

INTEGRABILITY AND  $L^1$ - CONVERGENCE  
OF REES-STANOJEVIĆ SUMS WITH GENERALIZED  
SEMI-CONVEX COEFFICIENTS OF NON-INTEGRAL ORDERS

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ABSTRACT. Integrability and  $L^1$ -convergence of modified cosine sums introduced by Rees and Stanojević under a class of generalized semi-convex null coefficients are studied by using Cesàro means of non-integral orders.

1. INTRODUCTION

Let

$$(1.1) \quad g(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

be the Fourier cosine series.

The problem of  $L^1$ -convergence of the Fourier cosine series (1.1) has been settled for various special classes of coefficients. Young [9] found that  $a_n \log n = o(1)$ ,  $n \rightarrow \infty$  is a necessary and sufficient condition for the  $L^1$ -convergence of the cosine series with convex ( $\Delta^2 a_n \geq 0$ ) coefficients, and Kolmogorov [8] extended this result to the cosine series with quasi-convex ( $\sum_{n=1}^{\infty} n |\Delta^2 a_n| < \infty$ ) coefficients and proved the following well known theorem:

**Theorem 1.1.** *If  $\{a_n\}$  is a quasi-convex null sequence, then for the  $L^1$ -convergence of the cosine series (1.1), it is necessary and sufficient that  $\lim_{n \rightarrow \infty} a_n \log n = 0$ .*

**Definition** ([6]). A sequence  $\{a_n\}$  is said to be semi-convex, if

$$a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

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2000 *Mathematics Subject Classification*: 42A20, 42A32.

*Key words and phrases*:  $L^1$ -convergences, Cesàro means, conjugate Cesàro mean, semi-convex null coefficients, generalized semi-convex null coefficients, Fourier cosine series.

Received March 9, 2004.

and

$$(1.2) \quad \sum_{n=1}^{\infty} n |\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \quad (a_0 = 0)$$

where

$$\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}.$$

It may be remarked here that every quasi-convex null sequence is semi-convex.

In 1968, Kano [6] generalized Theorem 1.1 in the following form:

**Theorem 1.2.** *If  $\{a_k\}$  is semi-convex null sequence, then (1.1) is a Fourier series, or equivalently it represents an integrable function.*

Later on, Garrett and Stanojević [5] introduced modified cosine sums

$$(1.3) \quad g_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n (\Delta a_j) \cos kx$$

and proved the following theorem.

**Theorem 1.3.** *Let  $\{a_n\}$  be a null sequence of bounded variation. Then the sequence of modified cosine sums*

$$(1.4) \quad g_n(x) = S_n(x) - a_{n+1} D_n(x)$$

where  $S_n(x)$  are the partial sums of the cosine series (1.1) and  $D_n(x)$  is the Dirichlet kernel, converges in  $L^1$ -norm to  $g(x)$ , the pointwise sum of the cosine series, if and only if for every  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$ , independent of  $n$ , such that

$$(1.5) \quad \int_0^{\delta} \sum_{k=n+1}^{\infty} |\Delta a_k D_k(x)| dx < \epsilon, \quad \text{for every } n.$$

This result contains as a special case a number of classical and neo-classical results. In particular, in [5] the following corollary to Theorem 1.3 is proved.

**Theorem 1.4.** *Let  $\{a_n\}$  be a null sequence of bounded variation satisfying condition (1.5). Then the cosine series is the Fourier series of its sum  $g(x)$  and  $\|S_n - g\| = o(1)$ ,  $n \rightarrow \infty$  is equivalent to  $a_n \log n = o(1)$ ,  $n \rightarrow \infty$ .*

In [4] Garrett and Stanojević proved the following theorem:

**Theorem 1.5.** *If  $\{a_n\}$  is a null quasi-convex sequence, then  $g_n(x)$  converges to  $g(x)$  in the  $L^1$ -norm.*

We generalize semi-convexity of null sequence in the following way:

**Definition** ([7]). A sequence  $\{a_n\}$  is said to be generalized semi-convex, if

$$a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$(1.6) \quad \sum_{n=1}^{\infty} n^{\alpha} |(\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n)| < \infty, \quad \text{for } \alpha > 0 \quad (a_0 = 0).$$

For  $\alpha = 1$ , this class reduces to the class defined in [6].

In [7] Kaur, K. and Bhatia, S. S. proved the following theorem:

**Theorem 1.6.** *If*

$$a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$\sum_{n=1}^{\infty} n^\alpha |(\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n)| < \infty,$$

where  $\alpha > 0$  is an integral ( $a_0 = 0$ ), then  $g_n(x)$  converges to  $g(x)$  in  $L^1$ -metric if and only if  $\Delta a_n \log n = o(1)$ , as  $n \rightarrow \infty$ .

The object of this paper is to show that the above mentioned Theorem of Kaur, K. and Bhatia, S. S. holds good for non-integral values of  $\alpha > 0$ .

## 2. NOTATION AND FORMULAE

In what follows, the following notations are used [10]:

Given a sequence  $S_0, S_1, S_2, \dots$ , we define for every  $\alpha = 0, 1, 2, \dots$ , the sequence  $S_0^\alpha, S_1^\alpha, S_2^\alpha, \dots$ , by the conditions:

$$\begin{aligned} S_n^0 &= S_n \\ S_n^\alpha &= S_0^{\alpha-1} + S_1^{\alpha-1} + S_2^{\alpha-1} + \dots + S_n^{\alpha-1} \quad (\alpha = 1, 2, \dots, n = 0, 1, 2, \dots). \end{aligned}$$

Similarly for  $\alpha = 0, 1, 2, \dots$ , we define the sequence of numbers  $A_0^\alpha, A_1^\alpha, A_2^\alpha, \dots$ , by the conditions

$$\begin{aligned} A_n^0 &= 1, \\ A_n^\alpha &= A_0^{\alpha-1} + A_1^{\alpha-1} + A_2^{\alpha-1} + \dots + A_n^{\alpha-1} \quad (\alpha = 1, 2, \dots, n = 0, 1, 2, \dots). \end{aligned}$$

Consider  $\sum a_n$  be a given infinite series. For any real number  $\alpha$  the Cesàro sums of order  $\alpha$  of  $\sum a_n$  are defined by

$$S_n^\alpha(a_p) = S_n^\alpha = \sum_{p=0}^n A_{n-p}^\alpha a_p = \sum_{p=0}^n A_{n-p}^{\alpha-1} S_p,$$

where  $A_p^\alpha$  denotes the binomial coefficients and are given by the following relations

$$\sum_{p=0}^{\infty} A_p^\alpha x^p = (1-x)^{-\alpha-1}$$

and  $S_n$ 's are given by

$$(2.1) \quad \sum_{p=0}^{\infty} S_p^\alpha x^p = (1-x)^{-\alpha} \sum_{p=0}^{\infty} S_p x^p.$$

Also

$$A_n^\alpha = \sum_{p=0}^n A_p^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$$

$$A_n^\alpha = \binom{n+\alpha}{n} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)} \quad (\alpha \neq -1, -2, -3, \dots).$$

The Cesàro means  $T_k^\alpha$  of order  $\alpha$  are then defined by

$$T_k^\alpha = \frac{S_k^\alpha}{A_k^\alpha}.$$

Also for  $0 < x \leq \pi$ , let

$$\begin{aligned} \bar{D}_0(x) &= -\frac{1}{2} \cot \frac{x}{2} \\ \bar{S}_n(x) &= \bar{D}_0(x) + \sin x + \sin 2x + \dots + \sin nx \\ \bar{S}_n^1(x) &= \bar{S}_0(x) + \bar{S}_1(x) + \bar{S}_2(x) + \dots + \bar{S}_n(x) \\ \bar{S}_n^2(x) &= \bar{S}_0^1(x) + \bar{S}_1^1(x) + \bar{S}_2^1(x) + \dots + \bar{S}_n^1(x) \\ &\vdots \\ \bar{S}_n^k(x) &= \bar{S}_0^{k-1}(x) + \bar{S}_1^{k-1}(x) + \bar{S}_2^{k-1}(x) + \dots + \bar{S}_n^{k-1}(x). \end{aligned}$$

The conjugate Cesàro means  $\bar{T}_k^\alpha$  of order  $\alpha$  are defined by

$$(2.2) \quad \bar{T}_k^\alpha = \frac{\bar{S}_k^\alpha}{A_k^\alpha}.$$

The following formulae will also be needed:

$$(2.3) \quad \bar{S}_n^\alpha(\bar{S}_p^r) = \bar{S}_n^{\alpha+r+1},$$

$$(2.4) \quad \bar{S}_n^{\alpha+1} - \bar{S}_{n-1}^{\alpha+1} = \bar{S}_n^\alpha, \quad \sum_{p=0}^n A_{n-p}^\alpha A_p^\beta = A_n^{\alpha+\beta+1}.$$

For any positive integer  $\alpha$  the differences of order  $\alpha$  of the sequence  $\{a_n\}$  are defined by the equations

$$\begin{aligned} \Delta^1 a_n &= a_n - a_{n+1}, \\ \Delta^\alpha a_n &= \Delta(\Delta^{\alpha-1} a_n), \quad n = 0, 1, 2, \dots \end{aligned}$$

For these differences we have

$$(2.5) \quad \Delta^\alpha a_n = \sum_{m=0}^{\alpha} A_m^{-\alpha-1} a_{n+m} = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m},$$

since  $A_m^{-\alpha-1} = 0$  for  $m \geq \alpha + 1$ .

If the series (2.5) are convergent for some  $\alpha$  which is not a positive integer, then we denote the differences

$$(2.6) \quad \Delta^\alpha a_n = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}, \quad n = 0, 1, 2, \dots$$

The broken differences  $\Delta_n^\alpha a_p$  are defined by

$$(2.7) \quad \Delta_n^\alpha a_p = \sum_{m=0}^{n-p} A_m^{-\alpha-1} a_{p+m}.$$

By repeated partial summation of order  $\alpha$ ,

$$(2.8) \quad \sum_{p=0}^n a_p b_p = \sum_{p=0}^n \bar{S}_p^{\alpha-1}(a_p) \Delta_n^\alpha b_p.$$

If  $\alpha$  is positive integer then we have

$$(2.9) \quad \sum_{p=0}^n a_p b_p = \sum_{p=0}^{n-\alpha} \bar{S}_p^{\alpha-1}(a_p) \Delta^\alpha b_p + \sum_{p=n-\alpha+1}^n \bar{S}_p^{\alpha-1}(a_p) \Delta_n^\alpha b_p.$$

### 3. LEMMAS

We need the following lemmas for the proof of our result:

**Lemma 3.1** ([3]). *If  $\alpha \geq 0, p \geq 0$ ,*

- (i)  $\epsilon_n = o(n^{-p})$ ,
- (ii)  $\sum_{n=0}^\infty A_n^{\alpha+p} |\Delta^{\alpha+1} \epsilon_n| < \infty$ ,

then

- (iii)  $\sum_{n=0}^\infty A_n^{\lambda+p} |\Delta^{\lambda+1} \epsilon_n| < \infty$ , for  $-1 \leq \lambda \leq \alpha$  and
- (iv)  $A_n^{\lambda+p} \Delta^\lambda \epsilon_n$  is of bounded variation for  $0 \leq \lambda \leq \alpha$  and tends to zero as  $n \rightarrow \infty$ .

**Lemma 3.2** ([1]). *Let  $r$  be the real number  $\geq 0$ . If the sequence  $\{\epsilon_n\}$  satisfies the conditions:*

- (i)  $\epsilon_n = O(1)$  and
- (ii)  $\sum_{n=1}^\infty n^r |\Delta^{r+1} \epsilon_n| < \infty$ ,

then

$$\Delta^\beta \epsilon_n = \sum_{m=0}^\infty A_m^{r-\beta} \Delta^{r+1} \epsilon_{n+m}, \quad \text{for } \beta > 0.$$

**Lemma 3.3** ([2]). *If  $0 \leq \delta \leq 1$  and  $0 \leq m < n$ , then*

$$\left| \sum_{i=0}^m A_{n-i}^{\delta-1} S_i \right| \leq \max_{0 \leq p \leq m} |S_p^\delta|.$$

**Lemma 3.4** ([10]). *Let  $\bar{S}_n(x)$  and  $\bar{T}_n^\alpha$  be the  $n$ -th partial sum and conjugate Cesàro mean of order  $\alpha > 0$ , respectively, of the series*

$$\bar{D}_0(x) + \sin x + \sin 2x + \sin 3x + \dots + \sin nx + \dots$$

Then

- (i)  $\int_0^\pi |\bar{S}_n(x)| dx \sim \log n$ ,

(ii)  $\int_0^\pi |\bar{T}_n^\alpha| dx$  remains bounded for all  $n$ .

#### 4. MAIN RESULT

The main result of this paper is the following theorem:

**Theorem 4.1.** *If*

$$a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$\sum_{n=1}^{\infty} n^\alpha |(\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n)| < \infty,$$

where  $\alpha > 0$  is non-integral ( $a_0 = 0$ ), then  $g_n(x)$  converges to  $g(x)$  in  $L^1$ -metric if and only if  $\Delta a_n \log n = o(1)$ , as  $n \rightarrow \infty$ .

**Proof.** We have

$$\begin{aligned} (4.1) \quad g_n(x) &= \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n (\Delta a_j) \cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^n a_k \cos kx - a_{n+1} D_n(x) \\ &= \sum_{k=1}^n a_k \cos kx - a_{n+1} D_n(x) \quad (a_0 = 0) \\ &= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2 \sin x} + a_{n-1} \frac{\sin nx}{2 \sin x} \\ &\quad + a_n \frac{\sin(n+1)x}{2 \sin x} - a_{n+1} D_n(x), \end{aligned}$$

where

$$\begin{aligned} D_n(x) &= \frac{\sin nx + \sin(n+1)x}{2 \sin x}, \\ g_n(x) &= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2 \sin x} + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \\ &\quad - a_{n+1} \frac{\sin nx}{2 \sin x} - a_{n+1} \frac{\sin(n+1)x}{2 \sin x} \\ &= \frac{1}{2 \sin x} \sum_{k=1}^n (\Delta a_{k-1} + \Delta a_k) \sin kx + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}. \end{aligned}$$

Applying Abel’s transformation, we have

$$\begin{aligned}
 g_n(x) &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \sum_{v=1}^k \sin vx \right] \\
 &\quad + \frac{1}{2 \sin x} \left[ (\Delta a_{n-1} + \Delta a_n) \sum_{v=1}^n \sin vx \right] + \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \\
 &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) (\bar{S}_k^0(x) - \bar{S}_0(x)) \right] \\
 &\quad + \frac{1}{2 \sin x} \left[ (\Delta a_{n-1} + \Delta a_n) (\bar{S}_n^0(x) - \bar{S}_0(x)) \right] + \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \\
 &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_k^0(x) - \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_0(x) \right] \\
 &\quad + \frac{1}{2 \sin x} \left[ (\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) - (\Delta a_{n-1} + \Delta a_n) \bar{S}_0(x) \right] \\
 &\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \\
 (4.2) \quad &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_k^0(x) + (\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) + a_2 \bar{S}_0(x) \right] \\
 &\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.
 \end{aligned}$$

As  $\alpha > 0$  is non-integral. Let  $\alpha = r + \delta$ ,  $r$  is the integral part of  $\alpha$ , and  $\delta$  is the fractional part,  $0 < \delta < 1$ .

**Case (i).** Let  $r = 0$ .

Applying Abel’s transformation of order  $-\delta + 1$ , we have by (2.8)

$$\begin{aligned}
 \sum_{k=1}^{n-1} \bar{S}_k^{\delta-1}(x) (\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k) \\
 = \sum_{k=1}^{n-1} \bar{S}_k(x) \sum_{m=1}^{n-(k+1)} A_m^{\delta-2} (\Delta^{\delta+1} a_{m+k-1} + \Delta^{\delta+1} a_{m+k})
 \end{aligned}$$

Also by Lemma 3.2, we have

$$\begin{aligned}
 \sum_{k=1}^{n-1} \bar{S}_k^{\delta-1}(x) (\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k) &= \sum_{k=1}^{n-1} \bar{S}_k(x) \left\{ (\Delta^2 a_{k-1} + \Delta^2 a_k) \right. \\
 &\quad \left. - \sum_{m=n-k}^{\infty} A_m^{\delta-2} (\Delta^{\delta+1} a_{m+k-1} + \Delta^{\delta+1} a_{m+k}) \right\} \\
 &= \sum_{k=1}^{n-1} \bar{S}_k(x) (\Delta^2 a_{k-1} + \Delta^2 a_k) - R_n(x)
 \end{aligned}$$

where

$$R_n(x) = \sum_{k=1}^{n-1} \bar{S}_k(x) \{ A_{n-k}^{\delta-2} (\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n) \\ + A_{n-k+1}^{\delta-2} (\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1}) + \dots \}.$$

Therefore

$$\frac{1}{2 \sin x} \sum_{k=1}^{n-1} \bar{S}_k(x) (\Delta^2 a_{k-1} + \Delta^2 a_k) \\ = \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{n-1} \bar{S}_k^{\delta-1}(x) (\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k) + R_n(x) \right\}$$

and consequently by (4.2)

$$g_n(x) = \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{n-1} \bar{S}_k^{\delta-1}(x) (\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k) \right. \\ \left. + R_n(x) + (\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) + a_2 \bar{S}_0(x) \right\} + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.$$

When  $r = 0$ , then  $\alpha = \delta$  and

$$g(x) = \lim_{n \rightarrow \infty} g_n(x) \\ = \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{\infty} \bar{S}_k^{\delta-1}(x) (\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k) + a_2 \bar{S}_0(x) \right\}$$

Therefore,

$$\int_0^\pi |g(x) - g_n(x)| dx \leq \int_0^\pi \left| \frac{1}{2 \sin x} \left\{ \sum_{k=n}^{\infty} \bar{S}_k^{\delta-1}(x) (\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k) \right. \right. \\ \left. \left. - R_n(x) - (\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) \right\} \right| dx + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\ \leq C \left\{ \sum_{k=n}^{\infty} |(\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k)| \int_0^\pi |\bar{S}_k^{\delta-1}(x)| dx \right. \\ \left. + |(\Delta a_{n-1} + \Delta a_n)| \int_0^\pi |\bar{S}_n^0(x)| dx \right\} \\ + C \int_0^\pi |R_n(x)| dx + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx$$



$$\begin{aligned}
 &= C \left\{ \sum_{k=n}^{\infty} A_k^{\delta-1} |(\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k)| \int_0^{\pi} |\bar{T}_k^{\delta-1}(x)| dx \right. \\
 &\quad \left. + |(\Delta a_{n-1} + \Delta a_n)| \int_0^{\pi} |\bar{S}_n^0(x)| dx \right\} \\
 &\quad + C \int_0^{\pi} |R_n(x)| dx + \int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
 &\leq C_1 \sum_{k=n}^{\infty} A_k^{\delta-1} |(\Delta^{\delta+1} a_{k-1} + \Delta^{\delta+1} a_k)| + |(\Delta a_{n-1} + \Delta a_n)| \\
 &\quad + C \int_0^{\pi} |R_n(x)| dx + \int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
 (4.3) \quad &= o(1) + C_1 |(a_{n-1} - a_{n+1})| + C \int_0^{\pi} |R_n(x)| dx + \int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx,
 \end{aligned}$$

by Lemma 3.1 and 3.4.

Now for the estimate of  $\int_0^{\pi} |R_n(x)| dx$ , we have

$$\begin{aligned}
 \int_0^{\pi} |R_n(x)| dx &= \int_0^{\pi} \left| \left( \sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k}^{\delta-2} \right) (\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n) \right. \\
 &\quad \left. + \left( \sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k+1}^{\delta-2} \right) (\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1}) + \dots \right| dx \\
 &\leq \int_0^{\pi} |(\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n)| \left| \sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k}^{\delta-2} \right| dx \\
 &\quad + \int_0^{\pi} |(\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1})| \left| \sum_{k=1}^{n-1} \bar{S}_k(x) A_{n-k+1}^{\delta-2} \right| dx + \dots \\
 &\leq \int_0^{\pi} |(\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n)| \max_{1 \leq p \leq n-1} |\bar{S}_p^{\delta-1}(x)| dx \\
 &\quad + \int_0^{\pi} |(\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1})| \max_{1 \leq p \leq n} |\bar{S}_p^{\delta-1}(x)| dx + \dots,
 \end{aligned}$$

by Lemma 3.3,

$$\begin{aligned}
&= |\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n| A_n^{\delta-1} \int_0^\pi \max_{1 \leq p \leq n-1} |\bar{T}_p^{\delta-1}(x)| dx \\
&\quad + |(\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1})| A_{n+1}^{\delta-1} \int_0^\pi \max_{1 \leq p \leq n} |\bar{T}_p^{\delta-1}(x)| dx + \dots \\
&\leq C A_n^{\delta-1} |(\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n)| \\
&\quad + C A_{n+1}^{\delta-1} |(\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1})| + \dots \\
&\leq C A_n^\delta |(\Delta^{\delta+1} a_{n-1} + \Delta^{\delta+1} a_n)| \\
&\quad + C A_{n+1}^\delta |(\Delta^{\delta+1} a_n + \Delta^{\delta+1} a_{n+1})| + \dots \\
&= o(1) + o(1) + \dots \\
&= o(1), \quad \text{by Lemma 3.1 and 3.4.}
\end{aligned}$$

Moreover, since

$$\int_0^\pi \left| \frac{\sin(n+1)x}{2 \sin x} dx \right| \leq C \log n, \quad n \geq 2.$$

Therefore,

$$\int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} dx \right| \sim \Delta a_n \log n.$$

Thus, by (4.3),

$$\begin{aligned}
(4.4) \quad \lim_{n \rightarrow \infty} \int_0^\pi |g(x) - g_n(x)| dx &= o(1), \quad \text{if and only if} \\
\Delta a_n \log n &= o(1), \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

**Case (ii).** Let  $r \geq 1$ .

We have

$$g_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^n (\Delta a_{k-1} + \Delta a_k) \sin kx + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.$$

Applying Abel's transformation of order  $r$ ,

$$\begin{aligned}
(4.5) \quad g_n(x) &= \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{n-r} (\Delta^{r+1} a_{k-1} + \Delta^{r+1} a_k) \bar{S}_k^{r-1}(x) \right. \\
&\quad \left. + \sum_{k=1}^r (\Delta^k a_{n-k} + \Delta^k a_{n-k+1}) \bar{S}_{n-k+1}^{k-1}(x) + a_2 \bar{S}_0(x) \right\} \\
&\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.
\end{aligned}$$

Again applying Abel's transformation of order  $-\delta$ , we obtain

$$\begin{aligned} \sum_{k=1}^{n-1} \bar{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k) \\ = \sum_{k=1}^{n-1} \bar{S}_k^{r-1}(x) \sum_{m=0}^{n-(k+1)} A_m^{\delta-1}(\Delta^{\alpha+1}a_{m+k-1} + \Delta^{\alpha+1}a_{m+k}). \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} (4.6) \quad \frac{1}{2 \sin x} \sum_{k=1}^{n-1} \bar{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k) \\ = \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{n-1} \bar{S}_k^{r-1}(x)(\Delta^{r+1}a_{k-1} + \Delta^{r+1}a_k) - R_n(x) \right\}, \end{aligned}$$

where

$$\begin{aligned} R_n(x) &= \sum_{k=1}^{n-1} \bar{S}_k^{r-1}(x) \{ A_{n-k}^{\delta-1}(\Delta^{\alpha+1}a_{n-1} + \Delta^{\alpha+1}a_n) \\ &\quad + A_{n-k+1}^{\delta-1}(\Delta^{\alpha+1}a_n + \Delta^{\alpha+1}a_{n+1}) + \dots \} \\ &= \left( \sum_{k=1}^{n-1} \bar{S}_k^{r-1}(x) A_{n-k}^{\delta-1} \right) (\Delta^{\alpha+1}a_{n-1} + \Delta^{\alpha+1}a_n) \\ &\quad + \left( \sum_{k=1}^{n-1} \bar{S}_k^{r-1}(x) A_{n-k+1}^{\delta-1} \right) (\Delta^{\alpha+1}a_n + \Delta^{\alpha+1}a_{n+1}) + \dots \end{aligned}$$

Replacing  $n$  by  $n - r + 1$  in (4.6), we have

$$\begin{aligned} (4.7) \quad \frac{1}{2 \sin x} \sum_{k=1}^{n-r} \bar{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k) \\ = \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x)(\Delta^{r+1}a_{k-1} + \Delta^{r+1}a_k) - R_{n-r+1}(x) \right\}. \end{aligned}$$

Now by (4.5) and (4.7), we have

$$\begin{aligned} (4.8) \quad g_n(x) &= \frac{1}{2 \sin x} \left\{ \sum_{k=1}^{n-r} \bar{S}_k^{\alpha-1}(x)(\Delta^{\alpha+1}a_{k-1} + \Delta^{\alpha+1}a_k) + R_{n-r+1}(x) \right. \\ &\quad \left. + \sum_{k=1}^r (\Delta^k a_{n-k} + \Delta^k a_{n-k+1}) \bar{S}_{n-k+1}^{k-1}(x) + a_2 \bar{S}_0(x) \right\} \\ &\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}. \end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^\pi |g(x) - g_n(x)| dx &\leq \int_0^\pi \left| \frac{1}{2 \sin x} \left\{ \sum_{k=n-r+1}^\infty \bar{S}_k^{\alpha-1}(x) (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \right. \right. \\
&\quad \left. \left. - R_{n-r+1}(x) - \sum_{k=1}^r (\Delta^k a_{n-k} + \Delta^k a_{n-k+1}) \bar{S}_{n-k+1}^{k-1}(x) \right\} \right| dx \\
&\quad + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
&\leq C \sum_{k=n-r+1}^\infty |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| \int_0^\pi |\bar{S}_k^{\alpha-1}(x)| dx \\
&\quad + \int_0^\pi |R_{n-r+1}(x)| dx \\
&\quad + \sum_{k=1}^r |(\Delta^k a_{n-k} + \Delta^k a_{n-k+1})| \int_0^\pi |\bar{S}_{n-k+1}^{k-1}(x)| dx \\
&\quad + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
&= C \sum_{k=n-r+1}^\infty A_k^{\alpha-1} |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| \int_0^\pi |\bar{T}_k^{\alpha-1}(x)| dx \\
&\quad + \int_0^\pi |R_{n-r+1}(x)| dx \\
&\quad + \sum_{k=1}^r A_{n-k+1}^{k-1} |(\Delta^k a_{n-k} + \Delta^k a_{n-k+1})| \int_0^\pi |\bar{T}_{n-k+1}^{k-1}(x)| dx \\
&\quad + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
&\leq C_1 \sum_{k=n-r+1}^\infty A_k^\alpha |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| + C \int_0^\pi |R_{n-r+1}(x)| dx \\
&\quad + C_1 \sum_{k=1}^r A_{n-k+1}^{k-1} |(\Delta^k a_{n-k} + \Delta^k a_{n-k+1})| + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx
\end{aligned}$$

$$\begin{aligned}
 &= o(1) + o(1) + C \int_0^\pi |R_{n-r+1}(x)| dx + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
 (4.9) \quad &= o(1) + C \int_0^\pi |R_{n-r+1}(x)| dx + \int_0^\pi \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx
 \end{aligned}$$

by the hypothesis of the theorem and Lemma 3.1.

But

$$\begin{aligned}
 &\int_0^\pi |R_{n-r+1}(x)| dx \leq \int_0^\pi \left| \left( \sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x) A_{n-r-k+1}^{\delta-1} \right) (\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1}) \right| dx \\
 &+ \int_0^\pi \left| \left( \sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x) A_{n-r-k+2}^{\delta-1} \right) (\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2}) \right| dx \\
 &+ \int_0^\pi \left| \left( \sum_{k=1}^{n-r} \bar{S}_k^{r-1}(x) A_{n-r-k+3}^{\delta-1} \right) (\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3}) \right| dx \\
 &\vdots \\
 &\leq \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^{r-1} \int_0^\pi |\bar{T}_k^{r-1}(x)| dx |(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1})| \\
 &+ \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^{r-1} \int_0^\pi |\bar{T}_k^{r-1}(x)| dx |(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2})| \\
 &+ \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^{r-1} \int_0^\pi |\bar{T}_k^{r-1}(x)| dx |(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3})| \\
 &\vdots \\
 &\leq C_1 \sum_{k=1}^{n-r} A_{n-r-k+1}^{\delta-1} A_k^{r-1} |(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1})| \\
 &+ C_1 \sum_{k=1}^{n-r} A_{n-r-k+2}^{\delta-1} A_k^{r-1} |(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2})| \\
 &+ C_1 \sum_{k=1}^{n-r} A_{n-r-k+3}^{\delta-1} A_k^{r-1} |(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3})| \\
 &\vdots
 \end{aligned}$$

$$\begin{aligned}
&\leq C_1 \sum_{k=1}^{n+1-r} A_{n+1-r-k}^{\delta-1} A_k^{r-1} |(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1})| \\
&\quad + C_1 \sum_{k=1}^{n+2-r} A_{n+2-r-k}^{\delta-1} A_k^{r-1} |(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2})| \\
&\quad + C_1 \sum_{k=1}^{n+3-r} A_{n+3-r-k}^{\delta-1} A_k^{r-1} |(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3})| \\
&\quad \vdots \\
&\leq C_2 A_{n+1-r}^{r+\delta-1} |(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1})| \\
&\quad + C_2 A_{n+2-r}^{r+\delta-1} |(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2})| \\
&\quad + C_2 A_{n+3-r}^{r+\delta-1} |(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3})| \\
&\quad \vdots \\
&\leq C_2 A_{n+1-r}^{r+\delta} |(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1})| \\
&\quad + C_2 A_{n+2-r}^{r+\delta} |(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2})| \\
&\quad + C_2 A_{n+3-r}^{r+\delta} |(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+3})| \\
&\quad \vdots \\
&= C_2 A_{n+1-r}^{\alpha} |(\Delta^{\alpha+1} a_{n-r} + \Delta^{\alpha+1} a_{n-r+1})| \\
&\quad + C_2 A_{n+2-r}^{\alpha} |(\Delta^{\alpha+1} a_{n-r+1} + \Delta^{\alpha+1} a_{n-r+2})| \\
&\quad + C_2 A_{n+3-r}^{\alpha} |(\Delta^{\alpha+1} a_{n-r+2} + \Delta^{\alpha+1} a_{n-r+4})| \\
&\quad \vdots \\
&= o(1) + o(1) \\
&\quad \vdots \\
&= o(1),
\end{aligned}$$

by the hypothesis of the theorem.

Hence (4.9) implies

$$(4.10) \quad \lim_{n \rightarrow \infty} \int_0^{\pi} |g(x) - g_n(x)| dx = o(1),$$

if and only if  $\Delta a_n \log n = o(1)$ , as  $n \rightarrow \infty$ .

Thus by (4.4) and (4.10)

$$(4.11) \quad \lim_{n \rightarrow \infty} \int_0^{\pi} |g(x) - g_n(x)| dx = o(1),$$

if and only if  $\Delta a_n \log n = o(1)$ , as  $n \rightarrow \infty$ , where  $\alpha$  is non-integral.  $\square$

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