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PERIODIC SOLUTIONS FOR DIFFERENTIAL INCLUSIONS IN $\mathbb{R}^{\mathbb{N}}$

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Abstract. We consider first order periodic differential inclusions in $\mathbb{R}^N$. The presence of a subdifferential term incorporates in our framework differential variational inequalities in $\mathbb{R}^N$. We establish the existence of extremal periodic solutions and we also obtain existence results for the "convex" and "nonconvex" problems.

1. Introduction

In this paper, we study the following differential inclusions in $\mathbb{R}^N$:

$$\left\{ \begin{array}{l} -x'(t) \in \partial\varphi(x(t)) + \operatorname{ext} F(t, x(t)) \text{ a.e on } T = [0, b] \\ x(0) = x(b)\,, \end{array} \right\} \tag{1.1}$$

and

$$\left\{ \begin{array}{l} -x'(t) \in \partial\varphi(x(t)) + F(t, x(t)) \text{ a.e on } T = [0, b] \\ x(0) = x(b)\,. \end{array} \right\} \tag{1.2}$$

Here $\varphi \in \Gamma_0(\mathbb{R}^N) = \{\varphi : \mathbb{R}^N \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} : \varphi$ is proper, convex and lower semicontinuous$\}$, $\partial\varphi$ denotes the subdifferential in the sense of convex analysis and $\operatorname{ext} F(t, x)$ denotes the extreme points of the set $F(t, x)$. The solutions of (1.1) are known as "extremal periodic solutions". In the past all works on periodic differential inclusions assumed that $\varphi \equiv 0$ and most of them assumed that the multifunction F has convex values. The reason for this is that then the multivalued Poincare operator exhibits nice properties, which permit the use of suitable fixed point theorems. We mention the works of De Blasi-Gorniewicz-Pianigiani [4], Haddad-Lasry [6], Macki-Nistri-Zecca [14] and Plaskacz [15]. When F has nonconvex values, the situation is more involved and only recently there have been results

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in this direction by De Blasi-Gorniewicz-Pianigiani [4], Hu-Papageorgiou [10], Hu-Kandilakis-Papageorgiou [9] and Li-Xue [13]. Finally the problem of extremal periodic solutions, is far more difficult because $\operatorname{ext} F(t,x)$ need not be closed and $\operatorname{ext} F(t,\cdot)$ need not have any continuity properties, even if $F(t,\cdot)$ is regular enough. In this direction there are only the results of De Blasi-Pianigiani [5] and Li-Xue [13]. Our results here extend the aforementioned works in several ways.

Let X be a a Banach space. We use the following notations:

- $P_f(X) = \{A \subseteq X : \text{ nonempty, closed}\}$
- $P_{fc}(X) = \{A \in P_f(X) : \text{convex}\}$
- $P_k(X) = \{A \subseteq X : \text{nonempty, compact}\}$
- $P_{kc}(X) = \{A \in P_k(X) : \text{convex}\}$
- and $P_{wkc}(X) = \{A \subseteq X : \text{nonempty, weakly compact, convex}\}$

If Y is another Banach space and $G : Y \to P_f(X)$ is a multifunction, the graph of G is the set $\operatorname{Gr} G = \{(y,x) \in Y \times X : x \in G(y)\}$. We say that G is a lower semicontinuous (lsc), if for all $C \subseteq X$ closed, the set $G^+(C) = \{y \in Y : G(y) \subseteq C\}$ is closed. We say that G is h-continuous if it is continuous into the metric space $(P_f(X), h)$, with h being the Hausdorff metric. Finally if $K : T = [0,b] \to P_f(X)$, we say that K is graph measurable, if Gr $K \in \mathcal{L}_T \times B(X)$, with $\mathcal{L}_T$ being the Lebesgue σ-field of T and $B(X)$ the Borel σ-field of X.

Let $T = [0,b]$ and let X be a Banach space. A set $W \subseteq L^1(T,X)$ is said to have property (U) (see Bourgain [2]), if the following conditions hold

(a) W is bounded and uniformly integrable;
(b) for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq X$ such that for every $f \in W$, there is a measurable set $T(f,\varepsilon) \subseteq T$ with $|T \backslash T(f,\varepsilon)| \leq \varepsilon$ (by $|\cdot|$ we denote the Lebesgue measure on $\mathbb{R}$) and $f(t) \in K_\varepsilon$ for all $t \in T(f,\varepsilon)$.

On $L^1(T,X)$ we can consider the "weak norm" $\|\cdot\|_w$ defined by

$$\|f\|_w = \sup\left[\left\|\int_{t_1}^{t_2} f(s)\,ds\right\| : 0 \leq t_1 \leq t_2 \leq b\right]$$

or equivalently by

$$\|f\|_w = \sup\left[\left\|\int_0^t f(s)\,ds\right\| : 0 \leq t \leq b\right].$$

We know that if $W \subseteq L^1(T,X)$ has property (U), then the weak topology and the $\|\cdot\|_w$-topology on W coincide.

If $\{f_n\}_{n\geq 1} \subseteq L^p(T,X)$, $1 \leq p < \infty$, $f_n \to f$ in $L^p(T,X)$ and for almost all $t \in T$, $f_n(t) \in G(t)$ with $G(t)$ being weakly compact, then

$$f(t) \in \overline{\operatorname{conv}}\ w - \limsup_{n\to\infty}\{f_n(t)\} \quad \text{a.e. on} \quad T\,.$$

If by X^* we denote the dual of the Banach space X, a nonlinear operator $A : X \to X^*$ which is maximal monotone and coercive, it is surjective. Recall that a monotone and demicontinuous operator is maximal monotone.

For further details on these and related issues we refer to Hu-Papageorgiou [11], [12].

2. Extremal periodic solutions

In this section we deal with problem (1.1). The hypotheses on φ and F are the following:

$\underline{H(\varphi)}$**:** $\varphi \in \Gamma_0(\mathbb{R}^{\mathbb{N}})$ and $0 = \varphi(0) = \inf_{R^N} \varphi$.

Remark 2.1. This hypothesis incorporates in our framework differential variational inequalities which are important in the analysis of dynamic economic models (see Cornet [3] and Henry [8]).

$H(F)_1$**:** $F : T \times \mathbb{R}^{\mathbb{N}} \to P_{kc}(\mathbb{R}^{\mathbb{N}})$ is a multifunction such that

(i) for all $x \in \mathbb{R}$, $t \to F(t,x)$ is graph measurable;
(ii) for almost all $t \in T$, $x \to F(t,x)$ is h-continuous;
(iii) for every $r > 0$, there exists $a_r \in L^2(T)_+$ such that $\|u\| \le a_r(t)$ a.e. on T, for all $\|x\| \le r$ and all $u \in F(t,x)$;
(iv) there exists $M > 0$ such that for a.a. $t \in T$, all $x \in \mathbb{R}^{\mathbb{N}}$ with $\|x\| = M$ and all $u \in F(t,x)$, we have $(u,x)_{\mathbb{R}^{\mathbb{N}}} \ge 0$.

Remark 2.2. Hypothesis $H(F)_1$(iv) is more general than hypothesis $H(F)_3$(iii) of Li-Xue [13].

We introduce the following modification of $F : F_1(t,x) = F(t,p_M(x)) - p_M(x)$, where $p_M(x) = \begin{cases} x & \text{if } \|x\| \le M \\ \dfrac{Mx}{\|x\|} & \text{if } \|x\| > M \end{cases}$ (the M- radial retraction). Also for $\lambda > 0$, let $\varphi_\lambda(x) = \inf\,[\varphi(y) + \dfrac{1}{2\lambda}\|x-y\|^2 : y \in \mathbb{R}^{\mathbb{N}}]$ (the Moreau-Yosida regularization of φ.) We know that φ_λ is differentiable and $(\partial\varphi)_\lambda = \partial\varphi_\lambda$ for all $\lambda > 0$ (see Hu-Papageorgiou [11], p.350). Using F_1 and $\partial\varphi_\lambda$ we introduce the following auxiliary problem.

$$(2.1) \qquad -x'(t) \in \partial\varphi_\lambda(x(t)) + \operatorname{ext} F_1(t,x(t)) \quad \text{a.e. on } T, \quad x(0) = x(b)\,, \quad \lambda > 0\,.$$

Proposition 2.3. *If hypotheses $H(\varphi)$ and $H(F)_1$, hold, then for every $\lambda > 0$ problem* (2.1) *has a solution $x_0 \in W^{1,2}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$.*

Proof. Let $\widehat{a}_M(t) = a_M(t) + M$, $\widehat{a}_M \in L^2(T)_+$ (see $H(F)_1$(iii)). We introduce the following set

$$K = \{g \in L^2(T,\mathbb{R}^{\mathbb{N}}) : \ \|g(t)\| \le \widehat{a}_M(t) \ \text{ a.e. on } T\} \in P_{wkc}(L^2(T,\mathbb{R}^{\mathbb{N}}))\,.$$

Given $g \in L^2(T,\mathbb{R}^{\mathbb{N}})$, we consider the following problem

$$(2.2) \qquad -x'(t) = \partial\varphi_\lambda(x(t)) + x(t) + g(t) \quad \text{a.e. on } T, \quad x(0) = x(b), \quad \lambda > 0\,.$$

Let $L : D(L) = W^{1,2}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) \subseteq L^2(T,\mathbb{R}^{\mathbb{N}}) \to L^2(T,\mathbb{R}^{\mathbb{N}})$ be the unbounded linear operator defined by $L(x) = x'$. We know that L is maximal monotone (see Hu-Papageorgiou [12], p.84). Also let $G_\lambda : L^2(T,\mathbb{R}^{\mathbb{N}}) \to L^2(T,\mathbb{R}^{\mathbb{N}})$ be defined by $G_\lambda(x)(\cdot) = \partial\varphi_\lambda(x(\cdot))$. Evidently G_λ is maximal monotone. Then $S_\lambda : D(L) \subseteq L^2(T,\mathbb{R}^{\mathbb{N}}) \to L^2(T,\mathbb{R}^{\mathbb{N}})$ defined by $S_\lambda(x) = L(x)+x+G_\lambda(x)$ is maximal monotone, strictly monotone and coercive. Hence S_λ is surjective (see Section 2) and so for every $g \in L^2(T, \mathbb{R}^{\mathbb{N}})$, there exists unique $x \in D(L)$ such that $g = S_\lambda(x)$ (uniqueness follows from the strict monotonicity of S_λ). Then problem (2.2) has a unique solution $u(g) \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}}) \subseteq C(T, \mathbb{R}^{\mathbb{N}})$. Consider the solution map $u : L^2(T, \mathbb{R}^{\mathbb{N}}) \to C(T, \mathbb{R}^{\mathbb{N}})$.

Claim 1. $u(K) \subseteq C(T, \mathbb{R}^{\mathbb{N}})$ is compact.

Let $x \in u(K)$. Then $x = u(g)$, $g \in K$ and so $x' + x + G_\lambda(x) + g = 0$. Taking inner product in $L^2(T, \mathbb{R}^{\mathbb{N}})$ with x, we obtain

$$\frac{1}{2}\int_0^b \frac{d}{dt}\|x(t)\|^2 dt + \|x\|_2^2 + \int_0^b (\partial\varphi_\lambda(x(t)), x(t))_{\mathbb{R}^{\mathbb{N}}}\, dt + \int_0^b (g(t), x(t))_{\mathbb{R}^{\mathbb{N}}}\, dt = 0\,,$$

$$\Rightarrow \|x\|_2^2 \le \|g\|_2 \|x\|_2, \quad \text{i.e.} \quad \|x\|_2 \le \|\widehat{a}_M\|_2$$

(note that $\partial\varphi_\lambda(0) = 0$ and so $(\partial\varphi_\lambda(x), x) \ge 0$ for all $x \in \mathbb{R}^{\mathbb{N}}$).

So $u(K) \subseteq L^2(T, \mathbb{R}^{\mathbb{N}})$ is bounded. Then directly from (2.2), we see that $V = \{x' : x \in u(K)\} \subseteq L^2(T, \mathbb{R}^{\mathbb{N}})$ is bounded. Hence $u(K) \subseteq W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ is bounded. Because $W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ is embedded compactly in $C(T, \mathbb{R}^{\mathbb{N}})$, we have that $u(K)$ is relatively compact in $C(T, \mathbb{R}^{\mathbb{N}})$. Clearly it is closed in $C(T, \mathbb{R}^{\mathbb{N}})$ and so $u(K)$ is compact in $C(T, \mathbb{R}^{\mathbb{N}})$. Then $E = \overline{\mathrm{conv}}\, u(K)$ is compact in $C(T, \mathbb{R}^{\mathbb{N}})$ and convex. Let $B : E \to P_{wkc}(L^2(T, \mathbb{R}^{\mathbb{N}}))$ be defined by $B(x) = S^2_{F_1(\cdot, x(\cdot))} = \{f \in L^2(T, \mathbb{R}^{\mathbb{N}}) : f(t) \in F_1(t, x(t)) \text{ a.e. on } T\}$. From [11] (p.260) we know that we can find a continuous map $v : E \to L^1_w(T, \mathbb{R}^{\mathbb{N}})$ such that $v(x) \in \mathrm{ext}\, B(x)$ for all $x \in E$ (here $L^1_w(T, \mathbb{R}^{\mathbb{N}})$ denotes the Lebesgue space $L^1(T, \mathbb{R}^{\mathbb{N}})$ equipped with the weak norm $\|g\|_w = \max[\|\int_s^t g(\tau)\, d\tau\| : 0 \le s \le t \le b])$. We know that $\mathrm{ext}\, B(x) = \mathrm{ext}\, S^2_{F_1(\cdot, x(\cdot))} = S^2_{\mathrm{ext}\, F_1(\cdot, x(\cdot))}$ (see [11], p.192). Consider the map $\xi : E \to C(T, \mathbb{R}^{\mathbb{N}})$ defined by $\xi(x) = (S_\lambda^{-1} \circ (-v))(x)$. Note that $(-v)(E) \subseteq K$ and so $(S_\lambda^{-1} \circ (-v))(E) = \xi(E) \subseteq E$.

Claim 2. $\xi : E \to E$ is continuous.

Suppose $x_n \to x$ in $C(T, \mathbb{R}^{\mathbb{N}})$, $x_n \in E$. We have $v(x_n) \to v(x)$ in $L^1_w(T, \mathbb{R}^{\mathbb{N}})$. From the multivalued Scorza-Dragoni Theorem (see [11], p.232), given $\varepsilon > 0$ we can find $T_\varepsilon \subseteq T$ closed such that $|T \backslash T_\varepsilon|_1 < \varepsilon$ ($|\cdot|_1$ is the Lebesgue measure on $\mathbb{R}$) and $F\,|_{T_\varepsilon \times \mathbb{R}^{\mathbb{N}}}$ is h-continuous. So $F(T_\varepsilon \times \overline{B}_M(0)) \in P_k(\mathbb{R}^{\mathbb{N}})$. Note that for every $g \in (-v)(E)$, we have $g = (-v)(x)$ with $x \in E$ and so $g(t) \in F_1(t, x(t)) \in F(T_\varepsilon \times \overline{B}_M(0)) + \overline{B}_M(0) \in P_k(\mathbb{R}^{\mathbb{N}})$ for a.a. $t \in T_\varepsilon$. This means that $(-v)(E)$ has property (U) (see Section 2). Therefore on $(-v)(E)$ the $\|\cdot\|_w$-topology and the weak-$L^1(T, \mathbb{R}^{\mathbb{N}})$ topology coincide (see Section 2). So we have $v(x_n) \xrightarrow{w} v(x)$ in $L^1(T, \mathbb{R}^{\mathbb{N}})$. Moreover, because of $H(F)_1$(iii) we have $v(x_n) \xrightarrow{w} v(x)$ in $L^2(T, \mathbb{R}^{\mathbb{N}})$. Let $y_n = S_\lambda^{-1}((-v)(x_n))$ $n \ge 1$. We have $y_n' + y_n + G_\lambda(y_n) = (-v)(x_n)$ $n \ge 1$. From the proof of Claim 1, we know that $\{y_n\}_{n \ge 1} \subseteq W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ is bounded. So we may assume that $y_n \xrightarrow{w} y$ in $W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ and $y_n \to y$ in $C(T, \mathbb{R}^{\mathbb{N}})$. Hence by virtue of the maximal monotonicity of G_λ, we have $G_\lambda(y_n) \xrightarrow{w} G_\lambda(y)$ in $L^2(T, \mathbb{R}^{\mathbb{N}})$. Therefore in the limit as $n \to \infty$ we obtain $y' + y + G_\lambda(y) = (-v)(x)$ and so $y = S_\lambda^{-1}((-v)(x))$. Thus for the original sequence, we have $S_\lambda^{-1}((-v)(x_n)) \to S_\lambda^{-1}((-v)(x))$ in $C(T,\mathbb{R}^{\mathbb{N}})$ and this proves the continuity of ξ.

Since E is compact convex in $C(T,\mathbb{R}^{\mathbb{N}})$, Claim 2 and the Schauder fixed point theorem, give $x_0 \in E$ such that

$$x_0' + x_0 + G_\lambda(x_0) = -v(x_0)\,, \quad x_0(0) = x_0(b)\,. \tag{2.3}$$

Claim 3. $\|x_0(t)\| \le M$ for all $t \in T$.

First let us show that it can not happen that $\|x_0(t)\| > M$ for all $t \in T$. Suppose that $\|x_0(t)\| > M$ for all $t \in T$. Since $\int_0^b (x_0'(t), x_0(t))_{\mathbb{R}^{\mathbb{N}}}\,dt = \int_0^b \frac{1}{2}\frac{d}{dt}\|x_0(t)\|^2\,dt$ and $(\partial\varphi_\lambda(x_0(t)), x_0(t))_{\mathbb{R}^{\mathbb{N}}} \ge 0$ a.e. on T, we have

$$\|x_0\|_2^2 + \int_0^b (v(x_0)(t), x_0(t))_{\mathbb{R}^{\mathbb{N}}} \le 0 \quad \text{(see (2.3))}. \tag{2.4}$$

Because $v(x_0)(t) \in F_1\big(t, x_0(t)\big) = F\big(t, p_M(x_0(t))\big) - p_M\big(x_0(t)\big)$ a.e. on T, we have $v(x_0) = \widehat{v}(x_0) - p_M\big(x_0(\cdot)\big)$, with $\widehat{v}(x_0) \in L^2(T,\mathbb{R}^{\mathbb{N}})$, $\widehat{v}(x_0)(t) \in F\big(t, p_M(x_0(t))\big)$ a.e. on T. Therefore

$$\begin{aligned}\int_0^b \big(v(x_0)(t), x_0(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt &= \int_0^b \frac{\|x_0(t)\|}{M}(\widehat{v}(x_0)(t), p_M(x_0(t)))_{\mathbb{R}^{\mathbb{N}}}\,dt \\ &\quad - \int_0^b M\|x_0(t)\|\,dt \ge -M\int_0^b \|x_0(t)\|\,dt \end{aligned} \tag{2.5}$$

(see hypothesis $H(F)_1$(iv)).

Using (2.5) in (2.4), we obtain $0 \ge \int_0^b \big(\|x_0(t)\|^2 - M\|x_0(t)\|\big)\,dt > 0$, a contradiction. So it can not happen that $\|x_0(t)\| > M$ for all $t \in T$.

Now suppose that the claim is not true. Extend x_0 with b-periodicity on $\mathbb{R}_+$. Then we can find $\tau_1,\ \tau_2 \ge 0$ such that $\|x_0(\tau_1)\| = \|x_0(\tau_2)\| = M$ and $\|x_0(t)\| > M$ for all $t \in (\tau_1, \tau_2)$. From (2.3), we have

$$\begin{aligned}&x_0'(t) + x_0(t) + \partial\varphi_\lambda(x_0(t)) + \widehat{v}(x_0)(t) - p_M(x_0(t)) = 0 \quad \text{a.e. on } \mathbb{R}_+,\\ &x_0(0) = x_0(nb),\ n \ge 1\,.\end{aligned}$$

Taking inner product with $x_0(t)$ and integrating over $[\tau_1, \tau_2]$, we obtain

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \frac{1}{2}\frac{d}{dt}\|x_0(t)\|^2\,dt + \int_{\tau_1}^{\tau_2} \|x_0(t)\|^2 + \int_{\tau_1}^{\tau_2} (\partial\varphi_\lambda(x_0(t)), x_0(t))_{\mathbb{R}^{\mathbb{N}}}\,dt \\ &+ \int_{\tau_1}^{\tau_2} \frac{\|x_0(t)\|}{M}\big(\widehat{v}(x_0)(t), p_M(x_0(t))\big)_{\mathbb{R}^{\mathbb{N}}}\,dt - M\int_{\tau_1}^{\tau_2} \|x_0(t)\|\,dt = 0\,, \\ &\Rightarrow 0 < \int_{\tau_1}^{\tau_2} \|x_0(t)\|(\|x_0(t)\| - M)\,dt \le 0\end{aligned}$$

(see hypothesis $H(F)_1$(iv) and recall that $\|x_0(\tau_1)\| = \|x_0(\tau_2)\|$),

a contradiction. It follows that $\|x_0(t)\| \le M$ for all $t \in T$.

Because of Claim 3, we have $F_1(t, x_0(t)) = F(t, x_0(t)) - x_0(t)$. Using this in (2.3) and recalling that $v(x) \in S^2_{\operatorname{ext} F_1(\cdot, x(\cdot))}$, we conclude that $x_0 \in W^{1,2}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ is a solution of (2.1).

Next we pass to the limit as $\lambda \downarrow 0$ in (2.1) to obtain a solution of problem (1.1).

Theorem 2.4. *If hypotheses $H(\varphi)$ and $H(F)_1$ hold, then problem* (1.1) *has a solution $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}})$.*

Proof. Let $\lambda_n \downarrow 0$ and let $x_n \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ be solutions of (2.1) when $\lambda = \lambda_n$ (see Proposition 2.3). From the proof of Proposition 2.3 we know that these solutions were obtained via a fixed point argument and satisfy

(2.6)
$$x_n'(t) + x_n(t) + \partial\varphi_{\lambda_n}(x_n(t)) + v(x_n)(t) = 0 \quad \text{a.e. on } T, \quad x_n(0) = x_n(b),\ n \geq 1\,.$$

with $v(x_n)(\cdot)$ as before (see (2.3)). Also from the argument in Claim 3 of the proof of Proposition 2.3, we know that

$$\|x_n(t)\| \leq M \quad \text{for all } n \geq 1, \text{ all } t \in T\,.$$

We take the inner product with $x_n'(t)$ and integrate over T. So we obtain

$$\|x_n'\|_2^2 + \int_0^b (\partial\varphi_{\lambda_n}(x_n(t)), x_n'(t))_{\mathbb{R}^{\mathbb{N}}}\, dt + \int_0^b (v(x_n)(t), x_n'(t))_{\mathbb{R}^{\mathbb{N}}}\, dt = 0\,.$$

We know that $(\partial\varphi_{\lambda_n}(x_n(t)), x_n'(t))_{\mathbb{R}^{\mathbb{N}}} = \frac{d}{dt}\varphi_{\lambda_n}(x_n(t))$ a.e. on T (see [11], p.357). It follows that

$$\begin{aligned} &\|x_n'\|_2^2 \leq \|v(x_n)\|_2 \|x_n'\|_2 \leq \|a_M\|_2 \|x_n'\|_2 \\ \Rightarrow &\{x_n'\}_{n\geq 1} \subseteq L^2(T,\mathbb{R}^{\mathbb{N}}) \quad \text{is bounded and so} \\ &\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}}) \quad \text{is bounded.} \end{aligned}$$

We may assume that $x_n \xrightarrow{w} x$ in $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ and $x_n \to x$ in $C(T,\mathbb{R}^{\mathbb{N}})$. Also from (2.1) it is clear that $\{\partial\varphi_{\lambda_n}(x_n(\cdot))\}_{n\geq 1} \subseteq L^2(T,\mathbb{R}^{\mathbb{N}})$ is bounded. So we can say that $\partial\varphi_{\lambda_n}(x_n(\cdot)) \xrightarrow{w} u$ in $L^2(T,\mathbb{R}^{\mathbb{N}})$. For every $y \in L^2(T,\mathbb{R}^{\mathbb{N}})$ we have

$$(\partial\varphi_{\lambda_n}(x_n(t)), y(t) - x_n(t))_{\mathbb{R}^{\mathbb{N}}} \leq \varphi_{\lambda_n}(y(t)) - \varphi_{\lambda_n}(x_n(t)) \quad \text{a.e. on } T,\ n \geq 1$$

(2.7)
$$\Rightarrow \int_0^b (\partial\varphi_{\lambda_n}(x_n(t)), y(t) - x_n(t))_{\mathbb{R}^{\mathbb{N}}}\, dt \leq I_{\varphi_{\lambda_n}}(y) - I_{\varphi_{\lambda_n}}(x_n) \quad \text{for all } n \geq 1\,,$$

where $I_{\varphi_\lambda}(y) = \int_0^b \varphi_\lambda\big(y(t)\big)\, dt$ for all $y \in L^2(T,\mathbb{R}^{\mathbb{N}})$. We have $I_{\varphi_\lambda} \in C\big(L^2(T,\mathbb{R}^{\mathbb{N}})\big)$ for all $\lambda > 0$. Moreover, by virtue of the monotone convergence theorem

$$I_{\varphi_{\lambda_n}}(y) \uparrow I_\varphi(y) = \begin{cases} \int_0^b \varphi\big(y(t)\big)\, dt & \text{if } \ \varphi(y(\cdot)) \in L^1(T) \\ \infty & \text{otherwise} \end{cases} \quad \text{and } I_\varphi \in \Gamma_0\big(L^2(T,\mathbb{R}^{\mathbb{N}})\big)$$

(see [11], p.351). Also since $\varphi(x(t)) \leq \liminf_{n\to+\infty} \varphi_{\lambda_n}(x_n(t))$ for all $t \in T$ (see [11], p.351), from Fatou's lemma we obtain $I_\varphi(x) \leq \liminf_{n\to+\infty} I_{\varphi_{\lambda_n}}(x_n)$. Therefore, if we pass to the limit as $n \to \infty$ in (2.7), we have

$$\begin{aligned} &\int_0^b (u(t), y(t) - x(t))_{\mathbb{R}^{\mathbb{N}}}\, dt \leq I_\varphi(y) - I_\varphi(x) \quad \text{for all } y \in L^2(T,\mathbb{R}^{\mathbb{N}}), \\ \Rightarrow &u \in \partial I_\varphi(x) \quad \text{and so } u(t) \in \partial\varphi(x(t)) \text{ a.e. on } T \qquad \text{(see[11], p.349).} \end{aligned}$$

Also from the proof of Proposition 2.3, we know that $v(x_n) \xrightarrow{w} v(x)$ in $L^2(T, \mathbb{R}^{\mathbb{N}})$. So in the limit as $n \to \infty$ we have $x' + u + v(x) = 0$, $x(0) = x(b)$, hence $-x'(t) \in \partial\varphi(x(t)) + \operatorname{ext} F(t, x(t))$ a.e. on T, $x(0) = x(b)$ (since $v(x) \in S^2_{\operatorname{ext} F(\cdot, x(\cdot))}$). Therefore $x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ is a solution of problem (1.1). $\square$

3. Convex problem

In this section we study problem (1.2) under the assumption that the multifunction $F(t,x)$ has convex values. Our hypotheses on F are weaker than $H(F)_1$ and are the following:

$H(F)_2$**:** $F : T \times \mathbb{R}^{\mathbb{N}} \to P_{kc}(\mathbb{R}^{\mathbb{N}})$ is a multifunction such that
- (i) for all $x \in \mathbb{R}^{\mathbb{N}}$, $t \to F(t,x)$ is graph measurable;
- (ii) for almost all $t \in T$, $x \to F(t,x)$ has closed graph;
- (iii) and (iv) are the same as $H(F)_1$(iii) and (iv).

As before, first we consider the following auxiliary periodic problem

$$-x'(t) \in \partial\varphi_\lambda(x(t)) + F(t, x(t)) \text{ a.e. on } T,\ x(0) = x(b),\ \lambda > 0\,. \tag{3.1}$$

Proposition 3.1. *If hypotheses $H(\varphi)$ and $H(F)_2$ hold, then for every $\lambda > 0$ problem* (3.1) *has a solution $x_0 \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$.*

Proof. We introduce the multifunction $B : C(T, \mathbb{R}^{\mathbb{N}}) \to P_{wkc}\big(L^2(T, \mathbb{R}^{\mathbb{N}})\big)$ defined by $B(x) = S^2_{F_1(\cdot, x(\cdot))}$ ($F_1(t,x)$ as in the proof of Proposition 2.3). We know that B is usc from $C(T, \mathbb{R}^{\mathbb{N}})$ into $L^2(T, \mathbb{R}^{\mathbb{N}})_w$ (the space $L^2(T, \mathbb{R}^{\mathbb{N}})$ supplied with the weak topology; see Halidias-Papageorgiou [7]). Using the operator S_λ from the proof of Proposition 2.3, we consider the abstract fixed point problem $x \in (S_\lambda^{-1} \circ (-B))(x)$. From the argument in claim 1 in the proof of Proposition 2.3, we see that $S_\lambda^{-1} \circ (-B)$ maps bounded sets in $C(T, \mathbb{R}^{\mathbb{N}})$ onto bounded sets in $W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$, hence onto relatively compact sets in $C(T, \mathbb{R}^{\mathbb{N}})$. Consider the set $C = \{x \in C(T, \mathbb{R}^{\mathbb{N}}) : x \in \beta(S_\lambda^{-1} \circ (-B))(x),\ 0 < \beta < 1\}$. If $x \in C$, then $\frac{1}{\beta}x' + \frac{1}{\beta}x + G_\lambda(\frac{1}{\beta}x) \in (-B)(x)$ and then as in the proofs of Proposition 2.3 and Theorem 2.4, we can check that $C \subseteq L^2(T, \mathbb{R}^{\mathbb{N}})$ and $C' = \{x' : x \in C\} \subseteq L^2(T, \mathbb{R}^{\mathbb{N}})$ are both bounded, hence $C \subseteq C(T, \mathbb{R}^{\mathbb{N}})$ is relatively compact. So Theorem 8 of Bader [2] implies the existence of $x_0 \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ such that $x_0 \in (S_\lambda^{-1} \circ (-B))(x_0)$. As in Claim 3 in the proof of Proposition 2.3, we can show that $\|x_0(t)\| \le M$ for all $t \in T$. This means that $x_0 \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ is a solution of problem (3.1). $\square$

Passing to the limit as $\lambda \downarrow 0$ in (3.1), we obtain a solution for problem (1.2).

Theorem 3.2. *If hypotheses $H(\varphi)$ and $H(F)_2$ hold, then problem* (1.2) *has a solution $x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$.*

Proof. Let $\lambda_n \downarrow 0$ and consider $x_n \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^{\mathbb{N}})$ solutions of problem (3.1) when $\lambda = \lambda_n$ $n \ge 1$ (see Proposition 3.1). We have $-x_n'(t) = \partial\varphi_{\lambda_n}(x_n(t)) + f_n(t)$ a.e. on T, $x_n(0) = x_n(b)$, with $f_n \in S^2_{F(\cdot, x_n(\cdot))}$, $n \ge 1$. Since $\|x_n(t)\| \le M$ for all $t \in T$ and all $n \ge 1$, by virtue of $H(F)_2$(iii) $\{f_n\}_{n \ge 1} \subseteq L^2(T, \mathbb{R}^{\mathbb{N}})$ is bounded and so we may assume that $f_n \xrightarrow{w} f$ in $L^2(T, \mathbb{R}^{\mathbb{N}})$. Also as in the proof of Theorem

2.4, we obtain that $x_n \xrightarrow{w} x$ in $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ and $x_n \to x$ in $C(T,\mathbb{R}^{\mathbb{N}})$. Using Proposition 3.9, p.694, of [11], (see also Section 2) we obtain $f \in B(x)$. The rest of the proof is the same as that of Theorem 2.4. $\square$

4. Nonconvex problem

Here we study problem (1.2) without the assumption that the multifunction $F(t,x)$ has convex values. Our hypotheses on F are the following:

$H(F)_3$**:** $F : T \times \mathbb{R}^{\mathbb{N}} \to P_k(\mathbb{R}^{\mathbb{N}})$ is a multifunction such that
- (i) $(t,x) \to F(t,x)$, is graph measurable;
- (ii) for almost all $t \in T$, $x \to F(t,x)$ is lsc;
- (iii) and (iv) are the same as $H(F)_1$(iii) and (iv).

In this case $B : C(T,\mathbb{R}^{\mathbb{N}}) \to P_f(L^2(T,\mathbb{R}^{\mathbb{N}}))$ defined by $B(x) = S^2_{F_1(\cdot,x(\cdot))}$ is lsc (see Halidias-Papageorgiou [7]). So we can apply Theorem 8.7, p.245, of [11] and obtain a continuous map $v : C(T,\mathbb{R}^{\mathbb{N}}) \to L^2(T,\mathbb{R}^{\mathbb{N}})$ such that $v(x) \in B(x)$ for all $x \in C(T,\mathbb{R}^{\mathbb{N}})$. We consider the auxiliary problem

$$-x'(t) = \partial\varphi_\lambda(x(t)) + v(x)(t) \quad \text{a.e. on } T,\ x(0) = x(b),\ \lambda > 0\,. \tag{4.1}$$

Reasoning as in Proposition 2.3, we obtain:

Proposition 4.1. *If hypotheses $H(\varphi)$ and $H(F)_3$ hold, then for every $\lambda > 0$ problem* (4.1) *has a solution $x_0 \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}})$.*

Then we let $\lambda \downarrow 0$ and we argue as in the proof of Theorem 2.4 using this time the continuous map $v : C(T,\mathbb{R}^{\mathbb{N}}) \to L^2(T,\mathbb{R}^{\mathbb{N}})$. So we obtain:

Theorem 4.2. *If hypotheses $H(\varphi)$ and $H(F)_3$ hold, then problem* (1.2) *has a solution $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^{\mathbb{N}})$.*

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