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CONDITIONAL OSCILLATION OF HALF-LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. We show that the half-linear differential equation

$$(*)\qquad \big[r(t)\Phi(x')\big]' + \frac{s(t)}{t^p}\Phi(x) = 0$$

with α-periodic positive functions r, s is conditionally oscillatory, i.e., there exists a constant $K > 0$ such that $(*)$ with $\frac{\gamma s(t)}{t^p}$ instead of $\frac{s(t)}{t^p}$ is oscillatory for $\gamma > K$ and nonoscillatory for $\gamma < K$.

1. Introduction

In this paper we study oscillatory properties of the half-linear equation

$$(1.1)\qquad \big[r(t)\Phi(x')\big]' + s(t)\Phi(x) = 0\,, \quad \Phi(x) = x|x|^{p-2}\,,$$

where r and s are α-periodic ($\alpha > 0$) positive continuous functions and $p > 1$ is a real number conjugated with q, which means, that

$$\frac{1}{p} + \frac{1}{q} = 1\,.$$

Our research is motivated by the paper of K. M. Schmidt [2]. In that paper, the author studies oscillatory properties of the linear differential equation

$$(1.2)\qquad \big[r(t)x'\big]' + \frac{\gamma s(t)}{t^2}x = 0\,, \quad t > 0$$

where r, s are positive α-periodic functions and γ is a real parameter. The main result of [2] (after a minor reformulation) reads as follows.

Theorem 1.1. *Let*

$$K = \frac{1}{4}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{d\tau}{r}\Big)^{-1}\Big(\frac{1}{\alpha}\int_0^\alpha s\,d\tau\Big)^{-1}\,,$$

then (1.2) *is oscillatory for $\gamma > K$ and nonoscillatory for $\gamma < K$.*

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The result presented in the previous theorem is interesting from the following point of view. It is known that the Euler equation

$$x'' + \frac{\gamma}{t^2}x = 0 \tag{1.3}$$

is conditionally oscillatory (i.e. there exists a constant γ_0 such that equation is oscillatory for $\gamma > \gamma_0$ and nonoscillatory for $\gamma < \gamma_0$) with the oscillation constant $\gamma_0 = \frac{1}{4}$. Theorem 1.1 shows that constant coefficients in (1.3) can be replaced by periodic functions and resulting equation remains conditionally oscillatory.

In our paper we show that a similar situation we have for half-linear equations. The Euler type half-linear differential equation

$$\big[\Phi(x')\big]' + \frac{\gamma}{t^p}\Phi(x) = 0\,, \tag{1.4}$$

is conditionally oscillatory (with $\gamma_0 = \big(\frac{p-1}{p}\big)^p$). The main result of our paper shows that also in half-linear case constant coefficients can be replaced by periodic ones, i.e., the equation

$$\big[r(t)\Phi(x')\big]' + \frac{\gamma s(t)}{t^p}\Phi(x) = 0$$

with periodic functions r, s remains conditionally oscillatory.

The basic difference between linear and half-linear differential equations is the fact that the solution space of half-linear equations is not additive (but remains homogeneous). The missing additivity (more or less) induces further differences as the absence of Wronskian-type identity, transform theory or reduction of order formula. Despite that, many results from linear equations may be extended to (1.1) (see e.g. [1]).

2. Preliminary results

We start with elements of oscillation theory of half-linear equation (1.1). It is known, see e.g. [1], that the linear Sturmian theory extends verbatim to half-linear equations. In particular, we have the following statements.

Proposition 2.1 (Sturmian separation theorem)**.** *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1)*. Then any other solution of this equation, which is not proportional to x, has exactly one zero in (t_1, t_2).*

Proposition 2.2 (Sturmian comparison theorem)**.** *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1) *and suppose, that*

$$S(t) \geq s(t)\,, \quad r(t) \geq R(t) > 0 \tag{2.1}$$

for $t \in [t_1, t_2]$. Then any solution of the equation

$$\big[R(t)\Phi(x')\big]' + S(t)\Phi(x) = 0 \tag{2.2}$$

has a zero in (t_1, t_2) or it is a multiple of the solution x. The last possibility is excluded if one of the inequalities in (2.1) *is strict on a set of positive measure.*

If (2.1) are satisfied in a given interval I, then (2.2) is said to be the *majorant equation* of (1.1) on I and (1.1) is said to be the *minorant equation* of (2.2) on I.

Proposition 2.1 implies that (1.1) can be classified as *oscillatory* or *nonoscillatory*. Recall, that points $t_1, t_2 \in \mathbb{R}$ are said to be *conjugate* relative to equation (1.1), if there exists a nontrivial solution x of this equation, such that $x(t_1) = x(t_2) = 0$. Then, equation (1.1) is said to be *disconjugate* on an interval I, if this interval does not contain two points conjugate relative to equation (1.1). In the opposite case, equation (1.1) is said to be *conjugate on I*.

Now, let us recall the definition of oscillation and nonoscillation of equation (1.1) at zero and infinity.

Definition 1. Equation (1.1) is said to be *nonoscillatory at* 0, if there exists $\varepsilon > 0$ such that equation (1.1) is disconjugate on $[0, \varepsilon]$. In the opposite case, equation (1.1) is said to be *oscillatory at* 0.

Definition 2. Equation (1.1) is said to be *nonoscillatory at* ∞, if there exists $T_0 \in \mathbb{R}$ such that equation (1.1) is disconjugate on $[T_0, T]$ for every $T > T_0$. In the opposite case, equation (1.1) is said to be *oscillatory at* ∞.

If equation (1.1) is nonoscillatory at zero, then there exists a solution $v_{\max}$ of the Riccati equation

$$v' + s(t) + (p-1) r^{1-q}(t) |v|^q = 0 \tag{2.3}$$

associated to equation (1.1) such that $v_{\max}(t) > v(t)$ for t from a right neighbourhood of 0 for any other solution v of (2.3) which is defined in a right neighbourhood of 0. If equation (1.1) is nonoscillatory at infinity, then there exists a solution $v_{\min}$ of Riccati equation (2.3) such that $v_{\min}(t) < v(t)$ for any other solution for large t. We call $v_{\max}$ the *maximal solution* of (2.3) and $v_{\min}$ the *minimal solution* of (2.3).

Then, we define the principal solution of (1.1) at zero [infinity] as the nontrivial solution of the equation

$$x' = \Phi^{-1}\Big(\frac{v_{\max}(t)}{r(t)}\Big)\, x\,, \qquad \Big[x' = \Phi^{-1}\Big(\frac{v_{\min}(t)}{r(t)}\Big)\, x\Big]\,.$$

Now, let us briefly recall some basic facts concerning the half-linear Euler equation (1.4).

As mentioned in Introduction, equation (1.4) is conditionally oscillatory both at $t = 0$ and $t = \infty$ with the oscillation constant $\gamma_0 = \big(\frac{p-1}{p}\big)^p$ (see [1]).

Let $0 < \gamma < \gamma_0$, then (1.4) is not only nonoscillatory at 0 and ∞ but also disconjugated on $(0, \infty)$. Substituting $x(t) = t^\lambda$ into (1.4), we obtain an algebraic equation for λ

$$|\lambda|^p - \Phi(\lambda) + \frac{\gamma}{p-1} = 0\,.$$

and solving this equation, we find, that its roots $\lambda_2 < \lambda_1$ satisfy

$$0 < \lambda_2 < \frac{p-1}{p} < \lambda_1 < 1\,.$$

The principal solution of (1.4) at zero is t^{λ_1}, principal solution of (1.4) at infinity is t^{λ_2}, maximal and minimal solutions of the associated Riccati equation

$$w' + \frac{\gamma}{t^p} + (p-1)|w|^q = 0$$

are

$$w_{\max} = \Phi(\lambda_1)t^{1-p}, \quad w_{\min} = \Phi(\lambda_2)t^{1-p}\,,$$

respectively.

Using the change of independent variable $t = \mathrm{e}^s$, $s \in \mathbb{R}$, we convert equation (1.4) into the equation with constant coefficients

$$\big[\Phi(y')\big]' - (p-1)\Phi(y') + \gamma\Phi(y) = 0\,. \tag{2.4}$$

The corresponding Riccati equation is

$$u' - (p-1)u + (p-1)|u|^q + \gamma = 0\,. \tag{2.5}$$

Denote

$$F(u) := \gamma - (p-1)u + (p-1)|u|^q\,.$$

Following lemmas and theorems will be useful in the next section of our paper.

Lemma 2.1. *Consider the Riccati equation*

$$w' + \frac{\gamma}{t^p} + (p-1)|w|^p = 0\,, \quad \gamma < \Big(\frac{p-1}{p}\Big)^p \tag{2.6}$$

associated with the nonoscillatory Euler half-linear equation (1.4)*. If $w(T) \geq 1$ for some $T > 0$, then there exists $\tau \in \big(T\mathrm{e}^{-\int_1^\infty \frac{\mathrm{d}u}{F(u)}}, T\big)$ such that $w(\tau+) = \infty$.*

Proof. We convert equation (1.4) into equation (2.4) with associated Riccati equation (2.5). Suppose, by contradiction, that there exists a solution u of (2.5) extensible to $-\infty$ which satisfies $u(S) \geq 1$, where $S = \log T$, and integrate equation (2.5) on the interval $[s, S]$, where $S \in \mathbb{R}$ is fixed. Any solution, different from maximal and minimal ones (for which is $F(u) = 0$), is implicitly given by the formula

$$-\int_{u(s)}^{u(S)} \frac{\mathrm{d}u}{F(u)} = \int_{u(S)}^{u(s)} \frac{\mathrm{d}u}{F(u)} = S - s\,.$$

Hence

$$\int_1^\infty \frac{\mathrm{d}u}{F(u)} > S - s = \log T - \log t = -\log\frac{t}{T}\,,$$

i.e., $t > T\mathrm{e}^{-\int_1^\infty \frac{\mathrm{d}u}{F(u)}}$ which implies the existence of $\tau \in \big(T\mathrm{e}^{-\int_1^\infty \frac{\mathrm{d}u}{F(u)}}, T\big)$ such that $w(\tau+) = \infty$. □

Lemma 2.2. *Consider Riccati equation* (2.6) *associated with the nonoscillatory half-linear Euler equation* (1.4)*. If $v(T) \leq 0$ for some $T > 0$, then there exists $\tau \in \big(T, T\mathrm{e}^{\int_{-\infty}^0 \frac{\mathrm{d}u}{F(u)}}\big)$ such that $v(\tau-) = -\infty$.*

Proof. Similarly as in the Proof of Lemma 2.1, we use conversion to equations (2.4) and (2.5). Suppose the existence of a solution u of (2.5) extensible to ∞ that satisfies $u(S) \leq 0$ and integrate equation (2.5) on the interval $[S, s]$, where $S \in \mathbb{R}$ is fixed. Any solution, different from maximal and minimal ones, is implicitly

$$\int_{u(s)}^{u(S)} \frac{\mathrm{d}u}{F(u)} = s - S \,. \tag{2.7}$$

Again, this contradicts the existence of such a solution u, because the left hand side of equation (2.7) is bounded and the right hand side tends to infinity as $s \to \infty$. □

We finish this section with formulating a couple of lemmas and theorems without proofs (see e.g. [1]).

Lemma 2.3. *Consider a pair of equations*

$$v' + C(t) + (p-1)|v|^q = 0 \,, \tag{2.8}$$

$$w' + c(t) + (p-1)|w|^q = 0 \,, \tag{2.9}$$

where $C(t) \geq c(t) > 0$ for $t \in (a, b)$. If $\tau, T \in (a, b), \tau < T$, and a solution w of (2.9) *exists on $(\tau, T]$ and satisfies $w(\tau+) = \infty$, then there exists $\widetilde{\tau} \in [\tau, T)$ such that the solution v of* (2.8) *given by the initial condition $v(T) = w(T)$ satisfies $v(\widetilde{\tau}+) = \infty$.*

Lemma 2.4. *Consider a pair of equations* (2.8), (2.9). *If $\tau, T \in (a, b), T < \tau$, and a solution w of* (2.9) *exists on $[T, \tau)$ and satisfies $w(\tau-) = -\infty$, then there exists $\widetilde{\tau} \in (T, \tau]$ such that the solution v of* (2.8) *given by the initial condition $v(T) = w(T)$ satisfies $v(\widetilde{\tau}-) = -\infty$.*

Following theorems compare solutions of a pair of Riccati equations associated with nonoscillatory half-linear differential equations.

Theorem 2.1. *Consider a pair of half-linear differential equations*

$$\big[r(t)\Phi(x')\big]' + c(t)\Phi(x) = 0 \,, \tag{2.10}$$

$$\big[R(t)\Phi(y')\big]' + C(t)\Phi(y) = 0 \tag{2.11}$$

and suppose that (2.11) *is a Sturmian majorant of* (2.10) *for large t, i.e., there exists $T \in \mathbb{R}$ such that $0 < R(t) \leq r(t)$, $c(t) \leq C(t)$ for $t \in [T, \infty)$. Suppose that the majorant equation* (2.11) *is nonoscillatory and denote $v_{\min}, w_{\min}$ minimal solutions of*

$$v' + c(t) + (p-1)r^{1-q}(t)|v|^q = 0 \,, \tag{2.12}$$

$$w' + C(t) + (p-1)R^{1-q}(t)|w|^q = 0 \,, \tag{2.13}$$

respectively. Then $v_{\min}(t) \leq w_{\min}(t)$ for large t.

Theorem 2.2. *Consider a pair of half-linear differential equations* (2.10), (2.11) *and suppose that* (2.11) *is a Sturmian majorant of* (2.10) *for t from a right neighbourhood of* 0, *i.e., there exists $\varepsilon \in \mathbb{R}$ such that $0 < R(t) \leq r(t)$, $c(t) \leq$*

$C(t)$ for $t \in (0, \varepsilon]$. Suppose that the majorant equation (2.11) *is nonoscillatory and denote $v_{\max}$, $w_{\max}$ maximal solutions of* (2.12), (2.13)*, respectively. Then $v_{\max}(t) \geq w_{\max}(t)$ for t from a right neighbourhood of* 0.

3. Conditional oscillation of equations with periodic coefficients

The main result of our paper reads as follows.

Theorem 3.1. *Consider the equation*

$$\big[r(t)\Phi(x')\big]' + \gamma \frac{s(t)}{t^p}\Phi(x) = 0\,, \tag{3.1}$$

where r and s are α-periodic ($\alpha > 0$) positive continuous functions, and $\gamma \in \mathbb{R}$. Let

$$K := q^{-p}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{\mathrm{d}\tau}{r^{q-1}}\Big)^{1-p}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}. \tag{3.2}$$

Then equation (3.1) *is oscillatory if $\gamma > K$ and nonoscillatory if $\gamma < K$.*

Proof. Let $\gamma > K$. Suppose, by contradiction, that (3.1) is nonoscillatory. It means that the associated Riccati equation (2.3) has a solution, which exists on some interval $[T, \infty)$. Because r and s are α-periodic, positive and continuous, the equation

$$\big[r_{\max}\Phi(x')\big]' + \gamma \frac{s_{\min}}{t^p}\Phi(x) = 0\,,$$

where

$$r_{\max} = \max\big\{r(t), t \geq 0\big\}\,,$$

$$s_{\min} = \min\big\{s(t), t \geq 0\big\}\,.$$

is a minorant of (3.1), hence it is also nonoscillatory.

Denote $\mu := \frac{s_{\min}}{r_{\max}}$. Solving the Euler-type equation

$$\big[\Phi(x')\big]' + \gamma \frac{\mu}{t^p}\Phi(x) = 0\,, \tag{3.3}$$

with $\mu\gamma \leq \big(\frac{p-1}{p}\big)^p$ we find, that the principal solutions at zero and infinity are t^{λ_1}, t^{λ_2}, respectively, where $0 < \lambda_2 < \lambda_1 < 1$ are roots of the equation

$$|\lambda|^p - \Phi(\lambda) + \gamma\frac{\mu}{p-1} = 0\,,$$

see Section 2.

Denote the maximal solution near $t = 0$ of the Riccati equation associated to equation (3.3) by

$$v_{\max}(t) := t^{1-p}\Phi(\lambda_1)\,,$$

and the minimal solution for large t by

$$v_{\min}(t) := t^{1-p}\Phi(\lambda_2)\,.$$

Introducing the function $w = \frac{r\Phi(x')}{\Phi(x)}$, we may transform equation (3.1) to the Riccati equation

$$w' + \gamma \frac{s(t)}{t^p} + (p-1) r^{1-q}(t)|w|^q = 0$$

with the maximal solution (at $t = 0$) $w_{\max}$ and the minimal solution (at $t = \infty$) $w_{\min}$ and denote

$$\zeta(t) := -wt^{p-1}\,, \quad \xi(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \zeta(\tau)\,\mathrm{d}\tau\,. \tag{3.4}$$

First, suppose that there exists $t_n \to \infty$ such that $\zeta(t_n) \le -1$, i.e.,

$$w(t_n) = -t_n^{1-p}\zeta(t_n) \ge t_n^{1-p} > \Phi(\lambda_1) t_n^{1-p} = v_{\max}(t_n) \ge w_{\max}(t_n)\,.$$

Consider the solution of (3.3) given by the initial condition $v(t_n) = t_n^{1-p}$, i.e.,

$$v(t_n) - v_{\max}(t_n) = [1 - \Phi(\lambda_1)] t_n^{1-p}\,.$$

Then, by Lemma 2.1, there exists $\tau_n \to \infty, \tau_n < t_n$, such that $v(\tau_n+) = \infty$. But this means, by Lemma 2.3, that $w(\widetilde{\tau}_n+) = \infty$ for some $\tau_n \le \widetilde{\tau}_n < t_n$, which is a contradiction.

Next, suppose that there exists a sequence $\hat{t}_n \to \infty$ such that $\zeta(\hat{t}_n) \ge 0$, i.e.,

$$w(\hat{t}_n) \le 0 < v_{\min}(\hat{t}_n) = \Phi(\lambda_2)\hat{t}_n^{1-p} \le w_{\min}(\hat{t}_n)\,.$$

This means (from Lemma 2.2 and Lemma 2.4), that there exists $\widehat{\tau}_n > \hat{t}_n$ such that $w(\widehat{\tau}_n-) = \infty$, which contradicts the fact, that $w(t)$ exists on $[T, \infty)$.

Hence, there exists $T_0 > T$ such that

$$v_{\min} = \Phi(\lambda_2) t^{1-p} \le w \le \Phi(\lambda_1) t^{1-p} = v_{\max}$$

for $t \ge T_0$. Multiplying the last inequality by $-t^{p-1}$, we obtain

$$0 > -\Phi(\lambda_2) \ge \zeta(t) \ge -\Phi(\lambda_1) > -1\,.$$

Let us denote

$$A := (p-1)\Big(\frac{p}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big)^{-\frac{1}{q}}\,, \quad B := \big|\xi(t)\big|\Big(\frac{p}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big)^{\frac{1}{q}}\,.$$

We have

$$\begin{aligned}\zeta'(t) &= \big[-w(t)t^{p-1}\big]' = -\big[w'(t)t^{p-1} + (p-1)w(t)t^{p-2}\big]\\ &= \frac{1}{t}\Big[(p-1)\zeta(t) + s(t)\gamma + (p-1)\frac{|\zeta(t)|^q}{r^{q-1}(t)}\Big]\,.\end{aligned}$$

Next, for $t \ge T_0$

$$\begin{aligned}\int_t^{t+\alpha} \big|\zeta'(\tau)\big|\,\mathrm{d}\tau &\le \frac{1}{t}\int_t^{t+\alpha} \Big|(p-1)\zeta(\tau)+\gamma s(\tau)+(p-1)\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\Big|\,\mathrm{d}\tau \\ &\le \frac{1}{t}\int_t^{t+\alpha} \Big[(p-1)+\gamma s(\tau)+\frac{p-1}{r^{q-1}(\tau)}\Big]\,\mathrm{d}\tau = \frac{C}{t}\,,\end{aligned} \tag{3.5}$$

where

$$C := \int_t^{t+\alpha} \Big[(p-1)+\gamma s(\tau)+\frac{p-1}{r^{q-1}(\tau)}\Big]\,\mathrm{d}\tau\,.$$

Hence, for every $t > T_0$ and $\tau_1, \tau_2 \in [t, t+\alpha]$ we have

$$\big|\zeta(\tau_1)-\zeta(\tau_2)\big| \le \int_t^{t+\alpha} \big|\zeta'(\tau)\big|\,\mathrm{d}\tau \le \frac{C}{t}\,.$$

Due to the continuity of the function ζ, there exists $\tau_0 \in [t, t+\alpha]$ such that

$$\xi(t) = \zeta(\tau_0) \quad \Rightarrow \quad |\zeta(\tau)-\xi(t)| \le \frac{C}{t}\,,$$

where $\tau \in [t, t+\alpha]$.

Now, we estimate the value of the function ξ'.

$$\begin{aligned}\xi'(t) &= \frac{1}{\alpha}\big[\zeta(t+\alpha)-\zeta(t)\big] = \frac{1}{\alpha}\int_t^{t+\alpha} \zeta'(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{\tau}\Big[(p-1)\zeta(\tau)+s(\tau)\gamma+(p-1)\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\Big]\,\mathrm{d}\tau \\ &\ge \frac{1}{t+\alpha}\Big[(p-1)\xi(t)+\frac{\gamma}{\alpha}\int_0^{\alpha} s(\tau)\,\mathrm{d}\tau+\frac{p-1}{\alpha}\int_t^{t+\alpha}\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big] \\ &\quad + \frac{(p-1)\alpha}{t(t+\alpha)}\xi(t) \\ &= \frac{1}{t+\alpha}\Big[(p-1)\xi(t)+\frac{A^p}{p}+\frac{B^q}{q}+\frac{\gamma}{\alpha}\int_0^{\alpha} s(\tau)\,\mathrm{d}\tau-\frac{A^p}{p}+\frac{(p-1)\alpha}{t}\xi(t) \\ &\quad + \frac{p-1}{\alpha}\int_t^{t+\alpha}\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau-\frac{B^q}{q}\Big]\,.\end{aligned}$$

Denote

$$
(3.6)\qquad
\begin{aligned}
X &:= (p-1)\xi(t) + \frac{A^p}{p} + \frac{B^q}{q}\,,\\
Y &:= \frac{\gamma}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} + \frac{(p-1)\alpha}{t}\xi(t)\,,\\
Z &:= \frac{p-1}{\alpha}\int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\,.
\end{aligned}
$$

Next, we estimate quantities appearing in (3.6). It follows from Young's inequality, that $\frac{A^p}{p} + \frac{B^q}{q} - AB \ge 0$, so (using $\xi \le 0$)

$$
X = \frac{A^p}{p} + \frac{B^q}{q} + (p-1)\xi = \frac{A^p}{p} + \frac{B^q}{q} - (p-1)|\xi| = \frac{A^p}{p} + \frac{B^q}{q} - AB \ge 0\,.
$$

As for the term Y, we denote

$$
K_\gamma := Y = \frac{\gamma}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} + \frac{(p-1)\alpha}{t}\xi(t)
$$

and show, that $K_\gamma \ge 0$.

$$
\begin{aligned}
K_\gamma &= \frac{\gamma}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau - \frac{(p-1)^p}{p}\Big(\frac{p}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}} + \frac{(p-1)\alpha}{t}\xi\\
&= \frac{\gamma}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau - q^{-p}\frac{\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau}{\Big(\frac{1}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{\frac{p}{q}}\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau} + \frac{(p-1)\alpha}{t}\frac{1}{\alpha}\int_t^{t+\alpha}\zeta\,\mathrm{d}\tau\\
&\ge \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big[\gamma - q^{-p}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}\Big] - \frac{p-1}{t}\\
&= \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau(\gamma - K) - \frac{p-1}{t} > 0\,,
\end{aligned}
$$

for $t \ge T_1$, because $\gamma > K$.

Finally, to estimate the last expression in (3.6), let us introduce the function

$$
F(x,y) := \begin{cases} \frac{|x|^q - |y|^q}{|x|-|y|}\,, & x \ne y, [x,y] \in M\,,\\ q\Phi^{-1}(|x|)\,, & x = y\,, \end{cases}
$$

where $M := [-1,0] \times [-1,0]$.

Then, we have

$$Z = \frac{p-1}{\alpha}\int\limits_t^{t+\alpha}\frac{|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau - \frac{B^q}{q} = \frac{p-1}{\alpha}\int\limits_t^{t+\alpha}\frac{|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau - \frac{|\xi|^q}{q}\frac{q(p-1)}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{r^{q-1}}\,\mathrm{d}\tau$$
$$= -\frac{p-1}{\alpha}\int\limits_t^{t+\alpha}\frac{|\xi|^q-|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau \geq -\frac{p-1}{\alpha}\int\limits_t^{t+\alpha}|\xi-\zeta|\frac{|\xi|^q-|\zeta|^q}{|\xi|-|\zeta|}\frac{1}{r^{q-1}}\,\mathrm{d}\tau$$
$$\geq -\frac{(p-1)CD}{\alpha t}\int\limits_0^{\alpha}\frac{1}{r^{q-1}}\,\mathrm{d}\tau\,,$$

where we have used (3.5) and $D := \max_M F(\xi,\zeta) < \infty$.

Altogether for $t \geq T := \max\{T_0, T_1, T_2\}$, where

$$T_2 := \frac{2CD(p-1)}{\alpha K_\gamma}\int\limits_0^{\alpha}\frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\,,$$

we obtain

$$\xi'(t) \geq \frac{1}{t+\alpha}\Big[K_\gamma - \frac{CD(p-1)}{\alpha t}\int\limits_0^{\alpha}\frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big]$$
$$\geq \frac{1}{t+\alpha}\Big(K_\gamma - \frac{K_\gamma}{2}\Big) = \frac{K_\gamma}{2(t+\alpha)}\,,$$

which means, that

$$\xi(t) \geq \xi(T) + \frac{K_\gamma}{2}\log\frac{t+\alpha}{T+\alpha} \to \infty \quad \text{as} \quad t\to\infty\,,$$

which is a contradiction. Thus, equation (3.1) is oscillatory for $\gamma > K$.

In the next part of the proof, we show, that (3.1) is nonoscillatory for $\gamma < K$. Denote $\mu := \frac{s_{\max}}{r_{\min}}$. Equation (3.3) is now a majorant equation of equation (3.1). We show that the majorant equation (3.3) is nonoscillatory, which implies, that equation (3.1) is also nonoscillatory.

Denote

$$\xi_0 := -\Big[\frac{p}{\alpha(p-1)}\int\limits_0^{\alpha}\frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big]^{1-p}.$$

We will show that there exists T such that $\xi(t)$ defined by (3.4) in the previous part of the proof satisfies $\xi(t) \leq \xi_0$, $(t \geq T)$. By contradiction, assume that

$$t_0 := \sup\{t \geq T, \xi(\tau) \leq \xi_0, \tau \in [T,t]\} < \infty\,.$$

Then $\xi'(t_0) \geq 0$ and $\xi(t_0) = \xi_0$. We estimate the value of $\xi'(t_0)$. We obtain

$$\begin{aligned}
\xi'(t_0) &= \frac{1}{\alpha} \int\limits_{t_0}^{t_0+\alpha} \frac{1}{\tau}\Big[(p-1)\zeta(\tau) + \gamma s(\tau) + (p-1)\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\Big]\,\mathrm{d}\tau \\
&\leq \frac{1}{t_0}\Big[(p-1)\xi(t_0) + \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s(\tau)\,\mathrm{d}\tau + \frac{p-1}{\alpha}\int\limits_{t_0}^{t_0+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big] \\
&\quad - \frac{(p-1)\alpha}{t_0(t_0+\alpha)}\xi(t_0) \\
&= \frac{1}{t_0}\Big[(p-1)\xi(t_0) + \frac{A^p}{p} + \frac{B^q}{q} + \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} - \frac{(p-1)\alpha}{t_0+\alpha}\xi(t_0) \\
&\quad + \frac{p-1}{\alpha}\int\limits_{t_0}^{t_0+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\Big].
\end{aligned}$$

Again, we denote

$$\begin{aligned}
X &:= (p-1)\xi(t_0) + \frac{A^p}{p} + \frac{B^q}{q}\,, \\
Y &:= \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} - \frac{(p-1)\alpha}{t+\alpha}\xi(t_0)\,, \qquad (3.7)\\
Z &:= \frac{p-1}{\alpha}\int\limits_{t_0}^{t_0+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\,,
\end{aligned}$$

with A, B given by

$$A := (p-1)\Big(\frac{p}{\alpha}\int\limits_0^{\alpha} \frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big)^{-\frac{1}{q}}, \quad B := |\xi(t_0)|\Big(\frac{p}{\alpha}\int\limits_0^{\alpha} \frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big)^{\frac{1}{q}},$$

and we estimate quantities appearing in (3.7).

Since $A^p = B^q$, we have

$$\frac{A^p}{p} + \frac{B^q}{q} = A^p\Big(\frac{1}{p} + \frac{1}{q}\Big) = A^{1+\frac{p}{q}} = A(B^q)^{\frac{1}{q}} = AB = -(p-1)\xi\,,$$

which means, that $X = 0$ in (3.7).

Next, we denote

$$-K_\gamma := Y = \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s\,\mathrm{d}\tau - \frac{A^p}{p} - \frac{(p-1)\alpha}{t_0+\alpha}\xi\,.$$

Then

$$
\begin{aligned}
-K_\gamma &= \frac{\gamma}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau - \frac{(p-1)^p}{p}\Big(\frac{p}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}} - \frac{(p-1)\alpha}{t_0+\alpha}\xi \\
&\le \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big[\gamma - q^{-p}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}\Big] + \frac{p-1}{t_0+\alpha} \\
&= \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau(\gamma - K) + \frac{p-1}{t_0+\alpha} < 0\,,
\end{aligned}
$$

because $\gamma < K$, i.e., $K_\gamma > 0$.

Finally, similarly as in the previous computation, we have

$$
\begin{aligned}
Z &= \frac{p-1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau - \frac{B^q}{q} \\
&= \frac{p-1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau - \frac{|\xi|^q}{q}\,\frac{q(p-1)}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{1}{r^{q-1}}\,\mathrm{d}\tau \\
&= \frac{p-1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{|\zeta|^q - |\xi|^q}{r^{q-1}}\,\mathrm{d}\tau \le \frac{CD(p-1)}{\alpha t_0}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\,.
\end{aligned}
$$

Altogether for $t_0 \ge T := \max\{T_0, T_1, T_2\}$, where T_0, T_1, T_2 are defined earlier, we obtain

$$
\begin{aligned}
\xi'(t_0) &\le \frac{1}{t_0}\Big[-K_\gamma + \frac{CD(p-1)}{\alpha t_0}\int_0^\alpha \frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big] \\
&\le \frac{1}{t_0}\Big(-K_\gamma + \frac{K_\gamma}{2}\Big) = -\frac{K_\gamma}{2t_0} < 0\,,
\end{aligned}
$$

which is a contradiction.

□

Remark 1. It is still an open problem to decide whether equation (3.1) is oscillatory or not in the case, $\gamma = K$, with K given by (3.2).

Remark 2. For $r(t) \equiv 1 \equiv s(t)$, equation (3.1) reduces to Euler equation (1.4) and our oscillation constant K defined by (3.2) reduces to the well known constant $\gamma_0 = \big(\frac{p-1}{p}\big)^p$.

References

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