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# DEFORMATION THEORY
# (LECTURE NOTES)

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Abstract. First three sections of this overview paper cover classical topics of deformation theory of associative algebras and necessary background material. We then analyze algebraic structures of the Hochschild cohomology and describe the relation between deformations and solutions of the corresponding Maurer-Cartan equation. In Section 6 we generalize the Maurer-Cartan equation to strongly homotopy Lie algebras and prove the homotopy invariance of the moduli space of solutions of this equation. In the last section we indicate the main ideas of Kontsevich's proof of the existence of deformation quantization of Poisson manifolds.

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**Conventions.** All algebraic objects will be considered over a fixed field $\mathbf{k}$ of characteristic zero. The symbol $\otimes$ will denote the tensor product over $\mathbf{k}$. We will sometimes use the same symbol for both an algebra and its underlying space.

## 1. Algebras and modules

In this section we investigate modules (where module means rather a bimodule than a one-sided module) over various types of algebras.

**Example 1.1.** – The category $\mathsf{Ass}$ of associative algebras.
An associative algebra is a $\mathbf{k}$-vector space $A$ with a bilinear product $A \otimes A \to A$

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satisfying
$$a(bc) = (ab)c, \qquad \text{for all } a, b, c \in A.$$
Observe that at this moment we do not assume the existence of a unit $1 \in A$.

What we understand by a module over an associative algebra is in fact a bimodule, i.e. a vector space $M$ equipped with multiplications ("actions") by elements of $A$ from both sides, subject to the axioms
$$\begin{aligned} a(bm) &= (ab)m\,, \\ a(mb) &= (am)b\,, \\ m(ab) &= (ma)b\,, \qquad \text{for all } m \in M,\ a, b \in A\,. \end{aligned}$$

**Example 1.2.** – The category $\mathsf{Com}$ of commutative associative algebras.
In this case left modules, right modules and bimodules coincide. In addition to the axioms in $\mathsf{Ass}$ we require the commutativity
$$ab = ba\,, \qquad \text{for all } \ a, b \in A\,,$$
and for a module
$$ma = am\,, \qquad \text{for all } \ m \in M,\ a \in A\,.$$

**Example 1.3.** – The category $\mathsf{Lie}$ of Lie algebras.
The bilinear bracket $[-,-] : L \otimes L \to L$ of a Lie algebra $L$ is anticommutative and satisfies the Jacobi identity, that is
$$[a, b] = -[b, a]\,, \text{ and}$$
$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,, \qquad \text{for all } \ a, b, c \in L\,.$$
A left module (also called a representation) $M$ of $L$ satisfies the standard axiom
$$a(bm) - b(am) = [a, b]m\,, \qquad \text{for all } \ m \in M,\ a, b \in L\,.$$
Given a left module $M$ as above, one can canonically turn it into a right module by setting $ma := -am$. Denoting these actions of $L$ by the bracket, one can rewrite the axioms as
$$[a, m] = -[m, a]\,, \text{ and}$$
$$[a, [b, m]] + [b, [m, a]] + [m, [a, b]] = 0\,, \qquad \text{for all } \ m \in M,\ a, b \in L\,.$$

Examples 1.1–1.3 indicate how axioms of algebras induce, by replacing one instance of an algebra variable by a module variable, axioms for the corresponding modules. In the rest of this section we formalize, following [38], this recipe. The standard definitions below can be found for example in [29].

**Definition 1.4.** The *product* in a category $\mathsf{C}$ is the limit of a discrete diagram. The *terminal object* of $\mathsf{C}$ is the limit of an empty diagram, or equivalently, an object $T$ such that for every $X \in \mathsf{C}$ there exists a unique morphism $X \to T$.

**Remark 1.5.** The product of any object $X$ with the terminal object $T$ is naturally isomorphic to $X$,
$$X \times T \cong X \cong T \times X\,.$$

**Remark 1.6.** It follows from the universal property of the product that there exists the *swapping morphism* $X \times X \xrightarrow{s} X \times X$ making the diagram

$$\begin{tikzcd}
X \times X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "s"] & X \\
X & X \times X \arrow[l, "p_1"] \arrow[u, "p_2"']
\end{tikzcd}$$

in which $p_1$ (resp. $p_2$) is the projection onto the first (resp. second) factor, commutative.

**Example 1.7.** In the category of $A$-bimodules, the product $M_1 \times M_2$ is the ordinary direct sum $M_1 \oplus M_2$. The terminal object is the trivial module 0.

**Definition 1.8.** A category $\mathsf{C}$ has *finite products*, if every *finite* discrete diagram has a limit in $\mathsf{C}$.

By [29, Proposition 5.1], $\mathsf{C}$ has finite limits if and only if it has a terminal object and products of pairs of objects.

**Definition 1.9.** Let $\mathsf{C}$ be a category, $A \in \mathsf{C}$. The *comma category* (also called the *slice category*) $\mathsf{C}/A$ is the category whose

– objects $(X, \pi)$ are $\mathsf{C}$-morphisms $X \xrightarrow{\pi} A$, $X \in \mathsf{C}$, and

– morphisms $(X', \pi') \xrightarrow{f} (X'', \pi'')$ are commutative diagrams of $\mathsf{C}$-morphisms:

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "\pi'"'] & X'' \arrow[d, "\pi''"] \\
A \arrow[r, equal, "\mathrm{id}_A"'] & A
\end{tikzcd}$$

**Definition 1.10.** The *fibered product* (or *pullback*) of morphisms $X_1 \xrightarrow{f_1} A$ and $X_2 \xrightarrow{f_2} A$ in $\mathsf{C}$ is the limit $D$ (together with morphisms $D \xrightarrow{p_1} X_1$, $D \xrightarrow{p_2} X_2$) of the lower right corner of the digram:

$$\begin{tikzcd}
D \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] & X_1 \arrow[d, "f_1"] \\
X_2 \arrow[r, "f_2"'] & A
\end{tikzcd}$$

In the above situation one sometimes writes $D = X_1 \times_A X_2$.

**Proposition 1.11.** *If $\mathsf{C}$ has fibered products then $\mathsf{C}/A$ has finite products.*

**Proof.** A straightforward verification. The identity morphism $(A, \mathrm{id}_A)$ is clearly the terminal object of $\mathsf{C}/A$.

Let $(X_1, \pi_1)$ and $(X_2, \pi_2)$ be objects of $\mathsf{C}/A$. By assumption, there exists the fibered product

(1)
$$\begin{tikzcd} D \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] \arrow[dr, "\delta"] & X_1 \arrow[d, "\pi_1"] \\ X_2 \arrow[r, "\pi_2"'] & A \end{tikzcd}$$

in $\mathsf{C}$. In the above diagram, of course, $\delta := \pi_1 p_1 = \pi_2 p_2$. The maps $p_1 : D \to X_1$ and $p_2 : D \to X_2$ of the above diagram define morphisms (denoted by the same symbols) $p_1 : (D, \delta) \to (X_1, \pi_1)$ and $p_2 : (D, \delta) \to (X_2, \pi_2)$ in $\mathsf{C}/A$. The universal property of the pullback (1) implies that the object $(D, \delta)$ with the projections $(p_1, p_2)$ is the product of $(X_1, \pi_1) \times (X_2, \pi_2)$ in $\mathsf{C}/A$. □

One may express the conclusion of the above proof by

$$(X_1, \pi_1) \times (X_2, \pi_2) = X_1 \times_A X_2\,, \tag{2}$$

but one must be aware that the left side lives in $\mathsf{C}/A$ while the right one in $\mathsf{C}$, therefore (2) has only a symbolical meaning.

**Example 1.12.** In $\mathsf{Ass}$, the fibered product of morphisms $B_1 \xrightarrow{f_1} A$, $B_2 \xrightarrow{f_2} A$ is the subalgebra

$$B_1 \times_A B_2 = \{(b_1, b_2) \mid f_1(b_1) = f_2(b_2)\} \subseteq B_1 \oplus B_2 \tag{3}$$

together with the restricted projections. Hence for any algebra $A \in \mathsf{Ass}$, the comma category $\mathsf{Ass}/A$ has finite products.

**Definition 1.13.** Let $\mathsf{C}$ be a category with finite products and $T$ its terminal object. An *abelian group object* in $\mathsf{C}$ is a quadruple $(G, G \times G \xrightarrow{\mu} G, G \xrightarrow{\eta} G, T \xrightarrow{e} G)$ of objects and morphisms of $\mathsf{C}$ such that following diagrams commute:

– the *associativity* $\mu$:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

– the *commutativity* of $\mu$ (with $s$ the swapping morphism of Remark 1.6):

$$\begin{tikzcd} G \times G \arrow[rr, "s"] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}$$

– the *neutrality* of $e$:

$$\begin{tikzcd} T \times G \arrow[r, "e \times \mathrm{id}_G"] \arrow[dr, dash, "\cong"] & G \times G \arrow[d, "\mu"] & G \times T \arrow[l, "\mathrm{id}_G \times e"'] \arrow[dl, dash, "\cong"'] \\ & G & \end{tikzcd}$$

– the diagram saying that $\eta$ is a two-sided *inverse* for the multiplication $\mu$:

$$\begin{tikzcd} G \arrow[r, "\eta \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \eta"'] \arrow[dr] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

in which the diagonal map is the composition $G \to T \xrightarrow{e} G$.

Maps $\mu$, $\eta$ and $e$ above are called the *multiplication*, the *inverse* and the *unit* of the abelian group structure, respectively.

*Morphisms* of abelian group objects $(G', \mu', \eta', e') \xrightarrow{f} (G'', \mu'', \eta'', e'')$ are morphisms $G' \xrightarrow{f} G''$ in $\mathsf{C}$ which preserve all structure operations. In terms of diagrams this means that

$$\begin{tikzcd} G' \times G' \arrow[r, "f \times f"] \arrow[d, "\mu'"'] & G'' \times G'' \arrow[d, "\mu''"] \\ G \arrow[r, "f"'] & G' \end{tikzcd} \qquad \begin{tikzcd} G' \arrow[r, "f"] \arrow[d, "\eta'"'] & G'' \arrow[d, "\eta''"] \\ G \arrow[r, "f"'] & G' \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, equal, "\mathrm{id}_T"] \arrow[d, "e'"'] & T \arrow[d, "e''"] \\ G \arrow[r, "f"'] & G' \end{tikzcd}$$

commute. The category of abelian group objects of $\mathsf{C}$ will be denoted $\mathsf{C}_{ab}$.

Let $\mathsf{Alg}$ be any of the examples of categories of algebras considered above and $A \in \mathsf{Alg}$. It turns out that the category $(\mathsf{Alg}/A)_{ab}$ is precisely the corresponding category of $A$-modules. To verify this for associative algebras, we identify, in Proposition 1.15 below, objects of $(\mathsf{Ass}/A)_{ab}$ with trivial extensions in the sense of:

**Definition 1.14.** Let $A$ be an associative algebra and $M$ an $A$-module. The *trivial extension* of $A$ by $M$ is the associative algebra $A \oplus M$ with the multiplication given by

$$(a, m)(b, n) = (ab, an + mb), \ a, b \in A \text{ and } m, n \in M.$$

**Proposition 1.15.** *The category* $(\mathsf{Ass}/A)_{ab}$ *is isomorphic to the category of trivial extensions of* $A$.

**Proof.** Let $M$ be an $A$-module and $A \oplus M$ the corresponding trivial extension. Then $A \oplus M$ with the projection $A \oplus M \overset{\pi_A}{\to} A$ determines an object $G$ of $\mathsf{Ass}/A$ and, by (2) and (3), $G \times G = (A \oplus M \oplus M \overset{\pi_A}{\to} A)$. It is clear that $\mu : G \times G \to G$ given by $\mu(a, m_1, m_2) := (a, m_1 + m_2)$, $e$ the inclusion $A \hookrightarrow A \oplus M$ and $\eta : G \to G$ defined by $\eta(a, m) := (a, -m)$ make $G$ an abelian group object in $(\mathsf{Ass}/A)_{ab}$.

On the other hand, let $((B, \pi), \mu, \eta, e)$ be an abelian group object in $\mathsf{Ass}/A$. The diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, no head, "\mathrm{id}_A"'] & B \arrow[d, "\pi"'] \\ & A \end{tikzcd}$$

for the neutral element says that $\pi$ is a retraction. Therefore one may identify the algebra $A$ with its image $e(A)$, which is a subalgebra of $B$. Define $M := \operatorname{Ker} \pi$ so that there is a vector spaces isomorphism $B = A \oplus M$ determined by the inclusion $e : A \hookrightarrow B$ and its retraction $\pi$. Since $M$ is an ideal in $B$, the algebra $A$ acts on $M$ from both sides. Obviously, $M$ with these actions is an $A$-bimodule, the bimodule axioms following from the associativity of $B$ as in Example 1.1. It remains to show that $m'm'' = 0$ for all $m', m'' \in M$ which would imply that $B$ is a trivial extension of $A$. Let us introduce the following notation.

For a morphism $f : (B', \pi') \to (B'', \pi'')$ of $\mathbf{k}$-splitting objects of $\mathsf{Ass}/A$ (i.e. objects with specific $\mathbf{k}$-vector space isomorphisms $B' \cong A \oplus M'$ and $B'' \cong A \oplus M''$ such that $\pi'$ and $\pi''$ are the projections on the first summand) we denote by $\tilde{f} : M' \to M''$ the restriction $f|_{M'}$ followed by the projection $B'' \overset{\pi'}{\to} M''$. We call $\tilde{f}$ the *reduction* of $f$. Clearly, for every diagram of splitting objects in $\mathsf{Ass}/A$ there is the corresponding diagram of reductions in $\mathsf{Ass}$.

The fibered product $(A \oplus M, \pi) \times (A \oplus M, \pi)$ in $\mathsf{Ass}/A$ is isomorphic to $A \oplus M \oplus M$ with the multiplication

$$(a', m_1', m_2')(a'', m_1'', m_2'') = (a'a'', a'm_1'' + m_1'a'' + m_1'm_1'', a'm_2'' + m_2'a'' + m_2'm_2'')\,.$$

The neutrality of $e$ implies the following diagram of reductions

$$\begin{tikzcd} 0 \oplus M \arrow[r, "\tilde{e} \times \mathrm{id}_M"] \arrow[dr, dash, "\cong"'] & M \oplus M \arrow[d, "\tilde{\mu}"] & M \oplus 0 \arrow[l, "\mathrm{id}_M \times \tilde{e}"'] \arrow[dl, dash, "\cong"] \\ & M & \end{tikzcd}$$

which in turn implies

$$\tilde{\mu}(0,m) = \tilde{\mu}(m,0) = m\,, \qquad \text{for all } m \in M\,.$$

Since $\mu$ is a morphism in $\mathsf{Ass}$, it preserves the multiplication and so does its reduction $\tilde{\mu}$. We finally obtain

$$m' \cdot m'' = \tilde{\mu}(m',0) \cdot \tilde{\mu}(0,m'') = \tilde{\mu}((m',0)\cdot(0,m'')) = \tilde{\mu}(m'\cdot 0, 0 \cdot m'') = 0\,.$$

This finishes the proof. □

We have shown that objects of $(\mathsf{Ass}/A)_{ab}$ are precisely trivial extensions of $A$. Since there is an obvious equivalence between modules and trivial extensions, we obtain:

**Theorem 1.16.** *The category* $(\mathsf{Ass}/A)_{ab}$ *is isomorphic to the category of $A$-modules.*

**Exercise 1.17.** Prove analogous statements also for $(\mathsf{Com}/A)_{ab}$ and $(\mathsf{Lie}/L)_{ab}$.

**Exercise 1.18.** The only property of abelian group objects used in our proof of Proposition 1.15 was the existence of a neutral element for the multiplication. In fact, by analyzing our arguments we conclude that in $\mathsf{Ass}/A$, every object with a multiplication and a neutral element (i.e. a *monoid* in $\mathsf{Ass}/A$) is an abelian group object. Is this statement true in any comma category? If not, what special property of $\mathsf{Ass}/A$ makes it hold in this particular category?

## 2. Cohomology

Let $A$ be an algebra, $M$ an $A$-module. There are the following approaches to the "cohomology of $A$ with coefficients in $M$."

(1) *Abelian cohomology* defined as $H^*(Lin(R_*, M))$, where $R_*$ is a resolution of $A$ in the category of $A$-modules.
(2) *Non-abelian cohomology* defined as $H^*(Der(\mathcal{F}_*, M))$, where $\mathcal{F}_*$ is a resolution of $A$ in the category of algebras and $Der(-,M)$ denotes the space of derivations with coefficients in $M$.
(3) *Deformation cohomology* which is the subject of this note.

The adjective *(non)-abelian* reminds us that (1) is a derived functor in the abelian category of modules while cohomology (2) is a derived functor in the non-abelian category of algebras. Construction (1) belongs entirely into classical homological algebra [27], but (2) requires Quillen's theory of closed model categories [37]. Recall that in this note we work over a field of characteristics 0, over the integers one should take in (2) a suitable simplicial resolution [1]. Let us indicate the meaning of deformation cohomology in the case of associative algebras.

Let $V = \mathrm{Span}\{e_1, \dots, e_d\}$ be a $d$-dimensional $\mathbf{k}$-vector space. Denote $\mathit{Ass}(V)$ the set of all associative algebra structures on $V$. Such a structure is determined by a bilinear map $\mu : V \otimes V \to V$. Relying on Einstein's convention, we write $\mu(e_i, e_j) = \Gamma^l_{ij} e_l$ for some scalars $\Gamma^l_{ij} \in \mathbf{k}$. The associativity $\mu(e_i, \mu(e_j, e_k)) = \mu(\mu(e_i, e_j), e_k)$ of $\mu$ can then be expressed as

$$\Gamma^r_{il}\Gamma^l_{jk} = \Gamma^l_{ij}\Gamma^r_{lk}\,, \qquad i, j, k, r = 1, \dots, d\,.$$

These $d^4$ polynomial equations define an affine algebraic variety, which is just another way to view $\mathit{Ass}(V)$, since every point of this variety corresponds to an associative algebra structure on $V$. We call $\mathit{Ass}(V)$ the *variety of structure constants* of associative algebras.

The next step is to consider the quotient $\mathit{Ass}(V)/\,GL(V)$ of $\mathit{Ass}(V)$ modulo the action of the general linear group $GL(V)$ recalled in formula (10) below. However, $\mathit{Ass}(V)/\,GL(V)$ is no longer an affine variety, but only a (possibly singular) algebraic stack (in the sense of Grothendieck). One can remove singularities by replacing $\mathit{Ass}(V)$ by a smooth dg-scheme $\mathcal{M}$. Deformation cohomology is then the cohomology of the tangent space of this smooth dg-scheme [6, 7].

Still more general approach to deformation cohomology is based on considering a given category of algebras as the category of algebras over a certain PROP $\mathsf{P}$ and defining the deformation cohomology using a resolution of $\mathsf{P}$ in the category of PROPs [24, 31, 33]. When $\mathsf{P}$ is a Koszul quadratic operad, we get the *operadic cohomology* whose relation to deformations was studied in [3]. There is also an approach to deformations based on *triples* [10].

For associative algebras all the above approaches give the classical Hochschild cohomology (formula 3.2 of [27, §X.3]):

**Definition 2.1.** The *Hochschild cohomology* of an associative algebra $A$ with coefficients in an $A$-module $M$ is the cohomology of the complex:

$$0 \longrightarrow M \xrightarrow{\delta_{\mathrm{Hoch}}} C^1_{\mathrm{Hoch}}(A, M) \xrightarrow{\delta_{\mathrm{Hoch}}} \cdots \xrightarrow{\delta_{\mathrm{Hoch}}} C^n_{\mathrm{Hoch}}(A, M) \xrightarrow{\delta_{\mathrm{Hoch}}} \cdots$$

in which $C^n_{\mathrm{Hoch}}(A, M) := \mathit{Lin}(A^{\otimes n}, M)$, the space of $n$-multilinear maps from $A$ to $M$. The coboundary $\delta = \delta_{\mathrm{Hoch}} : C^n_{\mathrm{Hoch}}(A, M) \to C^{n+1}_{\mathrm{Hoch}}(A, M)$ is defined by

$$\begin{aligned}\delta_{\mathrm{Hoch}} f(a_0 \otimes \ldots \otimes a_n) := (-1)^{n+1} a_0 f(a_1 \otimes \ldots \otimes a_n) + f(a_0 \otimes \ldots \otimes a_{n-1}) a_n \\ + \sum_{i=0}^{n-1} (-1)^{i+n} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n)\,,\end{aligned}$$

for $a_i \in A$. Denote $H^n_{\mathrm{Hoch}}(A, M) := H^n(C^*_{\mathrm{Hoch}}(A, M), \delta)$.

**Exercise 2.2.** Prove that $\delta^2_{\mathrm{Hoch}} = 0$.

**Example 2.3.** A simple computation shows that

– $H^0_{\mathrm{Hoch}}(A, M) = \{m \in M \mid am - ma = 0 \text{ for all } a \in A\}$,

– $H^1_{\mathrm{Hoch}}(A,M) = Der(A,M)/\,IDer(A,M)$, where $IDer(A,M)$ denotes the subspace of internal derivations, i.e. derivations of the form $\vartheta_m(a) = am - ma$ for $a \in A$ and some fixed $m \in M$. Slightly more difficult is to prove that

– $H^2_{\mathrm{Hoch}}(A,M)$ is the space of isomorphism classes of singular extensions of $A$ by $M$ [27, Theorem X.3.1].

## 3. Classical deformation theory

As everywhere in this note, we work over a field $\mathbf{k}$ of characteristics zero and $\otimes$ denotes the tensor product over $\mathbf{k}$. By a *ring* we will mean a commutative associative $\mathbf{k}$-algebra. Let us start with necessary preliminary notions.

**Definition 3.1.** Let $R$ be a ring with unit $e$ and $\omega : \mathbf{k} \to R$ the homomorphism given by $\omega(1) := e$. A homomorphism $\epsilon : R \to \mathbf{k}$ is an *augmentation* of $R$ if $\epsilon\omega = \mathrm{id}_{\mathbf{k}}$ or, diagrammatically,

$$\begin{tikzcd} R \arrow[r, "\epsilon"] & \mathbf{k} \\ \mathbf{k} \arrow[u, "\omega"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

The subspace $\overline{R} := \operatorname{Ker}\epsilon$ is called the *augmentation ideal* of $R$. The *indecomposables* of the augmented ring $R$ are defined as the quotient $Q(R) := \overline{R}/\overline{R}^2$.

**Example 3.2.** The unital ring $\mathbf{k}[[t]]$ of formal power series with coefficients in $\mathbf{k}$ is augmented, with augmentation $\epsilon : \mathbf{k}[[t]] \to \mathbf{k}$ given by $\epsilon(\sum_{i\in\mathbb{N}_0} a_i t^i) := a_0$. The unital ring $\mathbf{k}[t]$ of polynomials with coefficients in $\mathbf{k}$ is augmented by $\epsilon(f) := f(0)$, for $f \in \mathbf{k}[t]$. The truncated polynomial rings $\mathbf{k}[t]/(t^n)$, $n \geq 1$, are also augmented, with the augmentation induced by the augmentation of $\mathbf{k}[t]$.

**Example 3.3.** Recall that the group ring $\mathbf{k}[G]$ of a finite group $G$ with unit $e$ is the space of all formal linear combinations $\sum_{g\in G} a_g g$, $a_g \in \mathbf{k}$, with the multiplication

$$(\textstyle\sum_{g\in G} a'_g g)(\sum_{g\in G} a''_g g) := \sum_{g\in G}\sum_{uv=g} a'_u a''_v g$$

and unit $1e$. The ring $\mathbf{k}[G]$ is augmented by $\epsilon : \mathbf{k}[G] \to \mathbf{k}$ given as

$$\epsilon(\textstyle\sum_{g\in G} a_g g) := \sum_{g\in G} a_g\,.$$

**Example 3.4.** A rather trivial example of a ring that does not admit an augmentation is provided by any proper extension $K \supsetneq \mathbf{k}$ of $\mathbf{k}$. If an augmentation $\epsilon : K \to \mathbf{k}$ exists, then $\operatorname{Ker}\epsilon$ is, as an ideal in a field, trivial, which implies that $\epsilon$ is injective, which would imply that $K = \mathbf{k}$ contradicting the assumption $K \neq \mathbf{k}$.

**Exercise 3.5.** If $\sqrt{-1} \notin \mathbf{k}$, then $\mathbf{k}[x]/(x^2+1)$ admits no augmentation.

In the rest of this section, $R$ will be an augmented unital ring with an augmentation $\epsilon : R \to \mathbf{k}$ and the unit map $\omega : \mathbf{k} \to R$. By a module we will understand a *left* module.

**Remark 3.6.** A unital augmented ring $R$ is a $\mathbf{k}$-bimodule, with the bimodule structure induced by the unit map $\omega$ in the obvious manner. Likewise, $\mathbf{k}$ is an $R$-bimodule, with the structure induced by $\epsilon$. If $V$ is a $\mathbf{k}$-module, then $R \otimes V$ is an $R$-module, with the action $r'(r'' \otimes v) := r'r'' \otimes v$, for $r', r'' \in R$ and $v \in V$.

**Definition 3.7.** Let $V$ be a $\mathbf{k}$-vector space and $R$ a unital $\mathbf{k}$-ring. The *free $R$-module* generated by $V$ is an $R$-module $R\langle V\rangle$ together with a $\mathbf{k}$-linear map $\iota : V \to R\langle V\rangle$ with the property that for every $R$-module $W$ and a $\mathbf{k}$-linear map $V \xrightarrow{\varphi} W$, there exists a unique $R$-linear map $\Phi : R\langle V\rangle \to W$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & R\langle V\rangle \arrow[d, dashed, "\Phi"] \\ & W \end{tikzcd}$$

This universal property determines the free module $R\langle V\rangle$ uniquely up to isomorphism. A concrete model is provided by the $R$-module $R \otimes V$ recalled in Remark 3.6.

**Definition 3.8.** Let $W$ be an $R$-module. The *reduction* of $W$ is the $\mathbf{k}$-module $\overline{W} := \mathbf{k} \otimes_R W$, with the $\mathbf{k}$-action given by $k'(k'' \otimes_R w) := k'k'' \otimes_R w$, for $k', k'' \in \mathbf{k}$ and $w \in W$.

One clearly has $\mathbf{k}$-module isomorphisms $\overline{W} \cong W/\overline{R}W$ and $\overline{R\langle V\rangle} \cong V$. The reduction clearly defines a functor from the category of $R$-modules to the category of $\mathbf{k}$-modules.

**Proposition 3.9.** *If $B$ is an associative $R$-algebra, then the reduction $\overline{B}$ is a $\mathbf{k}$-algebra, with the structure induced by the algebra structure of $B$.*

**Proof.** Since $\overline{B} \simeq B/\overline{R}B$, it suffices to verify that $\overline{R}B$ is a two-sided ideal in $B$. But this is simple. For $r \in \overline{R}$, $b', b'' \in B$ one sees that $\mu(rb', b'') = r\mu(b', b'') \in \overline{R}B$, which shows that $\mu(\overline{R}B, B) \subset \overline{R}B$. The right multiplication by elements of $\overline{R}B$ is discussed similarly. $\square$

**Definition 3.10.** Let $A$ be an associative $\mathbf{k}$-algebra and $R$ an augmented unital ring. An *$R$-deformation* of $A$ is an associative $R$-algebra $B$ together with a $\mathbf{k}$-algebra isomorphism $\alpha : \overline{B} \to A$.

Two $R$-deformations $(B', \overline{B}' \xrightarrow{\alpha'} A)$ and $(B'', \overline{B}'' \xrightarrow{\alpha''} A)$ of $A$ are *equivalent* if there exists an $R$-algebra isomorphism $\phi : B' \to B''$ such that $\overline{\phi} = \alpha''^{-1} \circ \alpha'$.

There is probably not much to be said about $R$-deformations without additional assumptions on the $R$-module $B$. In this note we assume that $B$ *is a free $R$-module* or, equivalently, that

$$B \cong R \otimes A \text{ (isomorphism of } R\text{-modules).} \tag{4}$$

The above isomorphism identifies $A$ with the $\mathbf{k}$-linear subspace $1 \otimes A$ of $B$ and $A \otimes A$ with the $\mathbf{k}$-linear subspace $(1 \otimes A) \otimes (1 \otimes A)$ of $B \otimes B$.

Another assumption frequently used in algebraic geometry [16, Section III.§9] is that the $R$-module $B$ is flat which, by definition, means that the functor $B \otimes_R -$ is left exact. One then speaks about *flat deformations*.

In what follows, $R$ will be either a power series ring $\mathbf{k}[[t]]$ or a truncation of the polynomial ring $\mathbf{k}[t]$ by an ideal generated by a power of $t$. All these rings are local Noetherian rings therefore a finitely generated $R$-module is flat if and only if it is free (see Exercise 7.15, Corollary 10.16 and Corollary 10.27 of [2]). It is clear that $B$ in Definition 3.10 is finitely generated over $R$ if and only if $A$ finitely generated as a $\mathbf{k}$-vector space. Therefore, for $A$ finitely generated over $\mathbf{k}$, free deformations are the same as the flat ones.

The $R$-linearity of deformations implies the following simple lemma. Recall that all deformations in this sections satisfy (4).

**Lemma 3.11.** *Let $B = (B, \mu)$ be a deformation as in Definition 3.10. Then the multiplication $\mu$ in $B$ is determined by its restriction to $A \otimes A \subset B \otimes B$. Likewise, every equivalence of deformations $\phi : B' \to B''$ is determined by its restriction to $A \subset B$.*

**Proof.** By (4), each element of $B$ is a finite sum of elements of the form $ra$, $r \in R$ and $a \in A$, and $\mu(ra, sb) = rs\mu(a, b)$ by the $R$-bilinearity of $\mu$ for each $a, b \in A$ and $r, s \in R$. This proves the first statement. The second part of the lemma is equally obvious. □

The following proposition will also be useful.

**Proposition 3.12.** *Let $B' = (B', \overline{B}' \stackrel{\alpha'}{\to} A)$ and $B'' = (B'', \overline{B}'' \stackrel{\alpha''}{\to} A)$ be $R$-deformations of an associative algebra $A$. Assume that $R$ is either a local Artinian ring or a complete local ring. Then every homomorphism $\phi : B' \to B''$ of $R$-algebras such that $\overline{\phi} = \alpha''^{-1} \circ \alpha'$ is an equivalence of deformations.*

**Proof** (Sketch of proof)**.** We must show that $\phi$ is invertible. One may consider a formal inverse of $\phi$ in the form of an expansion in the successive quotients of the maximal ideal. If $R$ is Artinian, this formal inverse has in fact only finitely many terms and hence it is an actual inverse of $\phi$. If $R$ is complete, this formal expansion is convergent. □

We leave as an exercise to prove that each $R$-deformation of $A$ in the sense of Definition 3.10 is equivalent to a deformation of the form $(B, \overline{B} \stackrel{can}{\to} A)$, with $B = R \otimes A$ (equality of $\mathbf{k}$-vector spaces) and *can* the canonical map $\overline{B} = \mathbf{k} \otimes_R (R \otimes A) \to A$ given by

$$can(1 \otimes_R (1 \otimes a)) := a\,, \quad \text{for} \quad a \in A\,.$$

Two deformations $(B, \mu')$ and $(B, \mu'')$ of this type are equivalent if and only if there exists an $R$-algebra isomorphism $\phi : (B, \mu') \to (B, \mu'')$ which reduces, under the identification $can : \overline{B} \to A$, to the identity $\mathrm{id}_A : A \to A$. Since we will be interested only in equivalence classes of deformations, we will assume that all deformations are of the above special form.

**Definition 3.13.** A *formal deformation* is a deformation, in the sense of Definition 3.10, over the complete local augmented ring $\mathbf{k}[[t]]$.

**Exercise 3.14.** Is $\mathbf{k}[x,y,t]/(x^2+txy)$ a formal deformation of $\mathbf{k}[x,y]/(x^2)$?

**Theorem 3.15.** *A formal deformation $B$ of $A$ is given by a family*

$$\{\mu_i : A\otimes A \to A \mid i \in \mathbb{N}\}$$

*satisfying $\mu_0(a,b) = ab$ (the multiplication in $A$) and*

$(D_k)$ $\quad \sum_{i+j=k,\ i,j\geq 0} \mu_i(\mu_j(a,b),c) = \sum_{i+j=k,\ i,j\geq 0} \mu_i(a,\mu_j(b,c))\ \forall a,b,c\in A$

*for each $k \geq 1$.*

**Proof.** By Lemma 3.11, the multiplication $\mu$ in $B$ is determined by its restriction to $A\otimes A$. Now expand $\mu(a,b)$, for $a,b\in A$, into the power series

$$\mu(a,b) = \mu_0(a,b) + t\mu_1(a,b) + t^2\mu_2(a,b) + \cdots$$

for some $\mathbf{k}$-bilinear functions $\mu_i : A\otimes A \to A$, $i \geq 0$. Obviously, $\mu_0$ must be the multiplication in $A$. It is easy to see that $\mu$ is associative if and only if $(D_k)$ are satisfied for each $k \geq 1$. $\square$

**Remark 3.16.** Observe that $(D_1)$ reads

$$a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

and says precisely that $\mu_1 \in \mathit{Lin}(A^{\otimes 2}, A)$ is a Hochschild cocycle, $\delta_{\mathrm{Hoch}}(\mu_1) = 0$, see Definition 2.1.

**Example 3.17.** Let us denote by $H$ the group

$$H := \{u = \mathrm{id}_A + \phi_1 t + \phi_2 t^2 + \cdots \mid \phi_i \in \mathit{Lin}(A,A)\}\,,$$

with the multiplication induced by the composition of linear maps. By Proposition 3.12, formal deformations $\mu' = \mu_0 + \mu_1' t + \mu_2' t^2 + \cdots$ and $\mu'' = \mu_0 + \mu_1'' t + \mu_2'' t^2 + \cdots$ of $\mu_0$ are equivalent if and only if

$$u \circ (\mu_0 + \mu_1' t + \mu_2' t^2 + \cdots) = (\mu_0 + \mu_1'' t + \mu_2'' t^2 + \cdots) \circ (u\otimes u)\,. \tag{5}$$

We close this section by formulating some classical statements [12, 13, 14] which reveal the connection between deformation theory of associative algebras and the Hochschild cohomology. As suggested by Remark 3.16, the first natural object to look at is $\mu_1$. This motivates the following

**Definition 3.18.** An *infinitesimal deformation* of an algebra $A$ is a $D$-deformation of $A$, where

$$D := \mathbf{k}[t]/(t^2)$$

is the local Artinian ring of *dual numbers*.

**Remark 3.19.** One can easily prove an analog of Theorem 3.15 for infinitesimal deformations, namely that there is a one-to-one correspondence between infinitesimal deformations of $A$ and $\mathbf{k}$-linear maps $\mu_1 : A\otimes A \to A$ satisfying $(D_1)$, that is, by Remark 3.16, Hochschild 2-cocycles of $A$ with coefficients in itself. But we can formulate a stronger statement:

**Theorem 3.20.** *There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of $A$ and the second Hochschild cohomology $H^2_{\mathrm{Hoch}}(A,A)$ of $A$ with coefficients in itself.*

**Proof.** Consider two infinitesimal deformations of $A$ given by multiplications $*'$ and $*''$, respectively. As we observed in Remark 3.19, these deformations are determined by Hochschild 2-cocycles $\mu_1', \mu_1'' : A \otimes A \to A$, via equations

$$
\begin{aligned}
a *' b &= ab + t\mu_1'(a,b) \\
a *'' b &= ab + t\mu_1''(a,b)\,, \quad a, b \in A\,.
\end{aligned}
\tag{6}
$$

Each equivalence $\phi$ of deformations $*'$ and $*''$ is determined by a $\mathbf{k}$-linear map $\phi_1 : A \to A$,

$$
\phi(a) = a + t\phi_1(a)\,, \quad a \in A\,,
\tag{7}
$$

the invertibility of such a $\phi$ follows from Proposition 3.12 but can easily be checked directly. Substituting (6) and (7) into

$$
\phi(a *' b) = \phi(a) *'' \phi(b)\,, \quad a, b \in A\,,
\tag{8}
$$

one obtains

$$
\phi(ab + t\mu_1'(a,b)) = (a + t\phi_1(a)) *'' (b + t\phi_1(b))
$$

which can be further expanded into

$$
ab + t\phi(\mu_1'(a,b)) = ab + t(a\phi_1(b)) + t(\phi_1(a)b) + t\mu_1''(a + t\phi_1(a), b + t\phi_1(b))
$$

so, finally,

$$
ab + t\mu_1'(a,b) = ab + t(a\phi_1(b) + \phi_1(a)b) + t\mu_1''(a,b)\,.
$$

Comparing the $t$-linear terms, we see that (8) is equivalent to

$$
\mu_1'(a,b) = \delta_{\mathrm{Hoch}}\phi_1(a,b) + \mu_1''(a,b)\,.
$$

We conclude that infinitesimal deformations given by $\mu_1', \mu_1'' \in C^2_{\mathrm{Hoch}}(A,A)$ are equivalent if and only if they differ by a coboundary, that is, if and only if $[\mu_1'] = [\mu_1'']$ in $H^2_{\mathrm{Hoch}}(A,A)$. □

Another classical result is:

**Theorem 3.21.** *Let $A$ be an associative algebra such that $H^2_{\mathrm{Hoch}}(A,A) = 0$. Then all formal deformations of $A$ are equivalent.*

**Proof** (Sketch of proof)**.** If $*', *''$ are two formal deformations of $A$, one can, using the assumption $H^2_{\mathrm{Hoch}}(A,A) = 0$, as in the proof of Theorem 3.20 find a $\mathbf{k}$-linear map $\phi_1 : A \to A$ defining an equivalence of $(B, *')$ to $(B, *'')$ modulo $t^2$. Repeating this process, one ends up with an equivalence $\phi = \mathrm{id} + t\phi_1 + t^2\phi_2 + \cdots$ of formal deformations $*'$ and $*''$. □

**Definition 3.22.** An *$n$-deformation* of an algebra $A$ is an $R$-deformation of $A$ for $R$ the local Artinian ring $\mathbf{k}[t]/(t^{n+1})$.

We have the following version of Theorem 3.15 whose proof is obvious.

**Theorem 3.23.** *An $n$-deformation of $A$ is given by a family*

$$\{\mu_i : A \otimes A \to A \mid 1 \leq i \leq n\}$$

*of* $\mathbf{k}$*-linear maps satisfying* $(D_k)$ *of Theorem 3.15 for* $1 \leq k \leq n$.

**Definition 3.24.** An $(n+1)$-deformation of $A$ given by $\{\mu_1, \ldots, \mu_{n+1}\}$ is called an *extension* of the $n$-deformation given by $\{\mu_1, \ldots, \mu_n\}$.

Let us rearrange $(D_{n+1})$ into

$$\begin{aligned} -a\mu_{n+1}(b,c) + \mu_{n+1}(ab,c) - \mu_{n+1}(a,bc) + \mu_{n+1}(a,b)c \\ = \sum_{i+j=n+1,\ i,j>0} \big(\mu_i(a, \mu_j(b,c)) - \mu_i(\mu_j(a,b),c)\big) \end{aligned}$$

Denote the trilinear function in the right-hand side by $\mathfrak{O}_n$ and interpret it as an element of $C^3_{\mathrm{Hoch}}(A,A)$,

$$\mathfrak{O}_n := \sum_{i+j=n+1,\ i,j>0} (\mu_i(a, \mu_j(b,c)) - \mu_i(\mu_j(a,b),c)) \in C^3_{\mathrm{Hoch}}(A,A)\,. \tag{9}$$

Using the Hochschild differential recalled in Definition 2.1, one can rewrite $(D_{n+1})$ as

$$\delta_{\mathrm{Hoch}}(\mu_{n+1}) = \mathfrak{O}_n\,.$$

We conclude that, if an $n$-deformation extends to an $(n+1)$-deformation, then $\mathfrak{O}_n$ is a Hochschild coboundary. In fact, one can prove:

**Theorem 3.25.** *For any $n$-deformation, the cochain $\mathfrak{O}_n \in C^3_{\mathrm{Hoch}}(A,A)$ defined in* (9) *is a cocycle, $\delta_{\mathrm{Hoch}}(\mathfrak{O}_n) = 0$. Moreover, $[\mathfrak{O}_n] = 0$ in $H^3_{\mathrm{Hoch}}(A,A)$ if and only if the $n$-deformation $\{\mu_1, \ldots, \mu_n\}$ extends into some $(n+1)$-deformation.*

**Proof.** Straightforward. □

**Geometric deformation theory.** Let us turn our attention back to the variety of structure constants $Ass(V)$ recalled in Section 2, page 340. Elements of $Ass(V)$ are associative $\mathbf{k}$-linear multiplications $\cdot : V \otimes V \to V$ and there is a natural left action $\cdot \mapsto \cdot_\phi$ of $GL(V)$ on $Ass(V)$ given by

$$a \cdot_\phi b := \phi\big(\phi^{-1}(a) \cdot \phi^{-1}(b)\big)\,, \tag{10}$$

for $a, b \in V$ and $\phi \in GL(V)$. We assume that $V$ is finite dimensional.

**Definition 3.26.** Let $A$ be an algebra with the underlying vector space $V$ interpreted as a point in the variety of structure constants, $A \in Ass(V)$. The algebra $A$ is called (geometrically) *rigid* if the $GL(V)$-orbit of $A$ in $Ass(V)$ is Zarisky-open.

The following classical statement whose proof can be found in [36, § 5] specifies the relation between the Hochschild cohomology and geometric rigidity, compare also Propositions 1 and 2 of [8].

**Theorem 3.27.** *Suppose that the ground field is algebraically closed.*
(i) *If $H^2_{\mathrm{Hoch}}(A,A) = 0$ then $A$ is rigid, and*
(ii) *if $H^3_{\mathrm{Hoch}}(A,A) = 0$ then $A$ is rigid if and only if $H^2_{\mathrm{Hoch}}(A,A) = 0$.*

**Remark 3.28.** An analysis parallel to the one presented in this section can be made for any class of "reasonable" algebras, where "reasonable" are algebras over quadratic Koszul operads [35, Section II.3.3] for which the deformation cohomology is given by a "standard construction." Let us emphasize that most of "classical" types of algebras (Lie, associative, associative commutative, Poisson, etc.) are "reasonable." See also [3, 4].

## 4. Structures of (co)associative (co)algebras

Let $V$ be a $\mathbf{k}$-vector space. In this section we recall, in Theorems 4.16 and 4.21, the following important correspondence between (co)algebras and differentials:

$$\{\text{coassociative coalgebra structures on the vector space } V\}$$
$$\updownarrow$$
$$\{\text{quadratic differentials on the free associative algebra generated by } V\}$$

and its dual version:

$$\{\text{associative algebras on the vector space } V\}$$
$$\updownarrow$$
$$\{\text{quadratic differentials on the "cofree" coassociative coalgebra cogenerated by } V\}.$$

The reason why we put 'cofree' into parentheses will become clear later in this section. Similar correspondences exist for any "reasonable" (in the sense explained in Remark 3.28) class of algebras, see [11, Theorem 8.2]. We will in fact need only the second correspondence but, since it relies on coderivations of "cofree" coalgebras, we decided to start with the first one which is simpler to explain.

**Definition 4.1.** The *free associative algebra* generated by a vector space $W$ is an associative algebra $\mathcal{A}(W) \in \mathsf{Ass}$ together with a linear map $W \to \mathcal{A}(W)$ having the following property:

For every $A \in \mathsf{Ass}$ and a linear map $W \xrightarrow{\varphi} A$, there exists a unique algebra homomorphism $\mathcal{A}(W) \to A$ making the diagram:

$$\begin{tikzcd} W \arrow[r] \arrow[dr, "\varphi"'] & \mathcal{A}(W) \arrow[d, dashed] \\ & A \end{tikzcd}$$

commutative.

The free associative algebra on $W$ is uniquely determined up to isomorphism. An example is provided by the *tensor algebra* $T(W) := \bigoplus_{n=1}^{\infty} W^{\otimes n}$ with the inclusion $W = W^{\otimes 1} \hookrightarrow T(W)$. There is a natural grading on $T(W)$ given by the number of tensor factors,

$$T(W) = \bigoplus\nolimits_{n=0}^{\infty} T^n(W)\,,$$

where $T^n(W) := W^{\otimes n}$ for $n \geq 1$ and $T^0(W) := 0$. Let us emphasize that the tensor algebra as defined above is *nonunital*, the unital version can be obtained by taking $T^0(W) := \mathbf{k}$.

**Convention 4.2.** We are going to consider graded algebraic objects. Our choice of signs will be dictated by the principle that whenever we commute two "things" of degrees $p$ and $q$, respectively, we multiply the sign by $(-1)^{pq}$. This rule is sometimes called the *Koszul sign convention.* As usual, non-graded (classical) objects will be, when necessary, considered as graded ones concentrated in degree 0.

Let $f' : V' \to W'$ and $f'' : V'' \to W''$ be homogeneous maps of graded vector spaces. The Koszul sign convention implies that the value of $(f' \otimes f'')$ on the product $v' \otimes v'' \in V' \otimes V''$ of homogeneous elements equals

$$(f' \otimes f'')(v' \otimes v'') := (-1)^{\deg(f'')\deg(v')} f'(v') \otimes f''(v'') .$$

In fact, the Koszul sign convention is determined by the above rule for evaluation.

**Definition 4.3.** Assume $V = V^*$ is a graded vector space, $V = \bigoplus_{i\in\mathbb{Z}} V^i$. The *suspension operator* $\uparrow$ assigns to $V$ the graded vector space $\uparrow V$ with $\mathbb{Z}$-grading $(\uparrow V)^i := V^{i-1}$. There is a natural degree $+1$ map $\uparrow\colon V \to \uparrow V$ that sends $v \in V$ into its suspended copy $\uparrow v \in \uparrow V$. Likewise, the *desuspension operator* $\downarrow$ changes the grading of $V$ according to the rule $(\downarrow V)^i := V^{i+1}$. The corresponding degree $-1$ map $\downarrow\colon V \to \downarrow V$ is defined in the obvious way. The suspension (resp. the desuspension) of $V$ is sometimes also denoted $sV$ or $V[-1]$ (resp. $s^{-1}V$ or $V[1]$).

**Example 4.4.** If $V$ is an un-graded vector space, then $\uparrow V$ is $V$ placed in degree $+1$ and $\downarrow V$ is $V$ placed in degree $-1$.

**Remark 4.5.** In the "superworld" of $\mathbb{Z}_2$-graded objects, the operators $\uparrow$ and $\downarrow$ agree and coincide with the *parity change operator*.

**Exercise 4.6.** Show that the Koszul sign convention implies $(\downarrow \otimes \downarrow) \circ (\uparrow \otimes \uparrow) = -\,\mathrm{id}$ or, more generally,

$$\downarrow^{\otimes n} \circ \uparrow^{\otimes n} = \uparrow^{\otimes n} \circ \downarrow^{\otimes n} = (-1)^{\frac{n(n-1)}{2}}\,\mathrm{id}$$

for an arbitrary $n \geq 1$.

**Definition 4.7.** A *derivation* of an associative algebra $A$ is a linear map $\theta : A \to A$ satisfying the *Leibniz rule*

$$\theta(ab) = \theta(a)b + a\theta(b)$$

for every $a, b \in A$. Denote $Der(A)$ the set of all derivations of $A$.

We will in fact need a graded version of the above definition:

**Definition 4.8.** A *degree $d$ derivation* of a $\mathbb{Z}$-graded algebra $A$ is a degree $d$ linear map $\theta : A \to A$ satisfying the graded Leibniz rule

$$\theta(ab) = \theta(a)b + (-1)^{d|a|} a\theta(b) \tag{11}$$

for every homogeneous element $a \in A$ of degree $|a|$ and for every $b \in A$. We denote $Der^d(A)$ the set of all degree $d$ derivations of $A$.

**Exercise 4.9.** Let $\mu : A \otimes A \to A$ be the multiplication of $A$. Prove that (11) is equivalent to

$$\theta\mu = \mu(\theta \otimes \mathrm{id}) + \mu(\mathrm{id} \otimes \theta)\,.$$

Observe namely how the signs in the right hand side of (11) are dictated by the Koszul convention.

**Proposition 4.10.** *Let $W$ be a graded vector space and $T(W)$ the tensor algebra generated by $W$ with the induced grading. For any $d$, there is a natural isomorphism*

$$Der^d\big(T(W)\big) \cong Lin^d\big(W, T(W)\big)\,, \tag{12}$$

*where $Lin^d(-,-)$ denotes the space of degree $d$ $\mathbf{k}$-linear maps.*

**Proof.** Let $\theta \in Der^d\big(T(W)\big)$ and $f := \theta|_W : W \to T(W)$. The Leibniz rule (11) implies that, for homogeneous elements $w_i \in W$, $1 \le i \le n$,

$$\begin{aligned}\theta(w_1 \otimes \cdots \otimes w_n) &= f(w_1) \otimes w_2 \otimes \cdots \otimes w_n + (-1)^{d|w_1|} w_1 \otimes f(w_2) \otimes \cdots \otimes w_n + \cdots \\ &= \sum_{i=1}^{n} (-1)^{d(|w_1|+\cdots+|w_{i-1}|)} w_1 \otimes \cdots \otimes f(w_i) \otimes \cdots \otimes w_n\end{aligned}$$

which reveals that $\theta$ is determined by its restriction $f$ on $W$. On the other hand, given a degree $d$ linear map $f : W \to T(W)$, the above formula clearly defines a derivation $\theta \in Der^d\big(T(W)\big)$. The correspondence

$$Der^d\big(T(W)\big) \ni \theta \longleftrightarrow f := \theta|_W \in Lin^d\big(W, T(W)\big)$$

is the required isomorphism (12). □

**Exercise 4.11.** Let $\theta \in Der^d\big(T(W)\big)$, $f := \theta|_V$ and $x \in T^2(W)$. Prove that

$$\theta(x) = (f \otimes \mathrm{id} + \mathrm{id} \otimes f)(x)\,.$$

**Definition 4.12.** A derivation $\theta \in Der^d\big(T(W)\big)$ is called *quadratic* if $\theta(W) \subset T^2W$. A degree 1 derivation $\theta$ is a *differential* if $\theta^2 = 0$.

**Exercise 4.13.** Prove that the isomorphism of Proposition 4.10 restricts to

$$Der^d_2\big(T(W)\big) \cong Lin^d\big(W, T^2(W)\big)\,,$$

where $Der^d_2\big(T(W)\big)$ is the space of all quadratic degree $d$ derivations of $T(W)$.

**Definition 4.14.** Let $V$ be a vector space. A *coassociative coalgebra* structure on $V$ is given by a linear map $\Delta : V \to V \otimes V$ satisfying

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

(the *coassociativity*).

We will need, in Section 6, also a cocommutative version of coalgebras:

**Definition 4.15.** A coassociative coalgebra $A = (V, \Delta)$ as in Definition 4.14 is *cocommutative* if

$$T\Delta = \Delta$$

with the swapping map $T : V \otimes V \to V \otimes V$ given by

$$T(v' \otimes v'') := (-1)^{|v'||v''|} v'' \otimes v'$$

for homogeneous $v', v'' \in V$.

**Theorem 4.16.** *Let $V$ be a (possibly graded) vector space. Denote $\mathit{Coass}(V)$ the set of all coassociative coalgebra structures on $V$ and $\mathit{Diff}^1_2(T(\uparrow V))$ the set of all quadratic differentials on the tensor algebra $T(\uparrow V)$. Then there is a natural isomorphism*

$$\mathit{Coass}(V) \cong \mathit{Diff}^1_2\big(T(\uparrow V)\big).$$

**Proof.** Let $\chi \in \mathit{Diff}^1_2(T(\uparrow V))$. Put $f := \chi|_{\uparrow V}$ so that $f$ is a degree $+1$ map $\uparrow V \to \uparrow V \otimes \uparrow V$. By Exercise 4.11 (with $W := \uparrow V$, $\theta := \chi$ and $x := f(\uparrow v)$),

$$0 = \chi^2(\uparrow v) = \chi(f(\uparrow v)) = (f \otimes \mathrm{id} + \mathrm{id} \otimes f)(f(\uparrow v))$$

for every $v \in V$, therefore

$$(f \otimes \mathrm{id} + \mathrm{id} \otimes f) f = 0\,. \tag{13}$$

We have clearly described a one-to-one correspondence between quadratic differentials $\chi \in \mathit{Diff}^1_2(T(\uparrow V))$ and degree $+1$ linear maps $f \in \mathit{Lin}^1(\uparrow V, T^2(\uparrow V))$ satisfying (13).

Given $f : \uparrow V \to \uparrow V \otimes \uparrow V$ as above, define the map $\Delta : V \to V \otimes V$ by the commutative diagram

$$\begin{tikzcd} \uparrow V \arrow[r, "f"] & \uparrow V \otimes \uparrow V \\ V \arrow[u, "\uparrow"] \arrow[r, "\Delta"] & V \otimes V \arrow[u, "\uparrow \otimes \uparrow"'] \end{tikzcd}$$

i.e., by Exercise 4.6,

$$\Delta := (\uparrow \otimes \uparrow)^{-1} \circ f \circ \uparrow = -(\downarrow \otimes \downarrow) \circ f \circ \uparrow\,.$$

Let us show that (13) is equivalent to the coassociativity of $\Delta$. We have

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta &= (-(\downarrow \otimes \downarrow) f \uparrow \otimes \mathrm{id})\,(-(\downarrow \otimes \downarrow) f \uparrow) = ((\downarrow \otimes \downarrow) f \uparrow \otimes \mathrm{id})\,(\downarrow \otimes \downarrow) f \uparrow \\ &= ((\downarrow \otimes \downarrow) f \otimes \downarrow) f \uparrow = -(\downarrow \otimes \downarrow \otimes \downarrow)(f \otimes \mathrm{id}) f \uparrow\,.\end{aligned}$$

The minus sign in the last term appeared because we interchanged $f$ (a “thing” of degree $+1$) with $\downarrow$ (a “thing” of degree $-1$). Similarly

$$\begin{aligned}(\mathrm{id} \otimes \Delta)\Delta &= (\mathrm{id} \otimes (-(\downarrow \otimes \downarrow)) f \uparrow)\,(-(\downarrow \otimes \downarrow) f \uparrow) = (\mathrm{id} \otimes (\downarrow \otimes \downarrow) f \uparrow)\,(\downarrow \otimes \downarrow) f \uparrow \\ &= (\downarrow \otimes (\downarrow \otimes \downarrow) f) f \uparrow = (\downarrow \otimes \downarrow \otimes \downarrow)(\mathrm{id} \otimes f) f \uparrow,\end{aligned}$$

so (13) is indeed equivalent to $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. This finishes the proof. $\square$

We are going to dualize Theorem 4.16 to get a description of associative algebras, not *co*algebras. First, we need a dual version of the tensor algebra:

**Definition 4.17.** The underlying vector space $T(W)$ of the tensor algebra with the comultiplication $\Delta : T(W) \to T(W) \otimes T(W)$ defined by

$$\Delta(w_1 \otimes \ldots \otimes w_n) := \sum_{i=1}^{n-1} (w_1 \otimes \ldots \otimes w_i) \otimes (w_{i+1} \otimes \ldots \otimes w_n)$$

is a coassociative coalgebra denoted ${}^cT(W)$ and called the *tensor coalgebra*.

**Warning.** Contrary to general belief, the coalgebra ${}^cT(W)$ with the projection ${}^cT(W) \to W$ is not cofree in the category of coassociative coalgebras! Cofree coalgebras (in the sense of the obvious dual of Definition 4.1) are surprisingly complicated objects [9, 40, 17]. The coalgebra ${}^cT(W)$ is, however, cofree in the subcategory of coaugmented nilpotent coalgebras [35, Section II.3.7]. This will be enough for our purposes.

In the following dual version of Definition 4.8 we use *Sweedler's convention* expressing the comultiplication in a coalgebra $C$ as $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, $c \in C$.

**Definition 4.18.** A *degree d coderivation* of a $\mathbb{Z}$-graded coalgebra $C$ is a linear degree $d$ map $\theta : C \to C$ satisfying the dual Leibniz rule

$$\Delta\theta(c) = \sum \theta(c_{(1)}) \otimes c_{(2)} + \sum (-1)^{d|c_{(1)}|} c_{(1)} \otimes \theta(c_{(2)})\,, \tag{14}$$

for every $c \in C$. Denote the set of all degree $d$ coderivations of $C$ by $\mathit{CoDer}^d(C)$.

As in Exercise 4.9 one easily proves that (14) is equivalent to

$$\Delta\theta = (\theta \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes \theta)\Delta.$$

Let us prove the dual of Proposition 4.10:

**Proposition 4.19.** *Let $W$ be a graded vector space. For any $d$, there is a natural isomorphism*

$$\mathit{CoDer}^d({}^cT(W)) \cong \mathit{Lin}^d(T(W), W). \tag{15}$$

**Proof.** For $\theta \in \mathit{CoDer}^d(T(W))$ and $s \geq 1$ denote $f_s \in \mathit{Lin}^d(T^s(W), W)$ the composition

$$f_s : T^s(W) \xrightarrow{\theta|_{T^s(W)}} {}^cT(W) \xrightarrow{\text{proj.}} W\,. \tag{16}$$

The dual Leibniz rule (14) implies that, for $w_1, \ldots, w_n \in W$ and $n \geq 0$,

$$\theta(w_1 \otimes \cdots \otimes w_n) := \sum_{s \geq 1} \sum_{i=1}^{n-s+1} (-1)^{d(|w_1| + \cdots + |w_{i-1}|)} w_1 \otimes \cdots \otimes f_s(w_i \otimes \cdots \otimes w_{i+s-1}) \otimes \cdots \otimes w_n\,,$$

which shows that $\theta$ is determined by $f := f_0 + f_1 + f_2 + \cdots \in \mathit{Lin}^d(T(W), W)$. On the other hand, it is easy to verify that for any map $f \in \mathit{Lin}^d(T(W), W)$ decomposed into the sum of its homogeneous components, the above formula defines a coderivation $\theta \in \mathit{CoDer}^d(T(W))$. This finishes the proof. $\square$

**Definition 4.20.** The composition $f_s : T^s(W) \to W$ defined in (16) is called the $s$th *corestriction* of the coderivation $\theta$. A coderivation $\theta \in CoDer^d(T(W))$ is *quadratic* if its $s$th corestriction is non-zero only for $s = 2$. A degree 1 coderivation $\theta$ is a *differential* if $\theta^2 = 0$.

Let us finally formulate a dual version of Theorem 4.16.

**Theorem 4.21.** *Let $V$ be a graded vector space. Denote $CoDiff^1_2({}^cT(\downarrow V))$ the set of all quadratic differentials on the tensor coalgebra ${}^cT(\downarrow V)$. One then has a natural isomorphism*

$$Ass(V) \cong CoDiff^1_2\left({}^cT(\downarrow V)\right). \tag{17}$$

**Proof.** Let $\chi \in CoDiff^1_2({}^cT(\downarrow V))$ and $f : \downarrow V \otimes \downarrow V \to \downarrow V$ be the 2nd corestriction of $\chi$. Define $\mu : V \otimes V \to V$ by the diagram

$$\begin{tikzcd} \downarrow V \otimes \downarrow V \arrow[r, "f"] & \downarrow V \\ V \otimes V \arrow[u, "\downarrow \otimes \downarrow"] \arrow[r, "\mu"] & V \arrow[u, "\downarrow"'] \end{tikzcd}$$

The correspondence $\chi \leftrightarrow \mu$ is then the required isomorphism. This can be verified by dualizing the steps of the proof of Theorem 4.16 so we can safely leave the details to the reader. $\square$

## 5. dg-Lie algebras and the Maurer-Cartan equation

**Definition 5.1.** A graded Lie algebra is a $\mathbb{Z}$-graded vector space

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}^n$$

equipped with a degree 0 bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (the *bracket*) which is graded antisymmetric, i.e.

$$[a,b] = -(-1)^{|a||b|}[b,a] \tag{18}$$

for all homogeneous $a, b \in \mathfrak{g}$, and satisfies the graded Jacobi identity:

$$\big[a,[b,c]\big] + (-1)^{|a|(|b|+|c|)}\big[b,[c,a]\big] + (-1)^{|c|(|a|+|b|)}\big[c,[a,b]\big] = 0 \tag{19}$$

for all homogeneous $a, b, c \in \mathfrak{g}$.

**Exercise 5.2.** Write the axioms of graded Lie algebras in an element-free form that would use only the bilinear map $l := [-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and its iterated compositions, and the operator of "permuting the inputs" of a multilinear map. Observe how the Koszul sign convention helps remembering the signs in (18) and (19).

**Definition 5.3.** A *dg-Lie algebra* (an abbreviation for *differential graded Lie algebra*) is a graded Lie algebra $L = \bigoplus_{n \in \mathbb{Z}} L^n$ as in Definition 5.1 together with a degree 1 linear map $d : L \to L$ which is

– a degree 1 derivation , i.e. $d[a,b] = [da,b] + (-1)^{|a|}[a,db]$ for homogeneous $a, b \in L$, and
– a differential, i.e. $d^2 = 0$.

Our next aim is to show that the Hochschild complex $(C^*_{\mathrm{Hoch}}(A,A), \delta_{\mathrm{Hoch}})$ of an associative algebra recalled in Definition 2.1 has a natural bracket which turns it into a dg-Lie algebra. We start with some preparatory material.

**Proposition 5.4.** *Let $C$ be a graded coalgebra. For coderivations $\theta, \phi \in CoDer(C)$ define*

$$[\theta, \phi] := \theta \circ \phi - (-1)^{|\theta||\phi|}\phi \circ \theta\,.$$

*The bracket $[-,-]$ makes $CoDer(C) = \bigoplus_{n\in\mathbb{Z}} CoDer^n(\mathcal{C})$ a graded Lie algebra.*

**Proof.** The key observation is that $[\theta, \phi]$ is a coderivation (note that *neither* $\theta \circ \phi$ *nor* $\phi \circ \theta$ are coderivations!). Verifying this and the properties of a graded Lie bracket is straightforward and will be omitted. □

**Proposition 5.5.** *Let $C$ be a graded coalgebra and $\chi \in CoDer^1(\mathcal{C})$ such that*

$$[\chi, \chi] = 0\,, \tag{20}$$

*where $[-,-]$ is the bracket of Proposition 5.4. Then*

$$d(\theta) := [\chi, \theta] \qquad \text{for } \theta \in CoDer(\mathcal{C})$$

*is a differential that makes $CoDer(\mathcal{C})$ a dg-Lie algebra.*

Observe that, since $|\chi| = 1$, (20) does not tautologically follow from the graded antisymmetry (18).

**Proof** (Proof of Proposition 5.5)**.** The graded Jacobi identity (19) implies that, for each homogeneous $\theta$,

$$\big[\chi, [\chi, \theta]\big] = -(-1)^{|\theta|+1}\big[\chi, [\theta, \chi]\big] - \big[\theta, [\chi, \chi]\big]\,.$$

Now use the graded antisymmetry $[\theta, \chi] = (-1)^{|\theta|+1}[\chi, \theta]$ and the assumption $[\chi, \chi] = 0$ to conclude from the above display that

$$\big[\chi, [\chi, \theta]\big] = -\big[\chi, [\chi, \theta]\big]\,,$$

therefore, since the characteristic of the ground field is zero,

$$d^2(\theta) = \big[\chi, [\chi, \theta]\big] = 0\,,$$

so $d$ is a differential. The derivation property of $d$ with respect to the bracket can be verified in the same way and we leave it as an exercise to the reader. □

In Proposition 5.5 we saw that coderivations of a graded coalgebra form a dg-Lie algebra. Another example of a dg-Lie algebra is provided by the Hochschild cochains of an associative algebra (see Definition 2.1). We need the following:

**Definition 5.6.** For $f \in Lin(V^{\otimes(m+1)}, V)$, $g \in Lin(V^{\otimes(n+1)}, V)$ and $1 \le i \le m+1$ define $f \circ_i g \in Lin(V^{\otimes(m+n+1)}, V)$ by

$$(f \circ_i g) := f\big(\,\mathrm{id}_V^{\otimes(i-1)} \otimes g \otimes \mathrm{id}_V^{\otimes(m-i+1)}\,\big)\,.$$

Define also

$$f \circ g := \sum_{i=1}^{m+1} (-1)^{n(i+1)} f \circ_i g$$

and, finally,

$$[f, g] := f \circ g - (-1)^{mn} g \circ f\,.$$

The operation $[-,-]$ is called the *Gerstenhaber bracket* (our definition however differs from the original one of [12] by the overall sign $(-1)^n$).

Let $A$ be an associative algebra with the underlying space $V$. Since, by Definition 2.1, $C^{*+1}_{\mathrm{Hoch}}(A,A) = Lin(V^{\otimes(*+1)}, V)$, the structure of Definition 5.6 defines a degree 0 operation $[-,-] : C^{*+1}_{\mathrm{Hoch}}(A,A) \otimes C^{*+1}_{\mathrm{Hoch}}(A,A) \to C^{*+1}_{\mathrm{Hoch}}(A,A)$ called again the Gerstenhaber bracket. We leave as an exercise the proof of

**Proposition 5.7.** *The Hochschild cochain complex of an associative algebra with the Gerstenhaber bracket form a dg-Lie algebra*

$$C^{*+1}_{\mathrm{Hoch}}(A,A) = (C^{*+1}_{\mathrm{Hoch}}(A,A), [-,-], \delta_{\mathrm{Hoch}})\,.$$

The following theorem gives an alternative description of the dg-Lie algebra of Proposition 5.7.

**Theorem 5.8.** *Let $A$ be an associative algebra with multiplication $\mu : V \otimes V \to V$ and $\chi \in CoDiff^{\,1}_2({}^cT(\downarrow V))$ the coderivation that corresponds to $\mu$ in the correspondence of Theorem 4.21. Let $d := [\chi, -]$ be the differential introduced in Proposition 5.5. Then there is a natural isomorphism of dg-Lie algebras*

$$\xi : \big(C^{(*+1)}_{\mathrm{Hoch}}(A,A), [-,-], \delta_{\mathrm{Hoch}}\big) \xrightarrow{\cong} \big(\, CoDer^*({}^cT(\downarrow V)), [-,-], d\big)\,.$$

**Proof.** Given $\phi \in C^{n+1}_{\mathrm{Hoch}}(A,A) = Lin(V^{\otimes(n+1)}, V)$, let $f : (\downarrow V)^{\otimes(n+1)} \to \downarrow V$ be the degree $n$ linear map defined by the diagram

$$\begin{tikzcd}
(\downarrow V)^{\otimes(n+1)} \arrow[r, "f"] & \downarrow V \\
V^{\otimes(n+1)} \arrow[u, "\downarrow^{\otimes(n+1)}"] \arrow[r, "\phi"] & V \arrow[u, "\downarrow"']
\end{tikzcd}$$

By Proposition 4.19, there exists a unique coderivation $\theta \in CoDer^n({}^cT(\downarrow V))$ whose $(n+1)$th corestriction is $f$ and other corestrictions are trivial.

The map $\xi : C^{(*+1)}_{\mathrm{Hoch}}(A,A) \to CoDer^*({}^cT(\downarrow V))$ defined by $\xi(\phi) := \theta$ is clearly an isomorphism. The verification that $\xi$ commutes with the differentials and brackets is a straightforward though involved exercise on the Koszul sign convention which we leave to the reader. $\square$

**Corollary 5.9.** *Let $\mu$ be the multiplication in $A$ interpreted as an element of $C^2_{\mathrm{Hoch}}(A,A)$, and $f \in C^*_{\mathrm{Hoch}}(A,A)$. Then $\delta_{\mathrm{Hoch}}(f) = [\mu, f]$.*

**Proof.** The corollary immediately follows from Theorem 5.8. Indeed, because $\xi$ commutes with all the structures, we have

$$\delta_{\mathrm{Hoch}}(f) = \xi^{-1}\xi\delta_{\mathrm{Hoch}}(f) = \xi^{-1}(d(\xi f)) = \xi^{-1}[\chi, \xi f] = [\mu, f]\,.$$

We however recommend as an exercise to verify the corollary directly, comparing $[\mu, f]$ to the formula for the Hochschild differential. □

**Proposition 5.10.** *A bilinear map $\kappa : V \otimes V \to V$ defines an associative algebra structure on $V$ if and only if $[\kappa, \kappa] = 0$.*

**Proof.** By Definition 5.6 (with $m = n = 1$),

$$\frac{1}{2}[\kappa,\kappa] = \frac{1}{2}\Big(\kappa \circ \kappa - (-1)^{mn}\kappa \circ \kappa\Big) = \kappa\circ\kappa = \kappa\circ_1\kappa - \kappa\circ_2\kappa = \kappa(\kappa\otimes \mathrm{id}_V) - \kappa(\mathrm{id}_V \otimes\kappa)\,,$$

therefore $[\kappa, \kappa] = 0$ is indeed equivalent to the associativity of $\kappa$. □

**Proposition 5.11.** *Let $A$ be an associative algebra with the underlying vector space $V$ and the multiplication $\mu : V \otimes V \to V$. Let $\nu \in C^2_{\mathrm{Hoch}}(A, A)$ be a Hochschild 2-cochain. Then $\mu + \nu \in C^2_{\mathrm{Hoch}}(A, A) = Lin(V^{\otimes 2}, V)$ is associative if and only if*

$$\delta_{\mathrm{Hoch}}(\nu) + \frac{1}{2}[\nu, \nu] = 0\,. \tag{21}$$

**Proof.** By Proposition 5.10, $\mu + \nu$ is associative if and only if

$$0 = \frac{1}{2}[\mu+\nu, \mu+\nu] = \frac{1}{2}\big\{[\mu,\mu] + [\nu,\nu] + [\mu,\nu] + [\nu,\mu]\big\} = \delta_{\mathrm{Hoch}}(\nu) + \frac{1}{2}[\nu,\nu]\,.$$

To get the rightmost term, we used the fact that, since $\mu$ is associative, $[\mu, \mu] = 0$ by Proposition 5.10. We also observed that $[\mu, \nu] = [\nu, \mu] = \delta_{\mathrm{Hoch}}(\nu)$ by Corollary 5.9. □

A bilinear map $\nu : V \otimes V \to V$ such that $\mu + \nu$ is associative can be viewed as a deformation of $\mu$. This suggests that (21) is related to deformations. This is indeed the case, as we will see later in this section. Equation (21) is a particular case of the Maurer-Cartan equation in a arbitrary dg-Lie algebra:

**Definition 5.12.** Let $L = (L, [-,-], d)$ be a dg-Lie algebra. A degree 1 element $s \in L^1$ is *Maurer-Cartan* if it satisfies the *Maurer-Cartan equation*

$$ds + \frac{1}{2}[s, s] = 0\,. \tag{22}$$

**Remark 5.13.** The Maurer-Cartan equation (also called the *Berikashvili equation*) along with its clones and generalizations is one of the most important equations in mathematics. For instance, a version of the Maurer-Cartan equation describes the differential of a left-invariant form, see [22, I.§4].

Let $\mathfrak{g}$ be a dg-Lie algebra over the ground field $\mathbf{k}$. Consider the dg-Lie algebra $L$ over the power series ring $\mathbf{k}[[t]]$ defined as

$$L := \mathfrak{g} \otimes (t)\,, \tag{23}$$

where $(t) \subset \mathbf{k}[[t]]$ is the ideal generated by $t$. Degree $n$ elements of $L$ are expressions $f_1 t + f_2 t^2 + \cdots$, $f_i \in \mathfrak{g}^n$ for $i \geq 1$. The dg-Lie structure on $L$ is induced from that of $\mathfrak{g}$ in an obvious manner. Denote by $\mathrm{MC}(\mathfrak{g})$ the set of all Maurer-Cartan elements in $L$. Clearly, a degree 1 element $s = f_1 t + f_2 t^2 + \cdots$ is Maurer-Cartan if its components $\{f_i \in \mathfrak{g}^1\}_{i \geq 1}$ satisfy the equation:

$$(MC_k) \qquad df_k + \frac{1}{2} \sum_{i+j=k} [f_i, f_j] = 0$$

for each $k \geq 1$.

**Example 5.14.** Let us apply the above construction to the Hochschild complex of an associative algebra $A$ with the multiplication $\mu_0$, that is, take $\mathfrak{g} := C^{*+1}_{\mathrm{Hoch}}(A, A)$ with the Gerstenhaber bracket and the Hochschild differential. In this case, one easily sees that $(MC_k)$ for $s = \mu_1 t + \mu_2 t^2 + \cdots$, $\mu_i \in C^2_{\mathrm{Hoch}}(A, A)$ is precisely equation $(D_k)$ of Theorem 3.15, $k \geq 1$, compare also calculations on page 346. We conclude that $\mathrm{MC}(\mathfrak{g})$ is the set of infinitesimal deformations of $\mu_0$.

Let us recall that each Lie algebra $\mathfrak{l}$ can be equipped with a group structure with the multiplication given by the *Hausdorff-Campbell formula*:

$$x \cdot y := x + y + \frac{1}{2}[x, y] + \frac{1}{12}\big([x, [x, y]] + [y, [y, x]]\big) + \cdots \tag{24}$$

assuming a suitable condition that guarantees that the above infinite sum makes sense in $\mathfrak{l}$, see [39, I.IV.§7]. We denote $\mathfrak{l}$ with this multiplication by $\exp(\mathfrak{l})$. Formula (24) is obtained by expressing the right hand side of

$$x \cdot y = \log\big(\exp(x)\exp(y)\big),$$

where

$$\exp(a) := 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots, \quad \log(1 + a) := a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \cdots,$$

in terms of iterated commutators of non-commutative variables $x$ and $y$.

Using this construction, we introduce the *gauge group* of $\mathfrak{g}$ as

$$\mathrm{G}(\mathfrak{g}) := \exp(L^0),$$

where $L^0 = \mathfrak{g}^0 \otimes (t)$ is the Lie subalgebra of degree zero elements in $L$ defined in (23). Let us fix an element $\chi \in \mathfrak{g}^1$. The gauge group then acts on $L^1 = \mathfrak{g}^1 \otimes (t)$ by the formula

$$x \cdot l := l + [x, \chi + l] + \frac{1}{2!}[x, [x, \chi + l]] + \frac{1}{3!}[x, [x, [x, \chi + l]]] + \cdots, \ x \in \mathrm{G}(\mathfrak{g}), l \in L^1, \tag{25}$$

obtained by expressing the right hand side of

$$x \cdot l = \exp(x)(\chi + l)\exp(-x) - \chi \tag{26}$$

in terms of iterated commutators. Denoting $d\chi := [\chi, \chi]$, formula (25) reads

$$\begin{aligned} x \cdot l = l + dx + [x, l] + \frac{1}{2}\big\{[x, dx] + [x, [x, l]]\big\} + \\ + \frac{1}{3}\big\{[x, [x, dx]] + [x, [x, [x, l]]]\big\} + \cdots \end{aligned} \tag{27}$$

**Lemma 5.15.** *Action* (27) *of* $\mathrm{G}(\mathfrak{g})$ *on* $L^1$ *preserves the space* $\mathrm{MC}(\mathfrak{g})$ *of solutions of the Maurer-Cartan equation.*

**Proof.** We will prove the lemma under the assumption that $\mathfrak{g}$ is a dg-Lie algebra whose differential $d$ has the form $d = [\chi, -]$ for some $\chi \in \mathfrak{g}^1$ satisfying $[\chi, \chi] = 0$ (see Proposition 5.5). The proof of the general case is a straightforward, though involved, verification.

It follows from (26) that $\chi + x \cdot l = \exp(x)(\chi + l)\exp(-x)$, i.e. $x$ transforms $\chi + l$ into $\exp(x)(\chi + l)\exp(-x)$. Under the assumption $d = [\chi, -]$, the Maurer-Cartan equation for $l$ is equivalent to $[\chi + l, \chi + l] = 0$. The Maurer-Cartan equation for the transformed $l$ then reads

$$[\exp(x)(\chi + l)\exp(-x), \exp(x)(\chi + l)\exp(-x)] = 0\,,$$

which can be rearranged into

$$\exp(x)[\chi + l, \chi + l]\exp(-x) = 0\,.$$

This finishes the proof. □

Thanks to Lemma 5.15, it makes sense to consider

$$\mathfrak{Def}(\mathfrak{g}) := \mathrm{MC}(\mathfrak{g})/\mathrm{G}(\mathfrak{g})\,,$$

the moduli space of solutions of the Maurer-Cartan equation in $L = \mathfrak{g} \otimes (t)$.

**Example 5.16.** Let us return to the situation in Example 5.14. In this case

$$\mathfrak{g}_0 = C^1_{\mathrm{Hoch}}(A, A) = Lin(A, A)\,,$$

with the bracket given by the commutator of the composition of linear maps. The gauge group $\mathrm{G}(\mathfrak{g})$ consists of elements $x = f_1 t + f_2 t^2 + \ldots$, $f_i \in Lin(A, A)$. It follows from the definition of the gauge group action that two formal deformations $\mu' = \mu_0 + \mu'_1 t + \mu'_2 t^2 + \cdots$ and $\mu'' = \mu_0 + \mu''_1 t + \mu''_2 t^2 + \cdots$ of $\mu_0$ define the same element in $\mathfrak{Def}(\mathfrak{g})$ if and only if

$$\exp(x)(\mu_0 + \mu'_1 t + \mu'_2 t^2 + \cdots) = (\mu_0 + \mu''_1 t + \mu''_2 t^2 + \cdots)\big(\exp(x) \otimes \exp(x)\big) \tag{28}$$

for some $x \in \mathrm{G}(\mathfrak{g})$. The above formula has an actual, not only formal, meaning – all power series make sense because of the completeness of the ground ring.

On the other hand, recall that in Example 3.17 we introduced the group

$$H := \big\{u = \mathrm{id}_A + \phi_1 t + \phi_2 t^2 + \cdots \mid \phi_i \in Lin(A, A)\big\}\,.$$

The exponential map $\exp : \mathrm{G}(\mathfrak{g}) \to H$ is a well-defined isomorphism with the inverse map $\log : H \to \mathrm{G}(\mathfrak{g})$. We conclude that the equivalence relation defined by (28) is the same as the equivalence defined by (5) in Example 3.17, therefore $\mathfrak{Def}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g})/\mathrm{G}(\mathfrak{g})$ is the moduli space of equivalence classes of formal deformations of $\mu_0$.

The above analysis can be generalized by replacing, in (23), $(t)$ by an arbitrary ideal $\mathfrak{m}$ in a local Artinian ring or in a complete local ring.

## 6. $L_\infty$-algebras and the Maurer-Cartan equation

We are going to describe a generalization of differential graded Lie algebras. Let us start by recalling some necessary notions.

Let $W$ be a $\mathbb{Z}$-graded vector space. We will denote by $\wedge W$ the *free graded commutative associative algebra* over $W$. It is characterized by the obvious analog of the universal property in Definition 4.1 with respect to graded commutative associative algebras. It can be realized as the tensor algebra $T(W)$ modulo the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$. If one decomposes

$$W = W^{\mathrm{even}} \oplus W^{\mathrm{odd}}$$

into the even and odd parts, then

$$\wedge W \cong \mathbf{k}[W^{\mathrm{even}}] \otimes E[W^{\mathrm{odd}}]\,,$$

where the first factor is the polynomial algebra and the second one is the exterior (Grassmann) algebra. The algebra $\wedge W$ can also be identified with the subspace of $T(W)$ consisting of graded-symmetric elements (remember we work over a characteristic zero field).

Denote the product of (homogeneous) elements $w_1, \ldots, w_n \in W$ in $\wedge W$ by $w_1 \wedge \ldots \wedge w_n$. For a permutation $\sigma \in \mathfrak{S}_k$ we define the *Koszul sign* $\varepsilon(\sigma) \in \{-1, +1\}$ by

$$w_1 \wedge \ldots \wedge w_k = \varepsilon(\sigma) w_{\sigma(1)} \wedge \ldots \wedge w_{\sigma(k)}$$

and the *antisymmetric Koszul sign* $\chi(\sigma) \in \{-1, +1\}$ by

$$\chi(\sigma) := \mathrm{sgn}(\sigma)\varepsilon(\sigma)\,.$$

**Exercise 6.1.** Express $\epsilon(\sigma)$ and $\chi(\sigma)$ explicitly in terms of $\sigma$ and the degrees $|w_1|, \ldots, |w_n|$.

Finally, a permutation $\sigma \in \mathfrak{S}_n$ is called an $(i, n-i)$-*unshuffle* if $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$. The set of all $(i, n-i)$-unshuffles will be denoted $\mathfrak{S}_{(i,n-i)}$.

**Definition 6.2.** An $L_\infty$-*algebra* (also called a *strongly homotopy Lie* or *sh Lie algebra*) is a graded vector space $V$ together with a system

$$l_k : \otimes^k V \to V, \quad k \in \mathbb{N}$$

of linear maps of degree $2-k$ subject to the following axioms.

– Antisymmetry: For every $k \in \mathbb{N}$, every permutation $\sigma \in \mathfrak{S}_k$ and every homogeneous $v_1, \ldots, v_k \in V$,

$$l_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \chi(\sigma) l_k(v_1, \ldots, v_k)\,. \tag{29}$$

– For every $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,

$$\sum_{i+j=n+1} (-1)^i \sum_{\sigma \in \mathfrak{S}_{i,n-i}} \chi(\sigma) l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0\,. \tag{$L_n$}$$

**Remark 6.3.** The sign in $(L_n)$ was taken from [15]. With this sign convention, all terms of the (generalized) Maurer-Cartan equation recalled in (31) below have $+1$-signs. Our sign convention is related to the original one in [25, 26] via the transformation $l_n \mapsto (-1)^{\binom{n+1}{2}} l_n$. We also used the opposite grading which is better suited for our purposes – the operation $l_k$ as introduced in [25, 26] has degree $k-2$.

Let us expand axioms $(L_n)$ for $n = 1, 2$ and 3.
**Case** $n = 1$**.** For $n = 1$ $(L_1)$ reduces to $l_1(l_1(v)) = 0$ for every $v \in V$, i.e. $l_1$ is a degree $+1$ differential.
**Case** $n = 2$**.** By (29), $l_2 : V \otimes V \to V$ is a linear degree 0 map which is graded antisymmetric,

$$l_2(v,u) = -(-1)^{|u||v|} l_2(u,v)$$

and $(L_n)$ for $n = 2$ gives

$$l_1\big(l_2(u,v)\big) = l_2\big(l_1(u),v\big) + (-1)^{|u|} l_2\big(u,l_1(v)\big) \tag{$L_2$}$$

meaning that $l_1$ is a graded derivation with respect to the multiplication $l_2$. Writing $d := l_1$ and $[u,v] := l_2(u,v)$, $(L_2)$ takes more usual form

$$d[u,v] = [du,v] + (-1)^{|u|}[u,dv]\,.$$

**Case** $n = 3$**.** The degree $-1$ graded antisymmetric map $l_3 : \otimes^3 V \to V$ satisfies $(L_3)$:

$$\begin{aligned}(-1)^{|u||w|}[[u,v],w] + (-1)^{|v||w|}[[w,u],v] + (-1)^{|u||v|}[[v,w],u]\\ = (-1)^{|u||w|}\big(dl_3(u,v,w) + l_3(du,v,w)\\ + (-1)^{|u|} l_3(u,dv,w) + (-1)^{|u|+|v|} l_3(u,v,dw)\big)\,.\end{aligned}$$

One immediately recognizes the three terms of the Jacobi identity in the left-hand side and the $d$-boundary of the trilinear map $l_3$ in the right-hand side. We conclude that the bracket $[-,-]$ satisfies the Jacobi identity modulo the homotopy $l_3$.

**Example 6.4.** If all structure operations of an $L_\infty$-algebra $L = (V, l_1, l_2, l_3, \ldots)$ except $l_1$ vanish, then $L$ is just a dg-vector space with the differential $d = l_1$. If all $l_k$'s except $l_1$ and $l_2$ vanish, then $L$ is our familiar dg-Lie algebra from Definition 5.3 with $d = l_1$ and the Lie bracket $[-,-] = l_2$. In this sense, dg-Lie algebras are particular cases of $L_\infty$-algebras.

**Example 6.5.** Let $L' = (V', l'_1, l'_2, l'_3, \ldots)$ and $L'' = (V'', l''_1, l''_2, l''_3, \ldots)$ be two $L_\infty$-algebras. Define their *direct sum* $L' \oplus L''$ to be the $L_\infty$-algebra $L' \oplus L''$ with the underlying vector space $V' \oplus V''$ and structure operations $\{l_k\}_{k \geq 1}$ given by

$$l_k(v'_1 \oplus v''_1, \ldots, v'_k \oplus v''_k) := l'_k(v'_1, \ldots, v'_k) + l''_k(v''_1, \ldots, v''_k)\,,$$

for $v'_1, \ldots, v'_k \in V'$, $v''_1, \ldots, v''_k \in V''$.

For a graded vector space $V$ denote $\vee_k(V)$ the quotient of $\bigotimes^k V$ modulo the subspace spanned by elements

$$v_1 \otimes \cdots \otimes v_k - \chi(\sigma)\; v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}\,.$$

The antisymmetry (29) implies that the structure operations of an $L_\infty$ algebra can be interpreted as maps

$$l_k : \vee_k(V) \to V, \quad k \geq 1\,.$$

We are going to give a description of the set of $L_\infty$-structures on a given graded vector space in terms of coderivations, in the spirit of Theorem 4.21. To this end, we need the following coalgebra which will play the role of ${}^cT(W)$.

**Proposition 6.6.** *The space* $\wedge(W)$ *with the comultiplication* $\Delta : \wedge(W) \to \wedge(W) \otimes \wedge(W)$ *defined by*

$$\Delta(w_1 \wedge \ldots \wedge w_n) := \sum_{i=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_{i,n-i}} \epsilon(\sigma)(w_{\sigma(1)} \wedge \ldots \wedge w_{\sigma(i)}) \otimes (w_{\sigma(i+1)} \wedge \ldots \wedge w_{\sigma(n)})$$

*is a graded coassociative cocommutative coalgebra. We will denote it* ${}^c\wedge(W)$.

**Proof.** A direct verification which we leave to the reader as an exercise. □

For the coalgebra ${}^c\wedge(W)$, the following analog of Proposition 4.19 holds.

**Proposition 6.7.** *Let $W$ be a graded vector space. For any $d$, there is a natural isomorphism*

$$\mathit{CoDer}^d({}^c\wedge(W)) \cong \mathit{Lin}^d\big({}^c\wedge(W), W\big)\,.$$

We leave the proof to the reader. Observe that the coalgebra ${}^c\wedge(W)$ is a direct sum

$${}^c\wedge(W) = \bigoplus_{n\geq 1} {}^c\wedge^n(W)$$

of subspaces ${}^c\wedge^n(W)$ spanned by $w_1 \wedge \ldots \wedge w_n$, for $w_1, \ldots, w_n \in W$. One may define the $s$th *corestriction* of a coderivation $\theta \in \mathit{CoDer}({}^c\wedge(W))$ as the composition

$$f_s : {}^c\wedge^s(W) \xrightarrow{\theta|_{\wedge^s(W)}} {}^c\wedge(W) \xrightarrow{\text{proj.}} W\,.$$

As in Definition 4.20, a coderivation $\theta \in \mathit{CoDer}^d({}^c\wedge(W))$ is *quadratic* if its $s$th corestriction is non-zero only for $s = 2$. A *differential* is a degree 1 coderivation $\theta$ such that $\theta^2 = 0$.

**Theorem 6.8.** *Denote by $L_\infty(V)$ the set of all $L_\infty$-algebra structures on a graded vector space $V$ and $\mathit{CoDiff}^1({}^c\wedge(\downarrow V))$ the set of differentials on ${}^c\wedge(\downarrow V)$. Then there is a bijection*

$$L_\infty(V) \cong \mathit{CoDiff}^1\left({}^c\wedge(\downarrow V)\right).$$

**Proof.** Let $\chi \in \mathit{CoDiff}^1({}^c\wedge(\downarrow V))$ and $f_n : {}^c\wedge^n(\downarrow V) \to \downarrow V$ the $n$th corestriction of $\chi$, $n \geq 1$. Define $\bar{l}_n : \vee_n(V) \to V$ by the diagram

$$\begin{tikzcd}
{}^c\wedge_n(\downarrow V) \arrow[r, "f_n"] & \downarrow V \\
\vee_n(V) \arrow[u, "\bigotimes^n \downarrow"] \arrow[r, "\bar{l}_n"] & V \arrow[u, "\downarrow"']
\end{tikzcd}$$

It is then a direct though involved verification that the maps

$$l_n := (-1)^{\binom{n+1}{2}} \bar{l}_n \tag{30}$$

define an $L_\infty$-structure on $V$ and that the correspondence $\chi \leftrightarrow (l_1, l_2, l_3, \ldots)$ is one-to-one. The reason for the sign change in (30) is explained in Remark 6.3. $\square$

**Remark 6.9.** By Theorem 6.8, $L_\infty$-algebras can be alternatively defined as differentials on "cofree" cocommutative coassociative coalgebras (the reason why we put 'cofree' into quotation marks is the same as in Section 4, see also the warning on page 351). Dual forms of these object are *Sullivan models* that have existed in rational homotopy theory since 1977 [42] though they were recognized as homotopy versions of Lie algebras much later [19, 26].

**Exercise 6.10.** Show that the isomorphism of Theorem 6.8 restricts to the isomorphism

$$\mathit{Lie}(V) \cong \mathit{CoDiff}_2^1\left({}^c\wedge(\downarrow V)\right)$$

between the set of Lie algebra structures on $V$ and quadratic differentials on the coalgebra ${}^c\wedge(\downarrow V)$. This isomorphism shall be compared to the isomorphism in Theorem 4.21.

Let us make a digression and see what happens when one allows in the right hand side of (17) all, not only quadratic, differentials. The above material indicates that one should expect a homotopy version of associative algebras. This is indeed so; one gets the following objects that appeared in 1963 [41] (but we use the sign convention of [30]).

**Definition 6.11.** An $A_\infty$-*algebra* (also called a *strongly homotopy associative algebra*) is a graded vector space $V$ together with a system

$$\mu_k : V^{\otimes k} \to V, \quad k \geq 1\,,$$

of linear maps of degree $k-2$ such that

$$(A_n) \sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} (-1)^{k+\lambda+k\lambda+k(|v_1|+\cdots+|v_\lambda|)} \mu_{n-k+1}\big(v_1, \ldots, \mu_k(v_{\lambda+1}, \ldots, v_{\lambda+k}), \ldots, v_n\big) = 0$$

for every $n \geq 1$, $v_1, \ldots, v_n \in V$.

One easily sees that $(A_1)$ means that $\partial := \mu_1$ is a degree $-1$ differential, $(A_2)$ that the bilinear product $\mu_2 \colon V \otimes V \to V$ commutes with $\partial$ and $(A_3)$ that $\mu_2$ is associative up to the homotopy $\mu_3$. $A_\infty$-algebras can also be described as algebras over the cellular chain complex of the non-$\Sigma$ operad $K = \{K_n\}_{n\geq 1}$ whose $n$th piece is the $(n-2)$-dimensional convex polytope $K_n$ called the *Stasheff associahedron* [35, Section II.1.6]. Let us mention at least that $K_2$ is the point, $K_3$ the closed interval and $K_4$ is the pentagon from Mac Lane's theory of monoidal categories [28]. A portrait of $K_5$ due to Masahico Saito is in Figure 1.

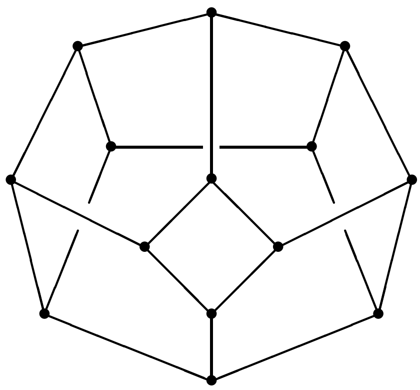

FIGURE 1. Saito's portrait of $K_5$.

**Theorem 6.12.** *For a graded vector space $V$ denote $A_\infty(V)$ the set of all $A_\infty$-algebra structures on $V$ and $CoDiff^1({}^cT(\downarrow V))$ the set of all differentials on ${}^cT(\downarrow V)$. Then there is a natural bijection*

$$A_\infty(V) \cong CoDiff^1\left({}^cT(\downarrow V)\right).$$

**Proof.** The isomorphism in the above theorem is of the same nature as the isomorphism of Theorem 6.8, but it also involves the 'flip' of degrees since we defined, following [30], $A_\infty$-algebras in such a way that the differential $\partial = \mu_1$ has degree $-1$. We leave the details to the reader. □

Let us return to the main theme of this section. Our next task will be to introduce morphisms of $L_\infty$-algebras. We start with a simple-minded definition.

Suppose $L' = (V', l'_1, l'_2, l'_3, \ldots)$ and $L'' = (V'', l''_1, l''_2, l''_3, \ldots)$ are $L_\infty$-algebras. A *strict morphism* is a degree zero linear map $f : V' \to V''$ which commutes with all structure operations, that is

$$f(l'_k(v_1, \ldots, v_k)) = l''_k\big(f(v_1), \ldots, f(v_k)\big),$$

for each $v_1, \ldots, v_k \in V'$, $k \geq 1$.

For our purposes we need, however, a subtler notion of morphisms. We give a definition that involves the isomorphism of Theorem 6.8.

**Definition 6.13.** Let $L'$ and $L''$ be $L_\infty$-algebras represented by dg-coalgebras $({}^c\wedge(\downarrow V'), \delta')$ and $({}^c\wedge(\downarrow V''), \delta'')$. A *(weak) morphism* of $L_\infty$-algebras is then a morphism of dg-coalgebras $F : ({}^c\wedge(\downarrow V'), \delta') \to ({}^c\wedge(\downarrow V''), \delta'')$.

Definition 6.13 can be unwrapped. Let $F_k : {}^c\wedge_k(\downarrow V') \to \downarrow V''$ be, for each $k \geq 1$, the composition

$${}^c\wedge^k(\downarrow V') \xrightarrow{F} {}^c\wedge(\downarrow V'') \xrightarrow{\text{proj.}} \downarrow V''.$$

Define the maps $f_k : \vee_k V' \to V''$ by the diagram

$$\begin{tikzcd} {}^c\wedge_k(\downarrow V') \arrow[r, "F_k"] & \downarrow V'' \\ \vee_k V' \arrow[u, "\bigotimes^k \downarrow"] \arrow[r, "f_k"] & V'' \arrow[u, "\downarrow"'] \end{tikzcd}$$

Clearly, $f_k$ is a degree $1-k$ linear map. The fact that $F$ is a dg-morphism can be expressed via a sequence of axioms $(M_n)$, $n \geq 1$, where $(M_n)$ postulates the vanishing of a combination of $n$-multilinear maps on $V'$ with values in $V''$ involving $f_i, l'_i$ and $l''_i$ for $i \leq n$.

We are not going to write $(M_n)$'s here. Explicit axioms for $L_\infty$-maps can be found in [21], see also [25, Definition 5.2] where the particular case when $L''$ is a dg-Lie algebra ($l''_k = 0$ for $k \geq 3$) is discussed in detail. The reader is however encouraged to verify that $(M_1)$ says that $f_1 : (V', l'_1) \to (V'', l''_1)$ is a chain map and that $(M_2)$ means that $f_1$ commutes with the brackets $l'_2$ and $l''_2$ modulo the homotopy $f_2$.

Morphisms of $L_\infty$-algebras $L'$ and $L''$ with underlying vector spaces $V'$ and $V''$ can therefore be equivalently defined as systems $f = \{f_k : \bigotimes^k V' \to V''\}_{k \geq 1}$, where $f_k$ is a degree $1-k$ graded antisymmetric linear map, and axioms $(M_n)$, $n \geq 1$, are satisfied. Let us denote by $\mathsf{L}_\infty$ the category of $L_\infty$-algebras and their morphisms in the sense of Definition 6.13.

**Exercise 6.14.** Show that the category $\mathsf{strL}_\infty$ of $L_\infty$-algebras and their strict morphisms can be identified with the (non-full) subcategory of $\mathsf{L}_\infty$ with the same objects and morphisms $f = (f_1, f_2, \ldots)$ such that $f_k = 0$ for $k \geq 2$.

Show that the obvious imbedding $\mathsf{dgLie} \hookrightarrow \mathsf{L}_\infty$ is not full. This means that there are more morphisms between dg-Lie algebras considered as elements of the category $\mathsf{L}_\infty$ than in the category of $\mathsf{dgLie}$. Observe finally that the forgetful functor $\Box : \mathsf{L}_\infty \to \mathsf{dgVect}$ given by forgetting all structure operations is not faithful.

## 7. Homotopy invariance of the Maurer-Cartan equation

Let us start with recalling some necessary definitions.

**Definition 7.1.** A morphism $f = (f_1, f_2, \ldots) : L' = (V', l'_1, l'_2, \ldots) \to L'' = (V'', l''_1, l''_2, \ldots)$ of $L_\infty$-algebras is a *weak equivalence* if the chain map $f_1 : (V', l'_1) \to (V'', l''_1)$ induces an isomorphism of cohomology.

**Definition 7.2.** An $L_\infty$-algebra $L = (V, l_1, l_2, \ldots)$ is *minimal* if $l_1 = 0$. It is *contractible* if $l_k = 0$ for $k \geq 2$ and if $H^*(V, l_1) = 0$.

**Proposition 7.3.** *Let $f$ be a weak equivalence of minimal $L_\infty$-algebras $\mathfrak{g}'$, $\mathfrak{g}''$ over the ground field $\mathbf{k}$. Let $\mathfrak{m}$ be the maximal ideal in a complete local $\mathbf{k}$-algebra $R$. Then the induced map $f \otimes \mathfrak{m} : L' \to L''$, where $L' := \mathfrak{g}' \otimes \mathfrak{m}$ and $L'' := \mathfrak{g}'' \otimes \mathfrak{m}$, is an isomorphism of $L_\infty$-algebras.*

**Proof.** It follows from the minimality of $\mathfrak{g}'$ and $\mathfrak{g}''$ that the linear part $f_1$ of the weak equivalence $f = (f_1, f_2, \ldots)$ is an isomorphism, thus the corresponding map $F : ({}^c\wedge(\downarrow V'), \delta') \to ({}^c\wedge(\downarrow V''), \delta'')$ induces an isomorphism of generators. Such maps can be formally inverted, and the extension of scalars by $\mathfrak{m}$ guarantees that the inversion formula converges. □

**Warning.** There seems to be general belief that a weak equivalence of minimal $L_\infty$-algebras is always an isomorphism, but simple examples show that this is not true. One needs to control the convergence of the inversion formula. This can be achieved either by extending the scalars as in the above proposition, or by imposing some restrictions on the grading, as the simple connectivity assumption in rational homotopy theory.

The following theorem, which can be found in [23], uses the direct sum of $L_\infty$-algebras recalled in Example 6.5.

**Theorem 7.4.** *Each $L_\infty$-algebra is the direct sum of a minimal and a contractible $L_\infty$-algebra.*

Let $L \cong L_m \oplus L_c$ be a decomposition of an $L_\infty$-algebra $L$ into a minimal $L_\infty$-algebra $L_m$ and a contractible $L_\infty$-algebra $L_c$. Since the inclusion $\iota : L_m \to L_m \oplus L_c \cong L$ is a weak equivalence, Theorem 7.4 implies:

**Corollary 7.5.** *Each $L_\infty$-algebra is weakly equivalent to a minimal one.*

Corollary 7.5 above can also be derived from homotopy invariance properties of strongly homotopy algebras proved in [32]. Suppose we are given an $L_\infty$-algebra $L = (V, l_1, l_2, \ldots)$. In characteristic zero, two cochain complexes have the same cochain homotopy type if and only if they have isomorphic cohomology. In particular, the cochain complex $(V, l_1)$ is homotopy equivalent to the cohomology $H^*(V, l_1)$ considered as a complex with trivial differential. Move **(M1)** on page 133 of [32] now implies that there exists an induced minimal $L_\infty$-structure on $H^*(V, l_1)$, weakly equivalent to $L$. Let us remark that an $A_\infty$-version of Corollary 7.5 was known to Kadeishvili already in 1985, see [20].

Remarkably, each $L_\infty$-algebra is, under some mild assumptions, weakly equivalent to a dg-Lie algebra. This can be proved as follows. Suppose $L$ is an $L_\infty$-algebra represented by a dg-coalgebra $({}^c\wedge(\downarrow V), \delta)$. The bar construction $B({}^c\wedge(\downarrow V), \delta)$ is a dg-Lie algebra and one may show, under an assumption that guarantees the convergence of a spectral sequence, that $B({}^c\wedge(\downarrow V), \delta)$ is weakly equivalent to $L$ in the category of $L_\infty$-algebras. This property is an algebraic analog of the *rectification principle* for $W\mathcal{P}$-spaces provided by the $M$-construction of Boardman and Vogt, see [35, Theorem II.2.9].

Let $\mathfrak{g}$ be an $L_\infty$-algebra over the ground field $\mathbf{k}$, with the underlying $\mathbf{k}$-vector space $V$. Then $V \otimes (t)$, where $(t) \subset \mathbf{k}[[t]]$ is the ideal generated by $t$, has a natural induced $L_\infty$-structure. Denote this $L_\infty$-algebra by $L := \mathfrak{g} \otimes (t) = (V \otimes (t), l_1, l_2, l_3, \ldots)$. Let $\mathrm{MC}(\mathfrak{g})$ be the set of all degree $+1$ elements $s \in L^1$ satisfying the *generalized Maurer-Cartan equation*

$$l_1(s) + \frac{1}{2} l_2(s, s) + \frac{1}{3!} l_3(s, s, s) + \cdots + \frac{1}{n!} l_n(s, \ldots, s) + \cdots = 0 . \tag{31}$$

When $\mathfrak{g}$ is a dg-Lie algebra, one recognizes the ordinary Maurer-Cartan equation (22).

At this moment one needs to introduce a suitable *gauge equivalence* between solutions of (31) generalizing the action of the gauge group $\mathrm{G}(\mathfrak{g})$ recalled in (25). Since in applications of Section 8 all relevant $L_\infty$-algebras are in fact dg-Lie algebras, we are not going to describe this generalized gauge equivalence here, and only refer to [23] instead. We denote $\mathfrak{Def}(\mathfrak{g})$ the set of gauge equivalence classes of solutions of (31). Let us, however, mention that there are examples, as bialgebras treated in [33], where deformations are described by a fully-fledged $L_\infty$-algebra.

**Example 7.6.** For $\mathfrak{g}$ contractible, $\mathfrak{Def}(\mathfrak{g})$ is the one-point set consisting of the class of the trivial solution of (31). Indeed,

$$\mathrm{MC}(\mathfrak{g}) = \{s = s_1t + s_2t^2 + \dots \mid ds_1 = ds_2 = \dots = 0\}$$

so, by acyclicity, $s_i = db_i$ for some $b_i \in \mathfrak{g}^0$, $i \geq 1$. Formula (27) (with $x = -b_1t_1 - b_2t_2 - \cdots$ and $l = s_1t + s_2t^2 + \cdots$) gives

$$(-b_1t_1 - b_2t_2 - \cdots) \cdot (s_1t + s_2t^2 + \cdots) = 0,$$

therefore $s = s_1t + s_2t^2 + \cdots$ is equivalent to the trivial solution.

**Example 7.7.** Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be two $L_\infty$-algebras. Then, for the direct product,

$$\mathfrak{Def}(\mathfrak{g}' \oplus \mathfrak{g}'') \cong \mathfrak{Def}(\mathfrak{g}') \times \mathfrak{Def}(\mathfrak{g}'').$$

Indeed, it follows from definition that $\mathrm{MC}(\mathfrak{g}' \oplus \mathfrak{g}'') \cong \mathrm{MC}(\mathfrak{g}') \times \mathrm{MC}(\mathfrak{g}'')$. This factorization is preserved by the gauge equivalence.

The central statement of this section reads:

**Theorem 7.8.** *The assignment $\mathfrak{g} \mapsto \mathfrak{Def}(\mathfrak{g})$ extends to a covariant functor from the category of $L_\infty$-algebras and their weak morphisms to the category of sets. A weak equivalence $f : \mathfrak{g}' \to \mathfrak{g}''$ induces an isomorphism $\mathfrak{Def}(f) : \mathfrak{Def}(\mathfrak{g}') \cong \mathfrak{Def}(\mathfrak{g}'')$.*

The above theorem implies that the deformation functor $\mathfrak{Def}$ descends to the localization $\mathsf{hoL}_\infty$ obtained by inverting weak equivalences in $\mathsf{L}_\infty$. By Quillen's theory [37], $\mathsf{hoL}_\infty$ is equivalent to the category of minimal $L_\infty$-algebras and homotopy classes (in an appropriate sense) of their maps. This explains the meaning of homotopy invariance in the title of this section.

**Proof** (Proof of Theorem 7.8)**.** For an $L_\infty$-morphism $f = (f_1, f_2, f_3, \dots) : \mathfrak{g}' \to \mathfrak{g}''$ define $\mathrm{MC}(f) : \mathrm{MC}(\mathfrak{g}') \to \mathrm{MC}(\mathfrak{g}'')$ by

$$\mathrm{MC}(f)(s) := f_1(s) + \frac{1}{2}f_2(s,s) + \cdots + \frac{1}{n!}f_n(s,\dots,s) + \cdots$$

It can be shown that $\mathrm{MC}(f)$ is a well-defined map that descends to the quotients by the gauge equivalence, giving rise to a map $\mathfrak{Def}(f) : \mathfrak{Def}(\mathfrak{g}') \to \mathfrak{Def}(\mathfrak{g}'')$.

Assume that $f : \mathfrak{g}' \to \mathfrak{g}''$ above is a weak equivalence. By Theorem 7.4, $\mathfrak{g}'$ decomposes as $\mathfrak{g}' = \mathfrak{g}'_m \oplus \mathfrak{g}'_c$, with $\mathfrak{g}'_m$ minimal and $\mathfrak{g}'_c$ contractible, and there is a similar decomposition $\mathfrak{g}'' = \mathfrak{g}''_m \oplus \mathfrak{g}''_c$ for $\mathfrak{g}''$. Define the map $\overline{f} : \mathfrak{g}'_m \to \mathfrak{g}''_m$ by the commutativity of the diagram

$$\begin{tikzcd} \mathfrak{g}'_m \oplus \mathfrak{g}'_c \arrow[d, "f"'] & \mathfrak{g}'_m \arrow[l, hook', "i"'] \arrow[d, "\overline{f}"] \\ \mathfrak{g}''_m \oplus \mathfrak{g}''_c \arrow[r, two heads, "p"] & \mathfrak{g}''_m \end{tikzcd}$$

in which $i$ is the natural inclusion and $p$ the natural projection. Observe that $\overline{f}$ is a weak equivalence. By Proposition 7.3, $\overline{f}$ becomes, after extending the scalars by $(t) \subset \mathbf{k}[[t]]$, an isomorphism. Therefore, in the following induced diagram, the map $\mathfrak{Def}(\overline{f})$ is an isomorphism, too:

$$\begin{tikzcd} \mathfrak{Def}(\mathfrak{g}'_m) \times \mathfrak{Def}(\mathfrak{g}'_c) \arrow[d, "\mathfrak{Def}(f)"'] & \mathfrak{Def}(\mathfrak{g}'_m) \arrow[l, hook', "\mathfrak{Def}(i)"'] \arrow[d, "\mathfrak{Def}(\overline{f})"] \\ \mathfrak{Def}(\mathfrak{g}''_m) \times \mathfrak{Def}(\mathfrak{g}''_c) \arrow[r, two heads, "\mathfrak{Def}(p)"] & \mathfrak{Def}(\mathfrak{g}''_m) \end{tikzcd}$$

Since, by Example 7.6, both $\mathfrak{Def}(\mathfrak{g}'_c)$ and $\mathfrak{Def}(\mathfrak{g}''_c)$ are points, the maps $\mathfrak{Def}(i)$ and $\mathfrak{Def}(p)$ are isomorphisms. We finish the proof by concluding that $\mathfrak{Def}(f)$ is also an isomorphism. $\square$

## 8. Deformation quantization of Poisson manifolds

In this section we indicate the main ideas of Kontsevich's proof of the existence of a deformation quantization of Poisson manifolds. Our exposition follows [23]. Let us recall some necessary notions.

**Definition 8.1.** A *Poisson algebra* is a vector space $V$ equipped with operations $\cdot : V \otimes V \to V$ and $\{-,-\} : V \otimes V \to V$ such that:
– $(V, \cdot\,)$ is an associative commutative algebra,
– $(V, \{-,-\})$ is a Lie algebra, and
– the map $v \mapsto \{u, v\}$ is a $\cdot$-derivation for any $u \in V$, i.e.

$$\{u, v \cdot w\} = \{u, v\} \cdot w + v \cdot \{u, w\}\,.$$

**Exercise 8.2.** Show that Poisson algebras can be equivalently defined as structures with only one operation $\bullet : V \otimes V \to V$ such that

$$u\bullet(v\bullet w) = (u\bullet v)\bullet w - \frac{1}{3}\big\{(u\bullet w)\bullet v + (v\bullet w)\bullet u - (v\bullet u)\bullet w - (w\bullet u)\bullet v)\big\}\,,$$

for each $u, v, w \in V$, see [34, Example 2].

Poisson algebras are 'classical limits' of associative deformations of commutative associative algebras. By this we mean the following. Let $A = (V, \cdot\,)$ be an associative algebra with multiplication $a, b \mapsto a \cdot b$. Consider a formal deformation $(\mathbf{k}[[t]] \otimes V, \star)$ of $A$ given, as in Theorem 3.15, by a family $\{\mu_i : A \otimes A \to A\}_{i \geq 1}$ by the formula

$$a \star b := a \cdot b + t\mu_1(a, b) + t^2\mu_2(a, b) + t^3\mu_3(a, b) + \cdots \tag{32}$$

for $a, b \in V$. We have the following:

**Proposition 8.3.** *Suppose $A = (V, \ \cdot\ )$ is a commutative associative algebra. Then, for an associative deformation* (32) *of $A$,*

$$\{a, b\} := \mu_1(a, b) - \mu_1(b, a), \quad a, b \in V\,,$$

*is a Lie bracket such that $P_\star := (V, \cdot, \{-,-\})$ is Poisson algebra.*

**Definition 8.4.** In the above situation, $P_\star$ is called the *classical limit* of the $\star$-product and $(\mathbf{k}[[t]] \otimes V, \star)$ a *deformation quantization* of the Poisson algebra $P_\star$.

**Proof** (Proof of Proposition 8.3)**.** Let us prove first that $\{-,-\}$ is a Lie bracket. The antisymmetry of $\{-,-\}$ is obvious, one thus only needs to verify the Jacobi identity. It is a standard fact that the antisymmetrization of an associative multiplication is a Lie product [39, Chapter I], therefore $[-,-]$ defined by $[x, y] := x \star y - y \star x$ for $x, y \in \mathbf{k}[[t]] \otimes A$, is a Lie bracket on $\mathbf{k}[[t]] \otimes A$. We conclude by observing that the Jacobi identity for $\{-,-\}$ evaluated at $a, b, c \in A$ is the term at $t^2$ of the Jacobi identify for $[-,-]$ evaluated at the same elements.

It remains to verify the derivation property. It is clearly equivalent to

$$\mu_1(ab, c) - \mu_1(c, ab) - a\mu_1(b, c) + a\mu_1(c, b) - \mu_1(a, c)b + \mu_1(c, a)b = 0 \tag{33}$$

where we, for brevity, omitted the symbol for the $\cdot$ -product. In Remark 3.16 we observed that $\mu_1$ is a Hochschild cocycle, therefore

$$\rho(a, b, c) := a\mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)c = 0\,.$$

A straightforward verification involving the commutativity of the $\cdot$ -product shows that the left hand side of (33) equals $-\rho(a, b, c) + \rho(a, c, b) - \rho(c, a, b)$. This finishes the proof. $\square$

Let us recall geometric versions of the above notions.

**Definition 8.5.** A *Poisson manifold* is a smooth manifold $M$ equipped with a Lie product $\{-,-\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ on the space of smooth functions such that $(C^\infty(M), \ \cdot\ , \{-,-\})$, where $\cdot$ is the standard pointwise multiplication, is a Poisson algebra.

Poisson manifolds generalize symplectic ones in that the bracket $\{-,-\}$ need not be induced by a nondegenerate 2-form. The following notion was introduced and physically justified in [5].

**Definition 8.6.** A *deformation quantization* (or a *star product*) of a Poisson manifold $M$ is a deformation quantization of the Poisson algebra $(C^\infty(M), \cdot, \{-,-\})$ such that all $\mu_i$'s in (32) are differential operators.

**Theorem 8.7** (Kontsevich [23])**.** *Every Poisson manifold admits a deformation quantization.*

**Proof** (Sketch of Proof)**.** Maxim Kontsevich proved this theorem in two steps. He proved first a 'local' version assuming $M = \mathbb{R}^d$, and then he globalized the result to an arbitrary $M$ using ideas of formal geometry and the language of superconnections. We are going to sketch only the first step of Kontsevich's proof.

The idea was to construct two weakly equivalent $L_\infty$-algebras $\mathfrak{g}'$, $\mathfrak{g}''$ such that $\mathfrak{Def}(\mathfrak{g}')$ contained the moduli space of Poisson structures on $M$ and $\mathfrak{Def}(\mathfrak{g}'')$ was the moduli space of star products, and then apply Theorem 7.8. In fact, $\mathfrak{g}'$ will turn out to be an ordinary graded Lie algebra and $\mathfrak{g}''$ a dg-Lie algebra.

– *Construction of* $\mathfrak{g}'$. It is the graded Lie algebra of *polyvector fields* with the structure given by the Shouten-Nijenhuis bracket. In more detail, $\mathfrak{g}' = \bigoplus_{n\geq 0} \mathfrak{g}'^n$ with

$$\mathfrak{g}'^n := \Gamma(M, \wedge^{n+1} TM), \quad n \geq 1\,,$$

where $\Gamma(M, \wedge^{n+1} TM)$ denotes the space of smooth sections of the $(n{+}1)$th exterior power of the tangent bundle $TM$. The bracket is determined by

$$[\xi_0 \wedge \ldots \wedge \xi_k, \eta_0 \wedge \ldots \wedge \eta_l] :=$$
$$:= \sum_{i=0}^{k}\sum_{j=0}^{l} (-1)^{i+j+k} [\xi_i, \eta_j] \wedge \xi_0 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k \wedge \eta_0 \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_l\,,$$

where $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_l \in \Gamma(M, TM)$ are vector fields, $\hat{}$ indicates the omission and $[\xi_i, \eta_j]$ in the right hand side denotes the classical Lie bracket of vector fields $\xi_i$ and $\eta_j$ [22, I.§1].

Recall that Poisson structures on $M$ are in one-to-one correspondence with smooth sections $\alpha \in \Gamma(M, \wedge^2 TM)$ satisfying $[\alpha, \alpha] = 0$. The corresponding bracket of smooth functions $f, g \in C^\infty(M)$ is given by $\{f, g\} = \alpha(f \otimes g)$. Since $\mathfrak{g}'$ is just a graded Lie algebra,

$$\mathrm{MC}(\mathfrak{g}') = \{s = s_1 t + s_2 t^2 + \ldots \in \mathfrak{g}'^1 \otimes (t) \mid [s, s] = 0\}$$

therefore clearly $s := \alpha t \in \mathrm{MC}(\mathfrak{g}')$ for each $\alpha \in \Gamma(M, \wedge^2 TM)$ defining a Poisson structure. We see that $\mathfrak{Def}(\mathfrak{g}')$ contains the moduli space of Poisson structures on $M$.

– *Construction of* $\mathfrak{g}''$. It is the dg Lie algebra of polydiffenential operators,

$$\mathfrak{g}'' = \bigoplus_{n\geq 0} D^n_{\mathrm{poly}}(M)\,,$$

where

$$D^n_{\mathrm{poly}}(M) \subset C^{n+1}_{\mathrm{Hoch}}\big(C^\infty(M), C^\infty(M)\big)$$

consists of Hochschild cochains (Definition 2.1) of the algebra $C^\infty(M)$ that are given by polydifferential operators. It is clear that $D^*_{\mathrm{poly}}(M)$ is closed under the Hochschild differential and the Gerstenhaber bracket, so the dg-Lie structure of Proposition 5.7 restricts to a dg-Lie structure on $\mathfrak{g}''$. The analysis of Example 5.16 shows that $\mathfrak{Def}(\mathfrak{g}'')$ represents equivalence classes of star products.

– *The weak equivalence.* Consider the map $f_1 : \mathfrak{g}' \to \mathfrak{g}''$ defined by

$$f_1(\xi_0, \ldots, \xi_k)(g_0, \ldots, g_k) := \frac{1}{(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \operatorname{sgn}(\sigma) \prod_{i=0}^{k} \xi_{\sigma(i)}(g_i)\,,$$

for $\xi_0, \ldots, \xi_k \in \Gamma(M, TM)$ and $g_0, \ldots, g_k \in C^\infty(M)$. It is easy to show that $f_1 : (\mathfrak{g}', d = 0) \to (\mathfrak{g}'', \delta_{\mathrm{Hoch}})$ is a chain map. Moreover, a version of the Kostant-Hochschild-Rosenberg theorem for smooth manifolds proved in [23] states that $f_1$ is a cohomology isomorphism. Unfortunately, $f_1$ *does not* commute with brackets. The following central statement of Kontsevich's approach to deformation quantization says that $f_1$ is, however, the linear part of an $L_\infty$-map:

**Formality Theorem.** The map $f_1$ can be extended to an $L_\infty$-homomorphism $f = (f_1, f_2, f_3, \ldots) : \mathfrak{g}' \to \mathfrak{g}''$.

The formality theorem implies that $\mathfrak{g}'$ and $\mathfrak{g}''$ are weakly equivalent in the category of $L_\infty$-algebras. In other words, the dg-Lie algebra of polydifferential operators is weakly equivalent to its cohomology. The 'formality' in the name of the theorem is justified by rational homotopy theory where formal algebras are algebras having the homotopy type of their cohomology.

Kontsevich's construction of higher $f_i$'s involves coefficients given as integrals over compactifications of certain configuration spaces. An independent approach of Tamarkin [43] based entirely on homological algebra uses a solution of the Deligne conjecture, see also an overview [18] containing references to original sources. □

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