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ON THE EXISTENCE OF SOLUTIONS OF SOME SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. We consider a second order nonlinear difference equation

$$\Delta^2 y_n = a_n y_{n+1} + f(n, y_n, y_{n+1}), \quad n \in N. \tag{E}$$

The necessary conditions under which there exists a solution of equation (E) which can be written in the form

$$y_{n+1} = \alpha_n u_n + \beta_n v_n, \quad \text{are given.}$$

Here u and v are two linearly independent solutions of equation

$$\Delta^2 y_n = a_{n+1} y_{n+1}, \quad (\lim_{n\to\infty} \alpha_n = \alpha < \infty \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \beta < \infty).$$

A special case of equation (E) is also considered.

1. Introduction

Consider the difference equation

$$\Delta^2 y_n = a_n y_{n+1} + f(n, y_n, y_{n+1}), \quad n \in N, \tag{E}$$

where N denotes the set of positive integers. By N_0 we define the set $\{n_0, n_0 + 1, \dots\}$ where $n_0 \in N$, by R the set of real numbers and by R_+ the set of real nonnegative numbers. By a solution of equation (E) we mean a sequence (y_n) which satisfies equation (E) for sufficiently large n. The necessary conditions under which there exists a solution of equation (E) which can be written in the following form

$$y_{n+1} = \alpha_n u_n + \beta_n v_n \tag{1}$$

are given. Here u and v are two linearly independent solutions of equation

$$\Delta^2 y_n = a_{n+1} y_{n+1},$$

where

$$\lim_{n\to\infty} \alpha_n = \alpha < \infty \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \beta < \infty.$$

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In the last few years there has been an increasing interest in the study of asymptotic behavior of solutions of difference equations, in particular second order difference equations (see, for example [2]–[3], [6]–[13]).

The equation (E) was considered by Migda, Schmeidel and Zbąszyniak in [9], too. This equation was considered under assumption

$$\int_{\epsilon}^{\infty} \frac{ds}{F(s)} = \infty \,. \tag{2}$$

In [9], the authors proved that each solution of equation (E) can be written in the form (1). In presented paper, we will show that under assumption

$$\int_{0}^{\epsilon} \frac{ds}{F(s)} = \infty \,, \tag{3}$$

where ϵ is a positive constant, there exists a solution of equation (E), which can be written in the form (1). It is clear that there exist functions F which satisfy condition (3) and for which condition (2) is not fulfil, for example $F(x) = x^2$.

To prove the main result we start with the following Lemmas:

Lemma 1. *Assume that $F : R \to R$ is continuous, nondecreasing function, such that $F(x) \neq 0$ for $x \neq 0$ and condition (3) holds. Moreover, let the function $B : N \times R_+^2 \to R_+$ be continuous on R_+^2 for each $n \in N$ and such that*

$$B(n, z_1, z_2) \le B(n, y_1, y_2) \quad \text{for} \quad 0 \le z_k \le y_k \,, \quad k = 1, 2 \,,$$

and

$$B(n, a_n z_1, a_n z_2) \le F(a_n) B(n, z_1, z_2) \quad \text{for} \quad a : N \to R_+ \,.$$

Let (μ_n) and (ρ_n) are positive sequences which satisfy the following inequality

$$\mu_n \le \mu_{n_0} + c \sum_{j=n_0}^{n-1} \rho_j B(j, \rho_{j-1}\mu_{j-1}, \rho_j \mu_j)$$

for $n \ge n_0$, $n_0 \in N$ and some positive constant c, and the series

$$\sum_{j=n_0}^{\infty} \rho_j B(j, \rho_{j-1}, \rho_j) \tag{4}$$

is convergent. Then there exists a sequence (μ_n) such that $\mu_n \le M$ for some $M > 0$, for all $n \in N_0$.

Proof. Let positive sequences (μ_n) and (ρ_n) satisfy the inequality

$$\mu_n \le \mu_{n_0} + c \sum_{j=n_0}^{n-1} \rho_j B(j, \rho_{j-1}\mu_{j-1}, \rho_j \mu_j) \,.$$

We denote $b_n = \mu_{n_0} + c \sum\limits_{j=n_0}^{n-1} \rho_j B(j, \rho_{j-1}\mu_{j-1}, \rho_j \mu_j)$. Since

$$\mu_i \le b_i \,, \quad i \ge n_0 \tag{5}$$

and

$$\Delta b_i = b_{i+1} - b_i = c\rho_i B(i, \rho_{i-1}\mu_{i-1}, \rho_i \mu_i) \geq 0\,,$$

we see that the sequence (b_i) is nondecreasing. Therefore, by (5) we have

$$\Delta b_i \leq c\rho_i B(i, \rho_{i-1} b_{i-1}, \rho_i b_i) \leq c\rho_i B(i, \rho_{i-1} b_i, \rho_i b_i) \leq c\rho_i F(b_i) B(i, \rho_{i-1}, \rho_i)\,,$$

where $F(b_i) \geq 0$. This imply,

$$\frac{\Delta b_i}{F(b_i)} \leq c\rho_i B(i, \rho_{i-1}, \rho_i)\,. \tag{6}$$

Since the function F is nondecreasing, it follows that the function $\frac{1}{F}$ is nonincreasing. This yields

$$\frac{\Delta b_i}{F(b_i)} \geq \int\limits_{b_i}^{b_{i+1}} \frac{ds}{F(s)}\,. \tag{7}$$

From (6) and (7) we have

$$\int\limits_{b_i}^{b_{i+1}} \frac{ds}{F(s)} \leq c\rho_i B(i, \rho_{i-1}, \rho_i), \quad i \geq n_0\,.$$

By summation from $i = n_0$ to $i = n-1$ one yields

$$\int\limits_{b_{n_0}}^{b_n} \frac{ds}{F(s)} \leq c \sum_{i=n_0}^{n-1} \rho_i B(i, \rho_{i-1}, \rho_i)\,. \tag{8}$$

Denoting

$$\int\limits_{\epsilon}^{x} \frac{ds}{F(s)} = G(x)\,, \quad \text{where } \epsilon \text{ is a positive constant} \tag{9}$$

we obtain that

$$\int\limits_{b_{n_0}}^{b_n} \frac{ds}{F(s)} = G(b_n) - G(b_{n_0})\,.$$

From this and (8) we see

$$G(b_n) \leq G(b_{n_0}) + c \sum_{i=n_0}^{n-1} \rho_i B(i, \rho_{i-1}, \rho_i)\,. \tag{10}$$

From (9) and properties of function F, function G is increasing. We have two possibilities:

(i) $\lim\limits_{x\to\infty} G(x) = \infty$. Then $G(b_{n_0}) + c \sum\limits_{i=n_0}^{n-1} \rho_i B(i, \rho_{i-1}, \rho_i)$ belongs to the domain of function G^{-1}, for every $n \in N$.

(ii) $\lim_{x\to\infty} G(x) = g < \infty$. From (3) we can take b_{n_0} such that

$$G(b_{n_0}) + c \sum_{i=n_0}^{\infty} \rho_i B(i, \rho_{i-1}, \rho_i) < g\,.$$

Then there exists a sequence (μ_n) such that $G(b_{n_0}) + c \sum_{i=n_0}^{\infty} \rho_i B(i, \rho_{i-1}, \rho_i)$ belongs to domain of function G^{-1} in this case, too.

Hence G^{-1} exists and is increasing.

We conclude from (10), that

$$b_n \le G^{-1} \left\{ G(b_{n_0}) + c \sum_{i=n_0}^{n-1} \rho_i B(i, \rho_{i-1}, \rho_i) \right\}\,,$$

and finally from (5) and (4), that

$$\mu_n \le G^{-1} \left\{ G(b_{n_0}) + c \sum_{i=n_0}^{\infty} \rho_i B(i, \rho_{i-1}, \rho_i) \right\} \le M\,,$$

where $n \in N_0$. □

Lemma 2. *The equation*

$$\Delta^2 z_n = a_{n+1} z_{n+1}\,, \quad n \in N \tag{EL}$$

where $a : N \to R$, *has linearly independent solutions* $u, v : N \to R$ *such that*

$$\begin{vmatrix} u_n & v_n \\ \Delta u_n & \Delta v_n \end{vmatrix} = -1 \quad \text{for all} \quad n \in N\,. \tag{11}$$

Theorem 1. *Let* (u_n) *and* (v_n) *are linearly independent solutions of equation* (EL). *Assume that*

$$|f(n, x_1, x_2)| \le B(n, |x_1|, |x_2|) \tag{12}$$

for all $x_1, x_2 \in R$, *and any fixed* $n \in N$, *where* $f : N \times R^2 \to R$ *and function* B *fulfil conditions of Lemma 1. Let us denote*

$$U_j = \max\{|u_{j-1}|, |v_{j-1}|, |u_j|, |v_j|, |u_{j+1}|, |v_{j+1}|\}\,. \tag{13}$$

If

$$\sum_{j=2}^{\infty} U_j B(j, U_{j-1}, U_j) = K < \infty \tag{14}$$

for some positive constant K, *then there exists a solution* (y_n) *of equation* (E), *which can be written in the form*

$$y_{n+1} = \alpha_n u_n + \beta_n v_n \tag{15}$$

where $\lim_{n\to\infty} \alpha_n = \alpha$ *and* $\lim_{n\to\infty} \beta_n = \beta$, *(*$\alpha$, β*-constants).*

Proof. First we prove the theorem for two linearly independent solutions (u_n) and (v_n) of equation (EL) which fulfil the condition (11). Assume that (y_n) is an arbitrary solution of equation (E). Let us denote

$$A_n = v_n \Delta y_n - y_{n+1} \Delta v_{n-1} \tag{16}$$

$$B_n = -u_n \Delta y_n + y_{n+1} \Delta u_{n-1} \,. \tag{17}$$

From (11) we get

$$y_{n+1} = u_n A_n + v_n B_n \,. \tag{18}$$

Applying the difference operator Δ to (16) and (17) we obtain

$$\Delta A_n = v_n \Delta^2 y_n - y_{n+1} \Delta^2 v_{n-1}$$

$$\Delta B_n = -u_n \Delta^2 y_n + y_{n+1} \Delta^2 u_{n-1} \,.$$

Using (EL) and (E) we have

$$\Delta A_n = v_n f(n, y_n, y_{n+1})$$

$$\Delta B_n = -u_n f(n, y_n, y_{n+1}) \,.$$

From (18) we obtain

$$\Delta A_j = v_j f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j)$$

$$\Delta B_j = -u_j f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j) \,, \quad j > 1 \,.$$

By summation we get

$$\begin{aligned} A_n &= A_2 + \sum_{j=2}^{n-1} v_j f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j) \\ B_n &= B_2 - \sum_{j=2}^{n-1} u_j f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j) \,. \end{aligned} \tag{19}$$

Then

$$|A_n| \le |A_2| + \sum_{j=2}^{n-1} |v_j| \, |f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j)|$$

$$|B_n| \le |B_2| + \sum_{j=2}^{n-1} |u_j| \, |f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j)| \,.$$

Therefore, we have

$$\begin{aligned} |A_n| + |B_n| &\le |A_2| + |B_2| \\ &\quad + \sum_{j=2}^{n-1} (|v_j| + |u_j|) |f(j, u_{j-1} A_{j-1} + v_{j-1} B_{j-1}, u_j A_j + v_j B_j)| \,. \end{aligned} \tag{20}$$

Let us denote

$$h_n = |A_n| + |B_n| \,, \quad n \in N \,. \tag{21}$$

By the definition of U_j we see that

$$|v_{j-1}| \le U_j\,,\quad |u_{j-1}| \le U_j\,,\quad |v_j| \le U_j\,,\quad |u_j| \le U_j\,,\quad |v_{j+1}| \le U_j\,,\quad |u_{j+1}| \le U_j\,.$$

It is clear that

$$|A_j u_j + B_j v_j| \le |A_j|\,|u_j| + |B_j|\,|v_j| \le U_j(|A_j| + |B_j|) \le U_j h_j\,.$$

Hence, by (12) we get

$$|f(j, A_{j-1}u_{j-1} + B_{j-1}v_{j-1}, A_j u_j + B_j v_j)| \le B(j, U_{j-1}h_{j-1}, U_j h_j)\,.$$

Therefore, (20) and (21) yields

$$h_n \le h_2 + 2\sum_{j=2}^{n-1} U_j B(j, U_{j-1}h_{j-1}, U_j h_j)\,.$$

By Lemma 1, there exists a sequence (h_n) and a constant $M > 0$ such that $h_n \le M$. Properties of function B and (12) give the following inequalities

$$\begin{aligned}
|v_j f(j, A_{j-1}u_{j-1} + B_{j-1}v_{j-1}, A_j u_j + B_j v_j)|&\\
&\le U_j B(j, |A_{j-1}u_{j-1} + B_{j-1}v_{j-1}|, |A_j u_j + B_j v_j|)\\
&\le U_j B(j, U_{j-1}h_{j-1}, U_j h_j) \le U_j B(j, U_{j-1}M, U_j M)\\
&\le F(M) U_j B(j, U_{j-1}, U_j)\,.
\end{aligned}$$

This means by (14) that the series

$$\sum_{j=2}^{\infty} v_j f(j, A_{j-1}u_{j-1} + B_{j-1}v_{j-1}, A_j u_j + B_j v_j)$$

is absolutely convergent. By (19) finite limit $\lim_{n\to\infty} A_n = \alpha$ exists. Analogously $\lim_{n\to\infty} B_n = \beta < \infty$ exists. Hence (18) holds, and there exist finite limits of sequences (A_n) and (B_n).

Now, we will prove this theorem for any two linearly independent solutions $(\tilde{u}_n)$ and $(\tilde{v}_n)$ of equation (EL). Let (u_n) and (v_n) be two linearly independent solutions of equation (EL) fulfilling condition (11). Then for some constants c_1, c_2, c_3 and c_4 we have

$$u_n = c_1\tilde{u}_n + c_2\tilde{v}_n\,,\quad v_n = c_3\tilde{u}_n + c_4\tilde{v}_n\,.$$

Now,

$$\tilde{U}_j = \max\left\{|\tilde{u}_{j-1}|, |\tilde{v}_{j-1}|, |\tilde{u}_j|, |\tilde{v}_j|, |\tilde{u}_{j+1}|, |\tilde{v}_{j+1}|\right\}\,.$$

We will show that the condition (14) holds. Let $\tilde{c} = \max\{|c_1|, |c_2|, |c_3|, |c_4|\}$. Hence

$$U_j \le \tilde{c}\max\left\{|\tilde{u}_{j-1}| + |\tilde{v}_{j-1}|, |\tilde{u}_j| + |\tilde{v}_j|, |\tilde{u}_{j+1}| + |\tilde{v}_{j+1}|\right\} \le 2\tilde{c}\tilde{U}_j\,.$$

Therefore, we obtain inequalities

$$U_j B(j, U_{j-1}, U_j) \le 2\tilde{c}\tilde{U}_j B(j, 2\tilde{c}\tilde{U}_{j-1}, 2\tilde{c}\tilde{U}_j) \le 2\tilde{c}\tilde{U}_j F(2\tilde{c}) B(j, \tilde{U}_{j-1}, \tilde{U}_j)\,,$$

and

$$\sum_{j=1}^{\infty} U_j B(j, U_{j-1}, U_j) < \infty\,.$$

We see that assumptions of the Theorem 1 hold for solutions (u_n) and (v_n), also. Then a solution of equation (E) can be written in the form

$$\begin{aligned} y_{n+1} &= A_n(c_1\tilde{u}_n + c_2\tilde{v}_n) + B_n(c_3\tilde{u}_n + c_4\tilde{v}_n) \\ &= (c_1A_n + c_3B_n)\tilde{u}_n + (c_2A_n + c_4B_n)\tilde{v}_n \\ &= \alpha_n\tilde{u}_n + \beta\tilde{v}_n\,, \end{aligned}$$

where $\alpha_n = c_1A_n + c_3B_n$, $\beta_n = c_2A_n + c_4B_n$, and $\lim_{n\to\infty}\alpha_n = \alpha$, $\lim_{n\to\infty}\beta_n = \beta$ (α, β-constants). This completes the proof of this Theorem. □

Example 1. *Consider the difference equation*

$$\Delta^2 y_n = \frac{y_n y_{n+1}}{(n^2+3n+2)2^{n+2} + 6n + 10 + 2^{1-n}}\,. \tag{22}$$

All conditions of Theorem 1 are satisfied with $B(n, x_1, x_2) = \frac{x_1x_2}{n^2 2^n}$ *and* $F(k) = k^2$. *Hence the equation* (22) *has a solution* (y_n) *which can be written in the form* (15). *In fact,* $y_n = n + (1 + \frac{1}{2^n})1$ *is such a solution, where* $\alpha_n = 1$ *and* $\beta_n = 1 + \frac{1}{2^n}$.

Note, that Theorem 1 is applicable to the equation (22), but Theorem 1 from [9] is not, because

$$\int_\epsilon^\infty \frac{ds}{F(s)} = \int_\epsilon^\infty \frac{ds}{s^2} = \frac{1}{\epsilon}$$

is convergent. So, condition (1) from [9] is not satisfied.

Theorem 2. *Assume that functions F and B fulfil conditions of Lemma 1 and function F fulfil condition* (12) *of Theorem 1. If*

$$\sum_{j=1}^{\infty} jB(j,j,j) = k < \infty\,, \tag{23}$$

then there exists a solution (y_n) *of equation*

$$\Delta^2 y_n = f(n, y_n, y_{n+1})\,, \quad n \in N\,, \tag{24}$$

which can be written in the form

$$y_{n+1} = an + b + \phi(n), \ \textit{where} \ \lim_{n\to\infty}\phi(n) = 0\,. \tag{25}$$

Proof. Equation $\Delta^2 z_n = 0$ has two linearly independent solution $u_n = n$ and $v_n = 1$. These solutions satisfy conditions (11) of Theorem 1. We will prove that condition (14) is also satisfied. From (13), $U_j = j + 1$. From properties of function B we obtain

$$\begin{aligned} U_jB(j, U_{j-1}, U_j) &= (j+1)B(j, j, j+1) \le (j+j)B(j, j+j, j+j) \\ &= (2j)B(j, 2j, 2j) \le 2F(2)jB(j,j,j)\,. \end{aligned}$$

Then, form (23)

$$\sum_{j=1}^{\infty} U_jB(j, U_{j-1}, U_j) \le 2F(2)k = K < \infty\,.$$

Since assumptions of Theorem 1 hold then we get the thesis of this Theorem. So, from (18)

$$y_{n+1} = A_n n + B_n \,, \tag{26}$$

where A_n and B_n are defined by (16) and (17), and finite limits of sequences (A_n), (B_n) exist. Let

$$\lim_{n\to\infty} A_n = a \,, \quad \lim_{n\to\infty} B_n = b \,. \tag{27}$$

From (19) we get

$$A_n = A_2 + \sum_{j=2}^{n-1} f(j, (j-1)A_{j-1} + B_{j-1}, jA_j + B_j) \,.$$

Hence, from (27) we obtain

$$a = A_2 + \sum_{j=2}^{\infty} f(j, (j-1)A_{j-1} + B_{j-1}, jA_j + B_j) \,.$$

Using properties of functions f and B we have

$$\begin{aligned} |A_n - a| &= \sum_{j=n}^{\infty} f\big(j, (j-1)A_{j-1} + B_{j-1}, jA_j + B_j\big) \\ &\le \sum_{j=n}^{\infty} B\big(j, (j-1)|A_{j-1}| + |B_{j-1}|, j|A_j| + |B_j|\big) \\ &\le \sum_{j=n}^{\infty} B\big(j, (j-1)(|A_{j-1}| + |B_{j-1}|), j(|A_j| + |B_j|)\big) \,. \end{aligned}$$

Therefore

$$n|A_n - a| \le \sum_{j=n}^{\infty} jB\big(j, (j-1)(|A_{j-1}| + |B_{j-1}|), j(|A_j| + |B_j|)\big) \,.$$

From (27) there exists a constant c such that

$$|A_n| + |B_n| \le c \quad \text{for} \quad n \in N \,.$$

Then

$$n|A_n - a| \le \sum_{j=n}^{\infty} jB(j, jc, jc) \le F(c) \sum_{j=n}^{\infty} jB(j, j, j)$$

and by (23) we have

$$\lim_{n\to\infty} F(c) \sum_{j=n}^{\infty} jB(j, j, j) = 0 \,,$$

what gives

$$\lim_{n\to\infty} n|A_n - a| = 0 \,.$$

Analogously we obtain $\lim_{n\to\infty} |B_n - b| = 0$. The solution (26) of equation (24) can be written in the form

$$y_{n+1} = an + b + (A_n - a)n + (B_n - b)\,.$$

Then

$$y_{n+1} = an + b + \phi(n)\,,$$

where

$$\phi(n) = (A_n - a)n + (B_n - b)\,,$$

and $\lim_{n\to\infty} \phi(n) = 0$. The proof is complete. □

Example 2. *Consider the difference equation*

$$\Delta^2 y_n = \frac{y_n + y_{n+1}}{2^{n+3}n + 3 \cdot 2^{n+2} + 6}\,. \tag{28}$$

All conditions of Theorem 2 are satisfied with $B(n, x_1, x_2) = \frac{1}{2^n}(x_1 + x_2)$ *and* $F(k) = k$. *Hence equation* (28) *has a solution* (y_n) *which can be written in* (25). *In fact* $y_n = n + 1 + \frac{1}{2^n}$ *is such a solution.*

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