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# FINELY DIFFERENTIABLE MONOGENIC FUNCTIONS

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Abstract. Since 1970's B. Fuglede and others have been studying finely holomorhic functions, i.e., 'holomorphic' functions defined on the so-called fine domains which are not necessarily open in the usual sense. This note is a survey of finely monogenic functions which were introduced in [12] like a higher dimensional analogue of finely holomorphic functions.

## 1. Introduction

At the beginning of the 20th century É. Borel worked on the idea that certain holomorphic functions can be continued beyond their classical maximal domain of existence to a larger (not necessarily open) domain, see [2]. A significant progress in the same direction was made not earlier than in 1970's when B. Fuglede extended the notion of holomorphic functions to those defined on domains from a topology finer than the Euclidean one, namely, the fine topology of potential theory. A very deep theory of the so-called finely holomorphic functions has been developed since then, see [4], [5] or [8].

The Clifford analysis may be considered as a higher dimensional analogue of classical complex analysis. In the Clifford analysis, functions called here monogenic are counterparts of holomorphic ones. In [12], finely monogenic functions were introduced. This note presents results on finely monogenic functions obtained in [11], [12] and [13]. In [11], a special case of dimension 4 is considered.

## 2. Monogenic functions

The Clifford analysis studies functions taking values in Clifford modules that we are going to introduce now, see e.g. [9, Chapter 2]. Consider a real (or complex) finite dimensional Hilbert space $\mathcal{H} = \big(\mathcal{H}, (\cdot,\cdot)\big)$ . Denote by $\mathcal{L}(\mathcal{H})$ the algebra of linear operators on $\mathcal{H}$ and by $\bar{a}$ the adjoint operator to $a \in \mathcal{L}(\mathcal{H})$ , i.e., $(au, v) = (u, \bar{a}v)$ , $u, v \in \mathcal{H}$ . In what follows, we suppose that $m \geq 1$ and $\mathcal{H}$ is a (left) $C\ell_m$-module, i.e., there are skew-adjoint operators $e_1, \dots, e_m$ in $\mathcal{L}(\mathcal{H})$ (i.e., $\bar{e}_j = -e_j$ ) such that

$$e_j^2 = -e_0\,, \quad e_j e_k = -e_k e_j$$

The paper is in final form and no version of it will be submitted elsewhere.

for $j, k = 1, \dots, m$, $j \neq k$ where $e_0$ is the identity operator on $\mathcal{H}$. Then the Euclidean space $\mathbb{R}^{m+1}$ can be naturally embedded into $\mathcal{L}(\mathcal{H})$ in the following way:

$$x = (x_0, \dots, x_m) \cong x_0 e_0 + \dots + x_m e_m \,.$$

Low dimensional examples of Clifford modules are the complex plane $\mathbb{C}$ and the skew field of real quaternions $\mathbb{H}$. Indeed, the complex plane $\mathbb{C}$ is a $C\ell_1$-module with $e_1(z) = iz$. As to the latter case, recall that the field $\mathbb{H}$ can be viewed as the Euclidean space $\mathbb{R}^4$ endowed with a non-commutative multiplication. A quaternion $x$ can be written in the form $x = x_0 + x_1 i + x_2 j + x_3 k$ where $x_0, x_1, x_2, x_3$ are real numbers and $i, j, k$ are the imaginary units such that

$$i^2 = j^2 = k^2 = -1 \,, \quad ij = -ji = k \,, \quad jk = -kj = i \,, \quad ki = -ik = j \,.$$

Then $\mathbb{H}$ is a $C\ell_3$-module with $e_1(x) = ix$, $e_2(x) = jx$ and $e_3(x) = kx$. In a general dimension, examples of Clifford modules are given by the corresponding real and complex Clifford algebras and spinor spaces, see e.g. [9, p. 60].

Now we are ready to introduce monogenic functions. Given an open set $G \subset \mathbb{R}^{m+1}$ and a $C\ell_m$-module $\mathcal{H}$, denote by $\mathcal{C}^1(G)$ the set of continuously differentiable functions $f : G \to \mathcal{H}$ and define the Cauchy-Riemann operator by

$$\mathbf{D} = \sum_{j=0}^{m} e_j \, \frac{\partial}{\partial x_j} \,.$$

**Definition 1.** A function $f \in \mathcal{C}^1(G)$ is called monogenic if $\mathbf{D} f = 0$ on $G$.

It is well known that a function $f$ is monogenic if and only if both $f$ and $xf(x)$ are harmonic where $xf(x) := (x_0 e_0 + \dots + x_m e_m) f(x)$. Let us remark that the Clifford analysis includes classical complex analysis. Indeed, in the complex case monogenic functions coincide with holomorphic ones. Furthermore, in the quaternionic case the so-called quaternionic analysis developed by R. Fueter in 1930's is obtained, see e.g. [9, Chapter 2] for details.

## 3. Fine topology

For an account of the fine topology, we refer to [1, Chapter 7]. The fine topology $\mathcal{F}$ in $\mathbb{R}^{m+1}$ is the weakest topology making all subharmonic functions in $\mathbb{R}^{m+1}$ continuous. It is strictly finer than the Euclidean topology in $\mathbb{R}^{m+1}$. For example, if $M$ is a dense countable subset of an open set $G \subset \mathbb{R}^{m+1}$, then $U := G \setminus M$ is a finely open set but it has no interior points in the usual sense.

Let $U \subset \mathbb{R}^{m+1}$ be finely open and $f : U \to \mathbb{R}^k$. Then we call $f$ finely continuous on $U$ if $f$ is continuous from $U$ endowed with the fine topology to $\mathbb{R}^k$ with the Euclidean topology. Denote by $\mathcal{F}_x$ the family of fine neighbourhoods of a point $x \in \mathbb{R}^{m+1}$. The fine limit of $f$ at a point $x \in U$ can be characterised as the usual limit along some fine neighbourhood of $x$, i.e., there is $V \in \mathcal{F}_x$ such that

$$\operatorname*{fine-lim}_{y \to x} f(y) = \lim_{y \to x, y \in V} f(y) \,,$$

see [1, p. 207]. Moreover, we call a linear map $L:\mathbb{R}^{m+1}\to\mathbb{R}^k$ the fine differential of $f$ at a point $x\in U$ if

$$\text{fine-}\lim_{y\to x}\frac{f(y)-f(x)-L(y-x)}{|y-x|}=0\,.$$

Here $|x|$ is the Euclidean norm of $x\in\mathbb{R}^{m+1}$. We write fine-$df(x)$ for $L$ and set, for $j=0,\dots,m$,

$$\text{fine-}\frac{\partial f}{\partial x_j}(x):=\text{fine-}df(x)(\mathbf{e}_j)$$

where the vectors $\mathbf{e}_0\dots,\mathbf{e}_m$ form the standard basis of $\mathbb{R}^{m+1}$.

In 1970-80's a very deep theory of finely holomorhic functions has been developed, see e.g. [4], [5] or [8]. Recall that, given a finely open subset $V$ of the complex plane $\mathbb{C}$, a function $f:V\to\mathbb{C}$ is called finely holomorphic provided that $f$ has a finely continuous fine derivative $f'$ on $V$. Here

$$f'(z)=\text{fine-}\lim_{w\to z}\frac{f(w)-f(z)}{w-z},\ z\in V\,.$$

Moreover, in [4], B. Fuglede proved the following theorem.

**Theorem 1.** *A function $f$ is finely holomorphic on a finely open set $V\subset\mathbb{C}$ if and only if, for each $z\in V$, there is $K\in\mathcal{F}_z$ and $F\in\mathcal{C}^1(\mathbb{C})$ such that $F=f$ on $K$ and $\bar\partial F=0$ on $K$ where $z=x+iy\in\mathbb{C}$ and*

$$\bar\partial F:=\frac{1}{2}\left(\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}\right)\,.$$

Finely holomorphic functions are closely related to finely harmonic ones. Indeed, a function $f$ is finely holomorphic on a finely open set $V\subset\mathbb{C}$ if and only if, the functions $f$ and $zf(z)$ are both finely harmonic on $V$. For an account of finely harmonic functions, we refer to [3]. Let us recall that a function $f$ is finely harmonic on a finely open set $U\subset\mathbb{R}^{m+1}$ if and only if, for every $x\in U$, there is $V\in\mathcal{F}_x$ such that $f|_V$, the restriction of $f$ to $V$, is a uniform limit of functions $f_n$ harmonic on open sets $V_n$ containing $V$.

## 4. Finely monogenic functions

Now we introduce finely monogenic functions. In what follows, we suppose that $\mathcal{H}$ is a $C\ell_m$-module, $U\subset\mathbb{R}^{m+1}$ be finely open and $f:U\to\mathcal{H}$ unless otherwise stated.

**Definition 2.** A function $f$ is called finely monogenic if $f$ and $xf(x)$ are both finely harmonic on $U$.

When $m=1$ we get nothing else but finely holomorphic functions introduced by B. Fuglede, see [4], [5] or [8]. A function $f$ is monogenic on a usual open set $G\subset\mathbb{R}^{m+1}$ if and only if $f$ is finely monogenic and locally bounded on $G$ because the same is true even for finely harmonic functions. Moreover, when $m=1$ we do not need to assume local boundedness of $f$. See [3, Theorem 10.16].

Recall that the Sobolev space $W^{1,2}(\mathbb{R}^{m+1})$ consists of (Lebesgue) measurable functions $F$ whose second power is integrable on $\mathbb{R}^{m+1}$ together with second powers of its first weak derivatives. Denote by $W^{1,2}_{\text{f-loc}}(U)$ the set of functions $f$ on $U$ satisfying that, for each $x \in U$, there exist $K \in \mathcal{F}_x$ and $F \in W^{1,2}(\mathbb{R}^{m+1})$ such that $F = f$ on $K$. For an account of the Sobolev spaces on fine domains, we refer to [10]. Now we are ready to state other characterisations of finely monogenic functions, see [12].

**Theorem 2.** *The following statements are equivalent to each other:*

(a) $f$ *is finely monogenic on* $U$,

(b) $f$ *is finely continuous on* $U$, $f \in W^{1,2}_{\text{f-loc}}(U)$ *and* $\mathbf{D}f = 0$ *on* $U$,

(c) $f$ *is finely harmonic on* $U$ *and* fine-$\mathbf{D}f = 0$ *almost everywhere on* $U$, *i.e., except for a Lebesgue null set. Here*

$$\text{fine-}\mathbf{D}f = \sum_{j=0}^{m} e_j \ \text{fine-}\frac{\partial f}{\partial x_j}$$

*at each point where* $f$ *is finely differentiable.*

To state our next result we need some notation. Let us denote by fine-$\mathcal{C}^1(U)$ the set of all functions $f$ finely differentiable everywhere on $U$ whose fine differential fine-$df$ is finely continuous on $U$. Moreover, $\mathcal{C}^1_{\text{f-loc}}(U)$ stands for the set of all functions $f$ on $U$ such that, for each $x \in U$, there is $K \in \mathcal{F}_x$ and $F \in \mathcal{C}^1(\mathbb{R}^{m+1})$ with $F = f$ on $K$. Then the following theorem is proved in [13].

**Theorem 3.** *Let* $m \geq 1$ *and* $U \subset \mathbb{R}^{m+1}$ *be finely open. Then*

$$\mathcal{C}^1_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^1(U) \cap W^{1,2}_{\text{f-loc}}(U)\,.$$

*In the case where* $m = 1$, *it is true even that* $\mathcal{C}^1_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^1(U)$.

Let us remark that Theorem 3 for $m = 1$ is essentially due to B. Fuglede. If $m \geq 2$, then it seems to be open whether

$$\text{fine-}\mathcal{C}^1(U) \subset W^{1,2}_{\text{f-loc}}(U)$$

or not. Since finely holomorphic functions are infinitely fine differentiable (in particular, they belong to fine-$\mathcal{C}^1(U)$) the following result generalises Theorem 1 mentioned above.

**Theorem 4.** *A function* $f$ *is finely monogenic and* $f \in \text{fine-}\mathcal{C}^1(U)$ *if and only if* $f \in \mathcal{C}^1_{\text{f-loc}}(U)$ *and* fine-$\mathbf{D}f = 0$ *on* $U$.

**Proof.** Let us notice that, by Theorem 2 (b), any finely monogenic function $f$ belongs to $W^{1,2}_{\text{f-loc}}(U)$. Now it is easy to see that Theorem 4 is a direct consequence of Theorem 3 stated above. □

In comparison with finely harmonic functions, finely holomorphic functions are infinitely fine differentiable everywhere and have the unique continuation property, i.e., any finely holomorphic function $f$ on a fine domain $U$ in $\mathbb{C}$ is uniquely determined by its values in some fine neighbourhood of a point of $U$, see [5] or [4]. It would be interesting to clear up to what extent these properties remain true for finely monogenic functions in higher dimensions.

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