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ON THE LIMIT POINTS OF THE FRACTIONAL PARTS OF POWERS OF PISOT NUMBERS

ARTŪRAS DUBICKAS

Abstract. We consider the sequence of fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, 3, \dots$, where $\alpha > 1$ is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$ is a positive number. We find the set of limit points of this sequence and describe all cases when it has a unique limit point. The case, where $\xi = 1$ and the unique limit point is zero, was earlier described by the author and Luca, independently.

1. Introduction

Suppose that $\alpha > 1$ is an arbitrary algebraic number, and suppose that ξ is an arbitrary positive number that lies outside the field $\mathbb{Q}(\alpha)$ if α is a Pisot number or a Salem number. For such pairs ξ, α, in [6] we proved a lower bound (in terms of α only) for the distance between the largest and the smallest limit points of the sequence of fractional parts $\{\xi\alpha^n\}_{n=1,2,3,\dots}$. More precisely, we showed that the distance between the largest and the smallest limit points of this sequence is at least $1/\inf L(PG)$, where $P(z) = a_d z^d + \dots + a_1 z + a_0 \in \mathbb{Z}[z]$ is the minimal polynomial of α and where G runs through polynomials with real coefficients having either leading or constant coefficient 1. (Here, L stands for the length of a polynomial.) For this result, we showed first that with the above conditions the sequence

$$\begin{aligned} s_n : &= a_d[\xi\alpha^{n+d}] + \dots + a_1[\xi\alpha^{n+1}] + a_0[\xi\alpha^n] \\ &= -a_d\{\xi\alpha^{n+d}\} - \dots - a_1\{\xi\alpha^{n+1}\} - a_0\{\xi\alpha^n\} \end{aligned}$$

is not ultimately periodic. Recall that s_n, $n = 0, 1, 2, \dots$, is called *ultimately periodic* if there is $t \in \mathbb{N}$ such that $s_{n+t} = s_n$ for all sufficiently large n. (In contrast, s_n, $n = 0, 1, 2, \dots$, is called *purely periodic* if there is $t \in \mathbb{N}$ such that $s_{n+t} = s_n$ for *all* $n \geqslant 0$.) For rational $\alpha = p/q > 1$, our result in [6] recovers the result of Flatto, Lagarias and Pollington [7]: the difference between the largest and the smallest limit points of the sequence $\{\xi(p/q)^n\}_{n=1,2,3,\dots}$ is at least $1/p$. (See also [1].)

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Moreover, the results of [6] imply that we always have

$$\limsup_{n\to\infty} \{\xi\alpha^n\} - \liminf_{n\to\infty} \{\xi\alpha^n\} \geqslant 1/L(P)\,,$$

unless s_n, $n = 1, 2, \dots$, is ultimately periodic with period of length 1. However, for some Pisot and Salem numbers α and for some $\xi \in \mathbb{Q}(\alpha)$, this can happen. As a result, no bound for the difference between the largest and the smallest limit points of the sequence $\{\xi\alpha^n\}_{n=1,2,3,\dots}$ can be obtained in terms of α only. More precisely, for Salem numbers α such that $\alpha - 1$ is not a unit, Zaimi [11] showed that for every $\varepsilon > 0$ there exist positive numbers $\xi \in \mathbb{Q}(\alpha)$ such that all fractional parts $\{\xi\alpha^n\}_{n=1,2,3,\dots}$ belong to an interval of length ε. In this context, the only pairs that remain to be considered are of the form ξ, α, where α is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$. The aim of this paper is to consider such pairs.

Recall that $\alpha > 1$ is a *Pisot number* if it is an algebraic integer (i.e. $a_d = 1$) and if all its conjugates over $\mathbb{Q}$ different from α itself lie in the open unit disc. The problem of finding all such pairs $\xi > 0$, $\alpha > 1$, where α is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$, for which the sequence $\{\xi\alpha^n\}_{n=1,2,3,\dots}$ has a unique limit point is also of interest in connection with the papers [3], [8] and [9]. In [8] Kuba asked whether there are algebraic numbers $\alpha > 1$ other than integers satisfying $\lim_{n\to\infty}\{\alpha^n\} = 0$. This was answered by the author [3] and by Luca [9] independently: the answer is 'no'.

2. Results

From now on, suppose that $\alpha = \alpha_1 > 1$ is a Pisot number with minimal polynomial

$$P(z) = z^d + a_{d-1}z^{d-1} + \dots + a_0 = (z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_d) \in \mathbb{Z}[z]\,.$$

Since $\xi \in \mathbb{Q}(\alpha)$, we can write $\xi = f(\alpha) > 0$, where f is a non-zero polynomial of degree at most $d - 1$ with rational coefficients

$$f(z) = (b_0 + b_1 z + \dots + b_{d-1}z^{d-1})/b\,. \tag{1}$$

Here $b_0, b_1, \dots, b_{d-1} \in \mathbb{Z}$ and b is the smallest positive integer for which $bf(z) \in \mathbb{Z}[z]$. Set $S_n := \alpha_1^n + \alpha_2^n + \dots + \alpha_d^n$ (which is a rational integer for each non-negative integer n) and

$$Y_n := b_0 S_n + b_1 S_{n+1} + \dots + b_{d-1} S_{n+d-1}\,.$$

Then $Y_n = b\mathrm{Trace}(f(\alpha)\alpha^n)$. By Newton's formulae, we have $S_{n+d} + a_{d-1}S_{n+d-1} + \dots + a_0 S_n = 0$ for every $n \geqslant 0$. It is easy to see that the sequence $Y_0, Y_1, Y_2, \dots$ satisfies the same linear recurrence

$$Y_{n+d} + a_{d-1}Y_{n+d-1} + \dots + a_0 Y_n = 0 \tag{2}$$

for every non-negative integer n. By Lemma 2 of [4], the sequence Y_n, $n = 0, 1, 2, \dots$, modulo b is ultimately periodic. Moreover, in case if $\gcd(b, a_0) = 1$, by Lemma 2 of [5], the sequence Y_n, $n = 0, 1, 2, \dots$, modulo b is purely periodic. (These statements both can be proved directly. Firstly, there are at most b^d different vectors for $(Y_{n+d-1}, \dots, Y_n)$ modulo b to occur, which implies the first statement by (2). Secondly, if $\gcd(b, a_0) = 1$, then Y_n modulo b is uniquely determined by $Y_{n+d}, \dots, Y_{n+1}$ modulo b. This shows that a respective sequence is purely periodic.)

Suppose that $\overline{B_1 B_2 \dots B_k}$, where $0 \leqslant B_j \leqslant b-1$, is the period of $Y_0, Y_1, Y_2, \dots$ modulo b. Some of B_j may be equal. Let $\mathcal{B}$ be the set $\{B_1, \dots, B_k\}$. In other words, $\mathcal{B} = \mathcal{B}_{\xi,\alpha}$ is the set of residues of the sequence Y_n, $n = 0, 1, 2, \dots$, modulo b which occur infinitely often. We can now state our results.

Theorem 1. *Let $\alpha > 1$ be a Pisot number and let $f(z)$ be a polynomial given in (1). Then $t \in (0,1)$ is a limit point of the sequence $\{f(\alpha)\alpha^n\}_{n=1,2,3,\dots}$ if and only if there is $c \in \mathcal{B}$ such that $t = c/b$. Furthermore, at least one of the numbers 0 and 1 is a limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,3,\dots}$ if and only if $0 \in \mathcal{B}$.*

Without loss of generality we can assume that the conjugates of α are labelled so that $\alpha = \alpha_1 > 1 > |\alpha_2| \geqslant |\alpha_3| \geqslant \dots \geqslant |\alpha_d|$. Then α is called a *strong Pisot number* if $d \geqslant 2$ and α_2 is positive [3]. By a result of Smyth [10] claiming that each circle $|z| = r$ contains at most two conjugates of a Pisot number α, the inequality $\alpha_2 > |\alpha_3|$ holds for every strong Pisot number α. Recall that a result of Pisot and Vijayaraghavan (see, e.g., [2]) implies that if the sequence $\{\xi\alpha^n\}_{n=1,2,3,\dots}$, where $\alpha > 1$ is algebraic and $\xi > 0$ is real, has a unique limit point, then α is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$. So our next result characterizes all possible cases when the sequence $\{\xi\alpha^n\}_{n=1,2,\dots}$ has a unique limit point and completes the results of the author [3] and of Luca [9].

Theorem 2. *Let $\alpha > 1$ be a Pisot number and let $f(z)$ be a polynomial given in (1). Then*

(i) $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = t$, *where $t \neq 0, 1$, if and only if $\mathcal{B} = \{c\}$, $c > 0$, $t = c/b$.*

(ii) $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 0$ *if and only if $\mathcal{B} = \{0\}$ and α is either an integer or a strong Pisot number and $f(\alpha_2) < 0$.*

(iii) $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 1$ *if and only if $\mathcal{B} = \{0\}$, α is a strong Pisot number and $f(\alpha_2) > 0$.*

The following theorem gives a simple practical criterion of determining whether the sequence $\{f(\alpha)\alpha^n\}_{n=1,2,\dots}$ has one or more than one limit point.

Theorem 3. *If $\mathcal{B} = \{c\}$, $c > 0$, then there is an integer r, where $1 \leqslant r \leqslant |P(1)|-1$, such that $c/b = r/|P(1)|$. Furthermore, if $\gcd(b, a_0) = 1$ then $\mathcal{B} = \{c\}$ is equivalent to $b \mid cP(1)$ and $b \mid (Y_n - c)$ for every $n = 0, 1, \dots, d-1$.*

Theorems 2 and 3 imply the following corollary.

Corollary. *Let ξ be an arbitrary positive number, and let α be a Pisot number which is not an integer or a strong Pisot number. If $P(1) = -1$, then $\{\xi\alpha^n\}_{n=1,2,3,\dots}$ has more than one limit point.*

Since $P(z)$ is the minimal polynomial of a Pisot number α, we have $P(1) < 0$ and $P'(\alpha) > 0$. Note that the condition $P(1) = -1$ is equivalent to the fact that

$\alpha - 1$ is a unit. Our final theorem describes all algebraic numbers $\alpha > 1$ for which there is a positive number ξ such that the sequence $\{\xi\alpha^n\}_{n=1,2,\dots}$ tends to a limit.

Theorem 4. *Suppose that $\alpha > 1$ is an algebraic number. Then there is a real number $\xi > 0$ such that the sequence $\{\xi\alpha^n\}_{n=1,2,3,\dots}$ tends to a limit if and only if α is either a strong Pisot number, or $\alpha = 2$, or α is a Pisot number whose minimal polynomial P satisfies $P(1) \leqslant -2$.*

In fact, we will show that if α is strong Pisot number or $\alpha = 2$ we can take $\xi = 1$, whereas in the third case of Theorem 4 we can take $\xi = 1/\big(P'(\alpha)(\alpha - 1)\big)$.

Some examples will be given in Section 4.

3. Proofs of the theorems

Proof of Theorem 1. Consider the trace of $f(\alpha)\alpha^n$:

$$f(\alpha_1)\alpha_1^n + f(\alpha_2)\alpha_2^n + \dots + f(\alpha_d)\alpha_d^n = Y_n/b\,.$$

Setting

$$L_n := f(\alpha_2)\alpha_2^n + \dots + f(\alpha_d)\alpha_d^n \tag{3}$$

(which is a real number), we have

$$\{f(\alpha)\alpha^n\} = Y_n/b - L_n - [f(\alpha)\alpha^n]\,. \tag{4}$$

Assume that $\mathcal{B}$ contains a non-zero integer c. Then $b \geqslant 2$. Since $1 \leqslant c \leqslant b-1$ and all $f(\alpha_j)\alpha_j^n$, where $j \geqslant 2$, tend to zero as $n \to \infty$, we get that $L_n \to 0$ as $n \to \infty$ and so $\{f(\alpha)\alpha^n\} = c/b - L_n$ for infinitely many n. Hence c/b is the limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,\dots}$ for each non-zero $c \in \mathcal{B}$. Suppose now that $t \in (0,1)$ is a limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,\dots}$. Since $L_n \to 0$ as $n \to \infty$, equality (4) implies that t is a limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,\dots}$ only if $t = c/b$, where $c \in \mathcal{B}$. This proves the first part of Theorem 1. The second part follows from (3) and (4) by a similar argument. □

Proof of Theorem 2. We begin with (i). As above, since $L_n \to 0$ as $n \to \infty$, (4) shows that the sequence $\{f(\alpha)\alpha^n\}_{n=1,2,\dots}$ has a unique limit point only if Y_n modulo b is ultimately periodic with period of length 1. Since the unique limit point is neither 0 nor 1, it follows that $\mathcal{B} = \{c\}$, where $c > 0$. For the converse, suppose that $\mathcal{B} = \{c\}$, where c is non-zero. Then $b \geqslant 2$ and $1 \leqslant c \leqslant b-1$. Furthermore, Y_n modulo b is c for each sufficiently large n. With these conditions, (4) implies that $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = c/b$. This proves (i).

If α is an integer, say $\alpha = g$, then $\{(b_0/b)g^n\} \to 0$ as $n \to \infty$ precisely when each prime divisor of b divides g, i.e. $\mathcal{B} = \{0\}$, because $Y_n = b_0 g^n$. This proves the subcase of (ii) corresponding to integer α. Suppose now that α is irrational. If $\mathcal{B} = \{0\}$, α is a strong Pisot number and $f(\alpha_2) < 0$, then L_n defined by (3) is negative for all sufficiently large n. So (4) implies that $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 0$.

For the converse, suppose that $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 0$. Then (4) shows immediately that $\mathcal{B} = \{0\}$, as otherwise the sequence of fractional parts has other limit points. We already know that one case when $\mathcal{B} = \{0\}$ and $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 0$

both occur is when α is an integer. Suppose it is not. Then, since 1 is not the limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,\dots}$, the sum L_n defined by (3) must be negative for all sufficiently large n. Recall that $\alpha_1 > |\alpha_2| \geqslant \dots \geqslant |\alpha_d|$.

We will consider three cases corresponding to α_2 being complex, negative and positive. By the above mentioned result of Smyth [10], if α_2 is complex, then α_2 and α_3 is the only complex conjugate pair on the circle $|z| = |\alpha_2|$. Since $\alpha_3 = \overline{\alpha_2}$, for each n sufficiently large, the sign of L_n is determined by the sign of $f(\alpha_2)\alpha_2^n + f(\alpha_3)\alpha_3^n$. Clearly, $f(\alpha_2) \neq 0$, because $\deg f < d$. Writing $\alpha_2 = \varrho e^{i\varphi}$ and $f(\alpha_2) = \varrho' e^{i\phi}$, where $\varrho, \varrho' > 0$ and $i = \sqrt{-1}$, we see that $\alpha_3 = \varrho e^{-i\varphi}$, $f(\alpha_3) = \varrho' e^{-i\phi}$. Hence $L_n < 0$ (for n sufficiently large) precisely when $\cos(n\varphi + \phi) < 0$. Note that φ/π is irrational, as otherwise there is a positive integer v such that $\alpha_2^v = \alpha_3^v$. Mapping α_2 to α_1 we get a contradiction, because α_1 is the only conjugate of α lying outside the unit circle. Hence, as the sequence $n\varphi/\pi + \phi/\pi$ modulo 1 has each point in $[0, 1]$ as its limit point, $\cos(n\varphi+\phi)$ will be both positive and negative for infinitely many n. This rules out the case of α_2 being complex. Similarly, if α_2 is negative then L_n is both positive and negative infinitely often, because so is $f(\alpha_2)\alpha_2^n$. This implies that α_2 must be positive, namely, α must be a strong Pisot number. Then $L_n < 0$ implies that $f(\alpha_2) < 0$. This proves (ii).

The case (iii) can be proved by the same argument as (ii). Indeed, if α is a strong Pisot number, $f(\alpha_2) > 0$, and $\mathcal{B} = \{0\}$, then (4) implies that $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 1$. For the converse, assume that $\lim_{n\to\infty}\{f(\alpha)\alpha^n\} = 1$. It is easy to see that then $\mathcal{B} = \{0\}$. Furthermore, α cannot be a rational integer. Now, (4) shows that L_n must be positive for all sufficiently large n. We already proved that this is impossible, unless α is a strong Pisot number. In case it is, (3) shows that $f(\alpha_2)$ must be positive too. This completes the proof of Theorem 2. □

Proof of Theorem 3. Suppose that $\mathcal{B} = \{c\}$, $c > 0$. Then (2) shows that b divides $c(1 + a_{d-1} + \dots + a_0) = cP(1)$, where $P(1) < 0$. It follows that there is $r \in \mathbb{N}$ such that $br = c|P(1)|$, giving $c/b = r/|P(1)|$. This proves the first statement of Theorem 3.

Now, let $\gcd(b, a_0) = 1$ and suppose again that $\mathcal{B} = \{c\}$, where c can be equal to zero. The above argument implies that $b \mid cP(1)$. Evidently, $\mathcal{B} = \{c\}$ is equivalent to the fact that Y_n modulo b is equal to c for every sufficiently large n. Suppose that there are $k \geqslant 0$ for which Y_k modulo b is different from c. Take the largest such k. Let Y_k modulo b be c', where $c' \neq c$. Then (2) with $n = k$ shows that $Y_{k+d} + \dots + a_1Y_{k+1} + a_0Y_k$ modulo b is $cP(1) + a_0(c' - c)$ which is divisible by b. Since $b \mid cP(1)$, we have that $b \mid a_0(c' - c)$. Since $\gcd(a_0, b) = 1$, we conclude that $c' = c$, a contradiction.

For the converse, suppose that $Y_0, Y_1, \dots, Y_{d-1}$ are all equal to c modulo b, and $b \mid cP(1)$. Evidently, (2) with $n = 0$ shows that $Y_d + a_{d-1}Y_{d-1} + \dots + a_0Y_0$ modulo b is zero. But it is equal to $Y_d + c(P(1) - 1) = Y_d - c + cP(1)$ modulo b. Since $b \mid cP(1)$, we obtain that Y_d is c modulo b. In the same manner (setting $n = 1$ into (2) and so on) we can see that Y_n is equal to c modulo b for every $n \geqslant 0$. Therefore, $\mathcal{B} = \{c\}$. Note that we were not using the condition $\gcd(a_0, b) = 1$ for this part of the proof. □

Proof of the Corollary. For $\xi \notin \mathbb{Q}(\alpha)$, the sequence $\{\xi\alpha^n\}_{n=1,2,\dots}$ has more than one limit point by the above mentioned result of Pisot and Vijayaraghavan (and by the results of [6] mentioned in Section 1 too). So suppose that $\xi \in \mathbb{Q}(\alpha)$, where α satisfies the conditions of the corollary. If $\{\xi\alpha^n\}_{n=1,2,\dots}$ has a unique limit point, then Theorem 2 implies that $\mathcal{B} = \{c\}$. Clearly, by the first part of Theorem 3, $|P(1)| = 1$ yields $c = 0$. Now, parts (ii) and (iii) of Theorem 2 show that α is either a rational integer or a strong Pisot number, a contradiction. $\square$

Proof of Theorem 4. Suppose that $\xi > 0$ and an algebraic number $\alpha > 1$ are such that $\{\xi\alpha^n\}_{n=1,2,\dots}$ has a unique limit point. Then (again by the theorem of Pisot and Vijayaraghavan) α is a Pisot number. The corollary shows that α must be either an integer, or a strong Pisot number, or a Pisot number whose minimal polynomial P satisfies $P(1) \leqslant -2$. Since all rational integers, except for $\alpha = 2$, are covered by the case $P(1) \leqslant -2$, the theorem is proved in one direction.

Now, if α is a strong Pisot number, then, with $\xi = 1$, we have $\lim_{n\to\infty}\{\alpha^n\} = 1$. (See, for instance, Theorem 2 (iii) with $b = 1$ and $f(z) = 1$.) If α is a rational integer, greater than or equal to 2, then, with $\xi = 1$, $\lim_{n\to\infty}\{\alpha^n\} = 0$.

Finally, suppose that α is a Pisot number of degree $d \geqslant 2$ whose minimal polynomial P satisfies $P(1) \leqslant -2$. Let us take $\xi = 1/\big(P'(\alpha)(\alpha-1)\big) > 0$. We will show that then $\lim_{n\to\infty}\{\xi\alpha^n\} = 1/|P(1)|$. Note that, for each $k = 0, 1, \dots, d-1$,

$$\frac{z^k}{P(z)} = \sum_{j=1}^{d} \frac{\alpha_j^k}{P'(\alpha_j)(z-\alpha_j)}\,. \tag{5}$$

Indeed, for each non-negative integer $k < d$, (5) is the identity, because multiplying both sides of (5) by $P(z)$ we obtain two polynomials, both of degree smaller than d, which are equal at d distinct points $z = \alpha_j$, $j = 1, 2, \dots, d$. Setting $z = 1$ into (5)), we deduce that the trace of $\alpha^k/\big(P'(\alpha)(\alpha-1)\big)$ is equal to $-1/P(1) = 1/|P(1)| < 1$ for every $k = 0, 1, \dots, d-1$. Of course, we can write $\xi = 1/\big(P'(\alpha)(\alpha-1)\big) = f(\alpha)$ for some polynomial f of the form (1). Then, as in the proof of Theorem 3, we will get that Y_n, $n = 0, 1, \dots, d-1$, modulo b are all equal to c, where $b = c|P(1)|$. Hence, as in the second part of the proof of Theorem 3 we obtain that Y_n modulo b is equal to c for every non-negative integer n. Consequently, $\mathcal{B} = \{c\}$, where $c/b = 1/|P(1)|$. Now, Theorem 2 (i) implies that

$$\lim_{n\to\infty}\{\alpha^n/\big(P'(\alpha)(\alpha-1)\big)\} = 1/|P(1)|$$

provided that α is a Pisot number whose minimal polynomial P satisfies $P(1) \leqslant -2$. (This result trivially holds for integer $\alpha \geqslant 3$ too.) The proof of Theorem 4 is completed. $\square$

4. Examples

We remark that the condition $\gcd(b, a_0) = 1$ of Theorem 3 cannot be removed. Take, for example, $\alpha = 3 + \sqrt{5}$. It is a strong Pisot number with other conjugate being $\alpha_2 = 3 - \sqrt{5}$. Its minimal polynomial is $P(z) = z^2 - 6z + 4$. Set $f(z) = (1+z)/4$. Here, $b = 4$ and $a_0 = 4$. Note that $S_0 = 2$, $S_1 = 6$, $S_2 = 28$, $S_3 = 144$, and so on. All S_n, $n = 2, 3, \dots$, are divisible by 4. Hence $Y_n = S_n + S_{n+1}$ modulo 4 is equal to 2 for $n = 1$ and to zero for all non-negative $n \neq 1$.

Suppose that $\theta > 1$ solves $z^3 - z - 1 = 0$. Then θ is a Pisot number having a pair complex conjugates inside the unit circle. Clearly, $P(1) = -1$. The corollary implies that there are no $\xi > 0$ (algebraic or transcendental) such that the sequence $\{\xi\theta^n\}_{n=1,2,\dots}$ tends to a limit with n tending to infinity.

Set, for instance, $f(z) = (2 + z)/3$. Let us find the set of limit points of $\{(2/3 + \theta/3)\theta^n\}_{n=1,2,\dots}$. Then $Y_n = 2S_n + S_{n+1}$, $b = 3$. We find that $S_0, S_1, S_2, S_3, S_4, \dots$ modulo 3 is purely periodic with period $\overline{0020222110212}$, so that $Y_0, Y_1, Y_2, Y_3, \dots$ modulo 3 is purely periodic with period $\overline{0212002022211}$. It follows that $\mathcal{B} = \{0, 1, 2\}$. Since θ has a pair of complex conjugates, on the arithmetical progression $n = 13m$, $m = 0, 1, 2, \dots$, the values of L_n, defined by (3) are positive and negative infinitely often. Hence the set of limit points of the sequence $\{(2/3+\theta/3)\theta^n\}_{n=1,2,\dots}$ is $\{0, 1/3, 2/3, 1\}$.

Finally, if, say, $\alpha > 1$ solves $z^2 - 7z + 2 = 0$ then $S_0, S_1, S_2, S_3, \dots$ modulo $b = 4$ is $2, 3, 1, 1, 1, \dots$. Taking, for example, $f(z) = (2 + 3z)/4$, we deduce that $Y_n = 2S_n + 3S_{n+1}$ modulo 4 is ultimately periodic, with $\mathcal{B} = \{1\}$. Consequently, $\lim_{n\to\infty}\{\frac{2+3\alpha}{4}\alpha^n\} = 1/4$.

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Department of Mathematics and Informatics
Vilnius University
Naugarduko 24, LT-03225 Vilnius, Lithuania
E-mail: arturas.dubickas@maf.vu.lt

Institute of Mathematics and Informatics
Akademijos 4, LT-08663 Vilnius, Lithuania