakly tight in $\sigma[M]$;*

(f) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly tight modules in* $\sigma[M]$*, where each* E_λ *is essential over* $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$*, is weakly* N*-tight, for every cyclic module* N *in* $\sigma[M]$*;*

(g) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where* $P_\lambda \in \mathcal{K}$*, is weakly* N*-tight, for every cyclic module* N *in* $\sigma[M]$*.*

Proof. (a)$\Longleftrightarrow$(b) follows from Lemma 3.1.

(b) $\Longrightarrow$ (c) Consider $X = \bigoplus_\Lambda E_\lambda$, where E_λ is injective in $\sigma[M]$ for every $\lambda \in \Lambda$ and $\mathrm{Soc}_{\mathcal{K}}(E_\lambda)$ is essential in E_λ. Let N be a finitely generated submodule of $\widehat{X}$. By the hypothesis, $\mathrm{Soc}_{\mathcal{K}}(N)$ is finitely generated, that is, $\mathrm{Soc}_{\mathcal{K}}(N) = P_1 \oplus \cdots \oplus P_n$ with $P_i \simeq P_i'$ for some $P_i' \in \mathcal{K}$ $(1 \le i \le n)$. So $\mathrm{Soc}_{\mathcal{K}}(N) \subseteq \mathrm{Soc}_{\mathcal{K}}(\widehat{X}) = \mathrm{Soc}_{\mathcal{K}}(X) \subseteq X$ and hence $\mathrm{Soc}_{\mathcal{K}}(N) \subseteq E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. This implies that $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ contains $\widehat{\mathrm{Soc}_{\mathcal{K}}(N)}$. Thus N embeds in X, proving that X is tight.

(c) $\Longrightarrow$ (d) Follows from Theorem 3.2 (b).

(d) $\Longrightarrow$ (e) Consider the module $X = \bigoplus_\Lambda M_\lambda$ a direct sum of weakly tight modules in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_{\mathcal{K}}(M_\lambda)$. Let N be a finitely generated submodule of $\widehat{X}$. By (d) the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is tight in $\sigma[M]$. Thus N embeds in $\bigoplus_\Lambda \widehat{M_\lambda}$ via a monomorphism, say, φ. Also $\varphi(N)$ is finitely generated and thus $N \subseteq \bigoplus_\Gamma \widehat{M_\lambda}$ for some finite $\Gamma \subseteq \Lambda$. Since $\bigoplus_\Gamma M_\lambda$ is weakly tight, $N \simeq \varphi(N)$ embeds in a finite direct sum of copies of $(\bigoplus_\Gamma M_\lambda)$, and thus embeds in a finite direct sum of copies of X, proving that X is weakly tight.

Clearly, (e) $\Longrightarrow$ (f) $\Longrightarrow$ (g).

$(g) \Longrightarrow (a)$ Let K be a cyclic submodule of M. If $\mathrm{Soc}_{\mathcal{K}}(K) = 0$, we are done. Suppose $0 \neq \mathrm{Soc}_{\mathcal{K}}(K) = \bigoplus_\Lambda P_\lambda$. We show that $\mathrm{Soc}_{\mathcal{K}}(K)$ is finitely generated. For this consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_\Lambda P_\lambda & \xrightarrow{\ \gamma\ } & K \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ & & \widehat{\bigoplus_\Lambda \widehat{P}_\lambda} & & \end{array}$$

where φ and γ are the inclusion homomorphisms. By M-injectivity of $\widehat{\bigoplus_\Lambda \widehat{P}_\lambda}$, there exists $\psi : K \to \widehat{\bigoplus_\Lambda \widehat{P}_\lambda}$ such that $\psi\gamma = \varphi$. By our hypothesis, $\bigoplus_\Lambda \widehat{P}_\lambda$ is weakly R-tight in $\sigma[M]$, hence $\mathrm{Im}\psi$ is embeddable in $(\bigoplus_\Lambda \widehat{P}_\lambda)^{(\aleph_0)}$. Therefore, $\mathrm{Soc}_{\mathcal{K}}(K)$ is embeddable in $\widehat{P}_{\lambda_1} \oplus \cdots \oplus \widehat{P}_{\lambda_m}$ for some natural number m and $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda$. Since each $\widehat{P}_{\lambda_i}$ is uniform, $\mathrm{Soc}_{\mathcal{K}}(K)$ has finite uniform dimension and is therefore finitely generated. $\square$

Taking $\mathcal{K}$ to be the class of all simple R-modules in $\sigma[M]$ in Theorem 3.3, we get [26, Theorem 2.6] as a corollary.

Corollary 3.4 ([26, Theorem 2.6])**.** *For a module* M_R*, the following conditions are equivalent:*

(a) *M is locally q.f.d.;*

(b) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of injectives in* $\sigma[M]$ *is weakly injective* (*or tight, weakly tight*) *in* $\sigma[M]$*;*
(c) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly injective in* $\sigma[M]$ *is weakly injective* (*or tight, weakly tight*) *in* $\sigma[M]$*;*
(d) *every direct sum of tight modules in* $\sigma[M]$ *is tight* (*or weakly tight*) *in* $\sigma[M]$*;*
(e) *every direct sum of weakly tight modules in* $\sigma[M]$ *is weakly tight* (*or weakly* R*-tight*) *in* $\sigma[M]$*;*
(f) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is simple, is weakly* N*-tight for every cyclic module* N *in* $\sigma[M]$*;*
(g) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is simple, is weakly* R*-tight in* $\sigma[M]$*.*

In case $M = R_R$ in Corollary 3.4, we obtain characterizations of q.f.d. rings that generalizes Theorem 2.6 and Corollary 2.7 in [30] and the main theorem in [2].

Corollary 3.5 ([26, Theorem 2.7])**.** *For a ring* R*, the following conditions are equivalent:*
(a) R *is q.f.d.;*
(b) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of injectives is weakly injective* (*or tight, weakly tight*)*;*
(c) *every direct sum* $\bigoplus_\Lambda E_\lambda$ *of weakly injective is weakly injective* (*or tight, weakly tight*)*;*
(d) *every direct sum of tight modules is tight* (*or weakly tight*)*;*
(e) *every direct sum of weakly tight module is weakly tight* (*or weakly* R*-tight*)*;*
(f) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is simple, is weakly* N*-tight for every cyclic module* N*;*
(g) *every direct sum* $\bigoplus_\Lambda \widehat{P_\lambda}$*, where each* P_λ *is simple, is weakly* R*-tight.*

Theorem 3.6. *A locally q.f.d. right* R*-module* M_R *over which every uniform cyclic right module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$ *is weakly semisimple.*

Proof. Let $N \in \sigma[M]$. Then N contains an essential submodule $X = \bigoplus_\Lambda X_\lambda$ which is a direct sum of cyclic uniform submodules. It follows by our hypothesis that each X_λ is weakly injective in $\sigma[M]$ and thus by Corollary 3.4, $\bigoplus_\Lambda X_\lambda$ is weakly injective in $\sigma[M]$. Thus N is weakly injective in $\sigma[M]$, proving that M is weakly semisimple. Now, the proofs of the other cases follow from Lemma 2.3, since every uniform weakly tight module in $\sigma[M]$ is weakly injective in $\sigma[M]$. □

A module X in $\sigma[M]$ is called *compressible* if it is embeddable in each of its essential submodules.

Theorem 3.7. *For a module* M_R*, the following conditions are equivalent:*
(a) M *is weakly semisimple;*
(b) M *is locally q.f.d. and every finitely generated module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$*;*
(c) M *is locally q.f.d. and every cyclic module in* $\sigma[M]$ *is weakly injective* (*tight, weakly tight*) *in* $\sigma[M]$*;*

(d) *M is locally q.f.d. and every uniform cyclic module in $\sigma[M]$ is weakly injective* (*tight, weakly tight*) *in $\sigma[M]$;*
(e) *M is locally q.f.d. and every finitely generated module in $\sigma[M]$ is compressible.*

Proof. (a) $\Longrightarrow$ (b) Follows from Corollary 3.4.

Clearly, (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d).

(d) $\Longrightarrow$ (e) Let N be a finitely generated module in $\sigma[M]$ and let $K \subseteq' N$. Since M is locally *q.f.d.*, N has finite uniform dimension. Thus there exists cyclic uniform submodules $U_i, i = 1, \ldots, n$, of N such that $\oplus_{i=1}^{i=n} U_i \subseteq' K \subseteq N$. Since each U_i is uniform it follows that each U_i is weakly injective in $\sigma[M]$ and thus by Lemma 2.2(i), $\oplus_{i=1}^{i=n} U_i$ is weakly injective in $\sigma[M]$. Thus, by Lemma 2.2(ii), K is weakly injective in $\sigma[M]$ and thus N embeds in K, proving that N is compressible.

(e) $\Longrightarrow$ (a) Let $0 \neq X$ in $\sigma[M]$ and let N be a finitely generated submodule of $\widehat{X}$. Since, $X \subseteq' \widehat{X}$, $X \cap N \subseteq' N$. Since M is locally *q.f.d.*, N has finite uniform dimension, and so there exists a finitely generated submodule F of $X \cap N$ which is essential in N. By our hypothesis N is compressible and thus N embeds in F and thus embeds in X, proving that X is tight in $\sigma[M]$. Thus, M is weakly semisimple by Theorem 3.6. □

As a consequence of Theorem 3.7 we get Theorem 3.1 in [9].

In case $M = R$ we obtain characterizations of weakly semisimple rings that generalizes those known results.

Corollary 3.8. *For a ring R, the following conditions are equivalent:*
(a) *R is weakly semisimple;*
(b) *R is q.f.d. and every finitely generated module is weakly injective* (*tight, weakly tight*)*;*
(c) *R is q.f.d. and every cyclic module is weakly injective* (*tight, weakly tight*)*;*
(d) *R is q.f.d. and every uniform cyclic module is weakly injective* (*tight, weakly tight*)*;*
(e) *R is q.f.d. and every finitely generated module is compressible.*

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