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PERIODIC SOLUTIONS FOR SYSTEMS WITH NONSMOOTH AND PARTIALLY COERCIVE POTENTIAL

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Abstract. In this paper we consider nonlinear periodic systems driven by the one-dimensional p-Laplacian and having a nonsmooth locally Lipschitz potential. Using a variational approach based on the nonsmooth Critical Point Theory, we establish the existence of a solution. We also prove a multiplicity result based on a nonsmooth extension of the result of Brezis-Nirenberg [3] due to Kandilakis-Kourogenis-Papageorgiou [13].

1. Introduction

The purpose of this paper is to prove an existence and a multiplicity result for nonlinear periodic systems driven by the one-dimensional p-Laplacian with nonsmooth Laplacian.

Recently there has been an increasing interest for problems involving the one-dimensional p-Laplacian and various solvability techniques were used. We mention the works of Dang-Oppenheimer [6], Del Pino-Manasevich-Murua [7], Fabry-Fayyad [8], Gasinski-Papageorgiou [9], Guo [10], Manasevich-Mawhin [16] and the references therein. From the above works Gasinski-Papageorgiou use a variational approach, while the others use degree theory combined with techniques from nonlinear analysis and the right hand side nonlinearity is continuous (i.e. the corresponding potential function is C^1). Also we should mention that in Dang-Oppenheimer, Guo and Manasevich-Mawhin the right hand side nonlinearity also depends on x' and consequently their hypotheses are stronger. Here the potential function $j(t,x)$ is only measurable in $t \in T$ and locally Lipschitz in $x \in \mathbb{R}^{\mathbb{N}}$ (not necessarily C^1). We assume that $j(t,\cdot)$ is only partially coercive, i.e. $j(t,x) \to +\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in E \subseteq T$, with $|E| > 0$ (here by $|\cdot|$ we denote the Lebesgue measure on $\mathbb{R}$). This way we extend the very recent work of Tang-Wu [18] where $p = 2$ (semilinear problem) and the potential function $j(t,\cdot)$ is

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C^1 (smooth problem). Initially semilinear problems with fully coercive potential, were studied by Berger-Schechter [2] and Mawhin-Willem [17].

Our approach is variational and it is based on the nonsmooth Critical Point Theory as this was formulated by Chang [4] and extended recently by Kourogenis-Papageorgiou [14]. The multiplicity result that we prove is based on a recent nonsmooth extension of the result of Brezis-Nirenberg [3] due to Kandilakis-Kourogenis-Papageorgiou [13].

2. Mathematical background

Let X be a Banach space, X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the *generalized directional derivative* of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) \overset{df}{=} \limsup_{\substack{x'\to x\\ \lambda\downarrow 0}} \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda}\,.$$

The function $h \to \varphi^0(x;h)$ is sublinear, continuous and so it is the support function of a nonempty, w^*-compact, convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) \overset{df}{=} \left\{x^* \in X^* :\ \langle x^*, h\rangle \le \varphi^0(x;h) \text{ for all } h \in X\right\}.$$

The multifunction $x \to \partial\varphi(x)$ is known as the *generalized* (or *Clarke*) *subdifferential* of φ. If φ is continuous convex (hence locally Lipschitz), then the generalized subdifferential and the subdifferential in the sense of convex analysis coincide. Also if $\varphi \in C^1(X)$ (hence it is locally Lipschitz), then $\partial\varphi = \{\varphi'(x)\}$.

A point $x \in X$ is a critical point of the locally Lipschitz function $\varphi : X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. A local extremum of φ is a critical point. The well-known Palais-Smale condition (PS-condition for short), in the present nonsmooth setting takes the following form:

> "A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the nonsmooth PS-condition, if every sequence $\{x_n\}_{n\ge1} \subseteq X$ such that $|\varphi(x_n)| \le M_1$ for some $M_1 > 0$, all $n \ge 1$ and $m(x_n) = \inf\left[\|x^*\| : x^* \in \partial\varphi(x_n)\right] \to 0$ as $n \to \infty$, has a strongly convergent subsequence."

3. Existence theorem

The nonlinear, nonsmooth periodic system under consideration is the following:

$$\begin{cases} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in \partial j(x(t)) & \text{a.e. on } T=[0,b] \\ x(0)=x(b)\,,\ x'(0)=x'(b)\,, & 2\le p<\infty\,. \end{cases} \tag{3.1}$$

Here by $\partial j(t,x)$ we denote the Clarke subdifferential of the locally Lipschitz potential function $j(t,\cdot)$. Our hypotheses on $j(t,x)$ are the following:

$H(j)_1$**:** $j : T \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a function such that $j = j_1 + j_2$ and for $i = 1, 2$;

(i) for all $x \in \mathbb{R}^{\mathbb{N}}$, $t \to j_i(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j_i(t,x)$ is locally Lipschitz;

(iii) for every $M > 0$, there exists $\alpha_M \in L^1(T)$ such that

$$\sup\big[|j(t,x)|,\ \|u\| : \|x\| \le M,\ u \in \partial j(t,x)\big] \le \alpha_M(t) \quad \text{a.e. on } T\,;$$

(iv) $j_1(t,x) \to +\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in E$, $|E| > 0$ and there exists $\xi \in L^1(T)$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$ $\xi(t) \le j_1(t,x)$;

(v) there exists $\theta \in L^1(T)$ such that for almost all $t \in T$, all $x \in \mathbb{R}^{\mathbb{N}}$ and all $u \in \partial j_2(t,x)$, $\|u\| \le \theta(t)$ and $\int_0^b j_2(t,x)\,dt \ge -c_0$ for all $x \in \mathbb{R}^{\mathbb{N}}$ with $c_0 > 0$.

In the proof of our existence theorem we shall need the following auxiliary result due to Tang-Wu [18] (see Lemma 3) relating uniform coercivity and subaddivity.

Lemma 3.1. *If $j : T \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a function such that for all $x \in \mathbb{R}^{\mathbb{N}}$, $t \to j(t,x)$ is measurable, for almost all $t \in T$ $x \to j(t,x)$ is continuous, for every $M > 0$ there exists $\alpha_M \in L^1(T)$ such that for almost all $t \in T$ and all $\|x\| \le M$, $|j(t,x)| \le \alpha_M(t)$ and $j(t,x) \to +\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in E$, $|E| > 0$, then there exist $g \in C(\mathbb{R}^{\mathbb{N}})_+$ subadditive function such that $g(x) \to +\infty$ as $\|x\| \to \infty$ and $g(x) \le \|x\| + 4$ and $\eta \in L^1(T)$ for which we have for almost all $t \in E$ and all $x \in \mathbb{R}^{\mathbb{N}}$ $j(t,x) \ge g(x) + \eta(t)$.*

Remark 3.2. Here by $|E|$ we denote the Lebesgue measure of $|E|$.

Theorem 3.3. *If hypotheses $H(j)_1$ hold, then problem* (3.1) *has a solution $x \in C^1(T,\mathbb{R}^{\mathbb{N}})$.*

Proof. Let $\varphi : W^{1,p}_{\text{per}}(T,\mathbb{R}^{\mathbb{N}}) \to \mathbb{R}$ be the energy functional defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p + \int_0^b j\big(t,x(t)\big)\,dt = \frac{1}{p}\|x'\|_p^p + \int_0^b j_1\big(t,x(t)\big)\,dt + \int_0^b j_2\big(t,x(t)\big)\,dt\,.$$

We know (see for example Chang [4] or Hu-Papageorgiou [12]) that φ is locally Lipschitz. By virtue of Lemma 3.1, we can find $E \subseteq T$, with $|E| > 0$ such that for almost all $t \in E$ and all $x \in \mathbb{R}^{\mathbb{N}}$ we have

$$j_1(t,x) \ge g(x) + \eta(t)$$

with $g \in C(\mathbb{R}^{\mathbb{N}})_+$ subadditive, coercive and $\eta \in L^1(T)$. We have

$$\begin{aligned}\int_0^b j_1\big(t,x(t)\big)\,dt &= \int_E j_1\big(t,x(t)\big)\,dt + \int_{T\setminus E} j_1\big(t,x(t)\big)\,dt \\ &\ge \int_E g\big(x(t)\big)\,dt + \int_E \eta(t)\,dt + \int_{T\setminus E} \xi(t)\,dt\,.\end{aligned}$$

Consider the following direct sum decomposition

$$W^{1,p}_{\text{per}}(T,\mathbb{R}^{\mathbb{N}}) = \mathbb{R}^{\mathbb{N}} \oplus V$$

with $V = \Big\{v \in W^{1,p}_{\text{per}}(T,\mathbb{R}^{\mathbb{N}}) : \int_0^b v(t) = 0\Big\}$. So if $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^{\mathbb{N}})$, we can write in a unique way $x = \overline{x} + \widehat{x}$, with $\overline{x} \in \mathbb{R}^{\mathbb{N}}$ and $\widehat{x} \in V$. Exploiting the subadditivity of g, we have

$$\begin{aligned} g(\overline{x}) =& g\big(x(t)-\widehat{x}(t)\big) \le g\big(x(t)\big)+g\big(-\widehat{x}(t)\big) \quad \text{for all} \quad t\in T\,,\\ &\Rightarrow g(\overline{x})-g\big(-\widehat{x}(t)\big)\le g\big(x(t)\big) \qquad \text{for all} \quad t\in T\,. \end{aligned}$$

Moreover, because of Lemma 3.1 we have

$$g\big(-\widehat{x}(t)\big)\le \|\widehat{x}(t)\|+4\le \|\widehat{x}\|_\infty+4\,.$$

We have

$$\begin{aligned} \int_E g\big(x(t)\big)\,dt &\ge \int_E g(\overline{x})\,dt-\int_E g\big(-\widehat{x}(t)\big)\,dt\\ &= g(\overline{x})|E|-(\|\widehat{x}\|_\infty+4)\,|E|\,. \end{aligned}$$

But from the Poincare-Wirtinger inequality (see Mawhin-Willem [17], p.8) we know that

$$\|\widehat{x}\|_\infty\le b^{\frac{1}{q}}\|\widehat{x}'\|_p=b^{\frac{1}{q}}\|x'\|_p\,.$$

So we obtain

$$\int_E g\big(x(t)\big)\,dt\ge g(\overline{x})|E|-\Big(b^{\frac{1}{q}}\|x'\|_p+4\Big)|E|\,.$$

Let $\Gamma(t)=\big\{(v,\lambda)\in\mathbb{R}^{\mathbb{N}}\times(0,1): v\in\partial j_2\big(t,\overline{x}+\lambda\widehat{x}(t)\big),\ j_2\big(t,\overline{x}+\widehat{x}(t)\big)-j_2(t,\overline{x}) = \big(v,\widehat{x}(t)\big)_{\mathbb{R}^{\mathbb{N}}}\big\}$. From the Mean Value Theorem (see for example Clarke [5],p.41), we know that for almost all $t\in T$, $\Gamma(t)\ne\emptyset$. By redefining $\Gamma(\cdot)$ on the exceptional Lebesgue-null set, we may assume without any loss of generality that $\Gamma(t)\ne\emptyset$ for all $t\in[0\cdot b]$. We claim that for every direction $h\in\mathbb{R}^{\mathbb{N}}$ the function $(t,\lambda)\to j_2^0\big(t,\overline{x}+\lambda\widehat{x}(t);h\big)$ is measurable. Indeed from the definition of the generalized derivative, we have

$$j_2^0\big(t,\overline{x}+\lambda\widehat{x}(t)\big)= \inf_{m\ge1}\ \sup_{r,s\in Q\cap(-\frac{1}{m},\frac{1}{m})}\frac{j_2(t,\overline{x}+\lambda\widehat{x}(t)+r+sh)-j_2(t,\overline{x}+\lambda\widehat{x}(t)+r)}{s}\,.$$

Since j_2 is jointly measurable (see Hu-Papageorgiou [11], p.142), it follows that $(t,\lambda)\to j_2^0\big(t,\overline{x}+\lambda\widehat{x}(t);h\big)$ is measurable. Set $S(t,\lambda)=\partial j_2\big(t,\overline{x}+\lambda\widehat{x}(t)\big)$ and let $\{h_m\}_{m\ge1}\subseteq\mathbb{R}^{\mathbb{N}}$ be a countable dense set. Because $j_2^0(t,\overline{x}+\lambda\widehat{x}(t);\cdot)$ is continuous, we have

$$\begin{aligned} GrS&=\big\{(t,\lambda,u)\in T\times(0,1)\times\mathbb{R}^{\mathbb{N}}: u\in S(t,\lambda)\big\}\\ &=\bigcap_{m\ge1}\big\{(t,\lambda,u)\in T\times(0,1)\times\mathbb{R}^{\mathbb{N}}:(u,h_m)_{\mathbb{R}^{\mathbb{N}}}\le j_2^0(t,\overline{x}+\lambda\widehat{x}(t);h_m)\big\}\\ &\Rightarrow GrS\in\mathcal{L}(T)\times B\big((0,1)\big)\times B(\mathbb{R}^{\mathbb{N}})\,, \end{aligned}$$

with $\mathcal{L}(T)$ being the Lebesgue σ-field of T and $B\big((0,1)\big)$ (resp. $B(\mathbb{R}^{\mathbb{N}})$) the Borel σ-field of $(0,1)$ (resp. of $\mathbb{R}^{\mathbb{N}}$). So we can apply the Yankon-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [11], p.158) to obtain measurable functions $v:T\to\mathbb{R}^{\mathbb{N}}$ and $\lambda:T\to(0,1)$ such that $\big(v(t),\lambda(t)\big)\in\Gamma(t)$ for all $t\in T$

and $j_2\big(t,\overline{x}+\widehat{x}(t)\big)-j_2(t,\overline{x})=\big(v(t),\widehat{x}(t)\big)_{\mathbb{R}^{\mathbb{N}}}$, $v(t)\in\partial j_2\big(t,\overline{x}+\lambda(t)\widehat{x}(t)\big)$ a.e. on T. Using hypothesis $H(j)_1(v)$ and the Poicare-Wirtinger inequality, we obtain

$$\begin{aligned}\int_0^b j_2\big(t,x(t)\big)\,dt &= \int_0^b j_2\big(t,\overline{x}+\widehat{x}(t)\big)\\ &\geq \int_0^b j_2(t,\overline{x})\,dt - b^{\frac{1}{p}}\|x'\|_p\|\theta\|_1\,.\end{aligned}$$

Thus finally we have

$$\varphi(x)\geq\frac{1}{p}\|x'\|_p^p+g(\overline{x})|E|-\big(b^{\frac{1}{q}}\|x'\|_p+4\big)|E|-\|\xi\|_1-c_0-b^{\frac{1}{q}}\|x'\|_p\|\theta\|_1\,.$$

From this inequality and the coercivity of g, it follows that φ is coercive. Exploiting the compact embedding of $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})$ into $C(T,\mathbb{R}^{\mathbb{N}})$, we can easily check that φ is weakly lower semicontinuous. So by the Weierstrass theorem we can find $x\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})$ such that $\varphi(x)=\inf\varphi$. Then we have $0\in\partial\varphi(x)$. Let A : $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})\to W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})^*$ be the nonlinear operator defined by

$$\langle A(x),y\rangle=\int_0^b -\|x'(t)\|^{p-2}\big(x'(t),y'(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt\,.$$

We have $A(x)=u$ with $u\in S^q_{\partial j\left(\cdot,x(\cdot)\right)}$. For every $\psi\in C_0^\infty\big((0,b),\mathbb{R}^{\mathbb{N}}\big)$ we have

$$\int_0^b -\|x'(t)\|^{p-2}\big(x'(t),\psi'(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt=\int_0^b\big(u(t),\psi(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt$$

Recalling that $\big(\|x'(\cdot)\|^{p-2}x'(\cdot)\big)\in W^{-1,q}(T,\mathbb{R}^{\mathbb{N}})=W_0^{1,p}(T,\mathbb{R}^{\mathbb{N}})^*$ (see Adams [1], p.50), we have that

$$\langle(\|x'\|^{p-2}x')',\psi\rangle_0=\int_0^b\big(u(t),\psi(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt=\langle u,\psi\rangle_0\,,$$

where $\langle\cdot,\cdot\rangle_0$ denotes the duality brackets for the pair $\big(W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}}),W^{-1,q}(T,\mathbb{R}^{\mathbb{N}})\big)$. Since $C_0^\infty\big((0,b),\mathbb{R}^{\mathbb{N}}\big)$ is dense in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})$ it follows that

$$\big(\|x'(t)\|^{p-2}x'(t)\big)'=u(t)\in\partial j\big(t,x(t)\big)\quad\text{a.e. on } T\,. \tag{3.2}$$

Also for every $y\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})$, using Green's identity (integration by parts), we obtain

$$\begin{aligned}\langle A(x),y\rangle &= \big(\|x'(b)\|^{p-2}x'(b),y(b)\big)_{\mathbb{R}^{\mathbb{N}}}-\big(\|x'(0)\|^{p-2}x'(0),y(0)\big)_{\mathbb{R}^{\mathbb{N}}}\\ &\quad-\int_0^b\big((\|x'(t)\|^{p-2}x'(t))',y(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt\quad\text{for all}\quad y\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})_{\mathbb{R}^{\mathbb{N}}}\,dt\,.\end{aligned}$$

Because $A(x)=u$, and using (3.2), we obtain

$$\begin{aligned}&\big(\|x'(b)\|^{p-2}x'(b),y(b)\big)_{\mathbb{R}^{\mathbb{N}}}=\big(\|x'(0)\|^{p-2}x'(0),y(0)\big)_{\mathbb{R}^{\mathbb{N}}}\quad\text{for all}\quad y\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^{\mathbb{N}})\,,\\ &\quad\Rightarrow\|x'(b)\|^{p-2}x'(b)=\|x'(0)\|^{p-2}x'(0)\,,\\ &\quad\Rightarrow x'(0)=x'(b)\,.\end{aligned}$$

Note that since $x \in W^{1,p}_{\text{per}}(T, \mathbb{R}^{\mathbb{N}})$, we have $(x(0) = x(b)$. Finally since $\|x'\|^{p-2}x' \in W^{1,q}_{\text{per}}(T, \mathbb{R}^{\mathbb{N}}) \Rightarrow \|x'(\cdot)\|^{p-2}x'(\cdot) \in C^1_{\text{per}}(T, \mathbb{R}^{\mathbb{N}})$. Because the map $y \to \|y\|^{p-2}y$ is a homeomorphism of $\mathbb{R}^{\mathbb{N}}$, we infer that $x' \in C_{\text{per}}(T, \mathbb{R}^{\mathbb{N}})$, hence $x \in C^1_{\text{per}}(T, \mathbb{R}^{\mathbb{N}})$ and it solves (3.1). □

4. Multiplicity result

Next by strengthening our hypotheses on $j(t, \cdot)$ with a condition about its behavior near zero, we obtain a multiplicity result for problem (3.1). For this we will need the following nonsmooth version of the Local Linking theorem due to Brezis-Nirenberg [3]. This theorem was proved recently by Kandilakis-Kourogenis-Papageorgiou [13].

Theorem 4.1. *If X is a reflexive Banach space such that $X = Y \oplus V$ with* $\dim Y < +\infty$, *$\varphi : x \to \mathbb{R}$ is a locally Lipschitz functional which satisfies the nonsmooth PS-condition, $\varphi(0) = 0$ and*

(a) *there exists $r > 0$ such that*

$$\varphi(y) \le 0 \quad \textit{for} \quad y \in Y,\ \|y\| \le r \quad \textit{and} \quad \varphi(v) \ge 0 \quad \textit{for} \quad v \in V,\ \|v\| \le r\,,$$

(ii) *φ is bounded below and* $\inf \varphi < 0$,

then φ has at least two nontrivial critical points.

Our hypotheses on the nonsmooth potential $j(t, x)$ are the following:

$H(j)_2$**:** $j : T \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a function which satisfies hypotheses $H(j)_1$ and

(vi) $\displaystyle\lim_{x \to 0} \frac{pj(t,x)}{\|x\|^p} = 0$ uniformly for almost all $t \in T$ and there exists $r_0 > 0$ such that for almost all $t \in T$ and all $\|x\| \le r_0$ we have $j(t, x) \le 0$.

Theorem 4.2. *If hypotheses $H(j)_2$ hold, then problem* (3.1) *has at least two nontrivial solutions in $C^1(T, \mathbb{R}^{\mathbb{N}})$.*

Proof. Let $\varphi : W^{1,p}_{\text{per}}(t, \mathbb{R}^{\mathbb{N}}) \to \mathbb{R}$ be the locally Lipschitz energy functional defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p + \int_0^b j\big(t, x(t)\big)\, dt\,.$$

From the proof of Theorem 3.3 we know that φ is coercive, hence it satisfies the nonsmooth PS-condition (see Kourogenis-Papageorgiou [15]). As before we consider the direct sum decomposition

$$W^{1,p}_{\text{per}}(T, \mathbb{R}^{\mathbb{N}}) = \mathbb{R}^{\mathbb{N}} \oplus V$$

with $V = \big\{v \in W^{1,p}_{\text{per}}(T, \mathbb{R}^{\mathbb{N}}) : \int_0^b v(t)\, dt = 0\big\}$. By virtue of hypothesis $H(j)_2(vi)$ given $\varepsilon > 0$, we can find $\delta > 0$ such that for almost all $t \in T$ and all $\|x\| \le \delta$ we have $-\dfrac{\varepsilon}{p}\|x\|^p \le j(t, x)$. Let $v \in V$ with $\|v'\|_p \le \dfrac{\delta}{b^{\frac{1}{q}}}$. From the Poincare-Wirtinger inequality we have that $\|v\|_\infty \le b^{\frac{1}{q}}\|v'\|_p \le \delta$. So if $v \in V$ with $\|v'\|_p \le \dfrac{\delta}{b^{\frac{1}{q}}} = \delta_1$, we have $\|v\|_\infty \le \delta$ and so

$$\begin{aligned}\varphi(v) &= \frac{1}{p}\|v'\|_p^p + \int_0^b j\big(t, v(t)\big)\,dt\\ &\ge \frac{1}{p}\|v'\|_p^p + \frac{\varepsilon}{p}\|v\|_p^p\\ &\ge \frac{1}{p}\Big(1 - \frac{\varepsilon}{\beta_1}\Big)\|v'\|_p^p \quad \text{for some} \quad \beta_1 > 0\,,\end{aligned}$$

from the Poincare-Wirtinger inequality. Choose $\varepsilon \le \beta_1$, to infer that for $\|v\| \le \delta_1$ we have $\varphi(v) \ge 0$.

Also if $y \in \mathbb{R}^{\mathbb{N}}$ and $\|y\| \le r_0$, then by hypothesis $H(j)_2$(vi) we have that

$$\varphi(y) = \int_0^b j(t, y)\,dt \le 0\,.$$

Note that φ being coercive, it is bounded below. If $\inf\varphi < 0$, then using $r = \min\{\delta_1, r_0\} > 0$ we can apply Theorem 4.1 and obtain two nontrivial critical points of φ, which we can check are two distinct nontrivial solutions of (3.1) in $C^1(T, \mathbb{R}^{\mathbb{N}})$.

If $\inf\varphi = 0$, then by virtue of hypothesis $H(j)_2(vi)$ for all $y \in \mathbb{R}^{\mathbb{N}}$ with $b^{\frac{1}{p}}\|y\|_{\mathbb{R}^{\mathbb{N}}} \le \delta_1$ we have $\inf\varphi = \varphi(y) = 0$ and so we conclude that φ has an infinity of critical points, therefore problem (3.1) has an infinity of solutions in $C^1(T, \mathbb{R}^{\mathbb{N}})$. □

The nonsmooth locally Lipschitz potential function

$$j(t,x) = \begin{cases} -\|x\|^p \ln\left(1 + \|x\|^p\right) & \text{if} \quad \|x\| \le 1 \\ \chi_E(t)\ln\|x\| + \chi_{E^c}(t)\sin\pi\|x\| - \ln 2 & \text{if} \quad \|x\| \ge 1 \end{cases},$$

with $|E| > 0$, satisfies hypotheses $H(j)_2$.

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