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ON RINGS ALL OF WHOSE MODULES ARE RETRACTABLE

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Abstract. Let R be a ring. A right R-module M is said to be *retractable* if $\operatorname{Hom}_R(M,N)\neq 0$ whenever N is a non-zero submodule of M. The goal of this article is to investigate a ring R for which every right R-module is retractable. Such a ring will be called right *mod-retractable*. We proved that
(1) The ring $\prod_{i\in\mathcal{I}} R_i$ is right mod-retractable if and only if each R_i is a right mod-retractable ring for each $i\in\mathcal{I}$, where $\mathcal{I}$ is an arbitrary finite set.
(2) If $R[x]$ is a mod-retractable ring then R is a mod-retractable ring.

Throughout this paper, R is an associative ring with unity and all modules are unital right R-modules.

Khuri [1] introduced the notion of retractable modules and gave some results for non-singular retractable modules when the endomorphism ring is (quasi-)continuous. For retractable modules, we direct the reader to the excellent papers [1],[2], [3] and [4] for nice introduction to this topic in the literature.

Let M be an R-module. M is said to be a *retractable module* if $\operatorname{Hom}_R(M,N)\neq 0$ whenever N is a non-zero submodule of M ([1]).
We give some examples.

(i) Free modules and semisimple modules are retractable.

(ii) Any direct sum of $\mathbb{Z}_{p^i}$ is retractable, where p is a prime number.

(iii) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not retractable.

(iv) Let R be an integral domain with quotient ring F and $F\neq R$. Then $R\oplus F$ is a retractable R-module, because $\operatorname{End}_R(M)=\begin{pmatrix} F & F\\ 0 & R\end{pmatrix}$.

(v) Assume that M_R is a finitely generated semisimple right R-module. Then the module M_R is retractable and $\operatorname{End}_R(M)$ is semisimple artinian By [3, Corollary 2.2]

(vi) Take an R-module M. Let $0\neq N\leq R\oplus M$; take $0\neq n\in N$ and construct the map $\varphi\colon R\oplus M\to N$ by $\varphi(1)=n$ and $\varphi(m)=0$ for all $m\in M$. Since $0\neq\varphi\in\operatorname{Hom}_R(R\oplus M,N)$, we have $\operatorname{Hom}_R(R\oplus M,N)\neq 0$, thus $R\oplus M$ is retractable.

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In this note, we deal with some ring extensions of a ring R for which every (right) R-module is retractable. Hence, such a ring will be called right *mod-retractable*. This will avoid a conflict of nomenclature with the existing concept of retractability. The following examples show that this definition is not meaningless.

We take $\mathbb{Z}$-modules $M = \mathbb{Q}$ and $N = \mathbb{Z}$. Note that $\mathbb{Q}$ is a divisible group, so every its homomorphic image is a divisible group as well. Since the only divisible subgroup of $\mathbb{Z}$ is 0, the only homomorphism of $\mathbb{Q}$ into $\mathbb{Z}$ is the zero homomorphism.

Let R, S be two rings and M be an R-S-bimodule. Then we consider the ring $R' = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Let $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $K = eR'$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We claim that $\mathrm{Hom}_{R'}(K, I) = 0$. Note that $I \nsubseteq K$. Let $f \in \mathrm{Hom}_{R'}(K, I)$. Then $f(K) = f(eR) = f(eeR) = f(e)eR = f(e)K \subseteq IK = 0$, i.e., R' is retractable.

A ring R is called (*finitely*) *mod-retractable* if all (finitely generated) right R-modules are retractable.

Example 1. (i) Any semisimple artinian ring is mod-retractable.

(ii) $\mathbb{Z}$ is a finitely mod-retractable ring but is not mod-retractable ring.

We start the Morita invariant property for (finitely) mod-retractable rings.

Theorem 2. (*Finite*) *mod-retractability is Morita invariant.*

Proof. Let R and S be two Morita equivalent rings. Assume that $f\colon$ Mod-$R \to$ Mod-S and g: Mod-$S \to$ Mod-R are two category equivalences. Let M be a retractable R-module. Then M is a retractable object in Mod-R. Let $0 \neq N \leq f(M)$. Then $\mathrm{Hom}_R\big(M, g(N)\big) \neq 0$ since $g(N)$ is isomorphic to a submodule of M. Thus, we have $0 \neq \mathrm{Hom}_S\big(f(M), fg(N)\big) \cong \mathrm{Hom}_S\big(f(M), N\big)$. This follows that $f(M)$ is a retractable object in Mod-S. □

Let R be a ring, n a positive integer and the ring $\mathbb{M}_n(R)$ of all $n \times n$ matrices with entries in R.

Corollary 3. *If R is* (*finitely*) *mod-retractable, then $\mathbb{M}_n(R)$ is* (*finitely*) *mod-retractable.*

Proof. By Theorem 2. □

Theorem 4. *The class of* (*finite*) *mod-retractable rings is closed under taking homomorphic images.*

Proof. Suppose R is a (finite) mod-retractable ring. It is well-known that

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R/I}(M, N)$$

for each ideal I of R and $M, N \in$ Mod-R/I. Now the proof is clear. □

Recall that a module M is said to be *e-retractable* if, for all every essential submodule N of M, $\mathrm{Hom}_R(M, N) \neq 0$ (see [1]).

Lemma 5. *The following statements are equivalent for a ring R.*

(1) *R is* (*finitely*) *mod-retractable.*

(2) *Every* (*finitely generated*) *R-module M is e-retractable.*

(3) *For every (finitely generated) R-module M and $N \leq M$,* $\mathrm{Hom}_R(M,N)=0$ *if and only if* $\mathrm{Hom}_R\big(M,E(N)\big)=0$*, where $E(N)$ is an injective hull of N.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.
(2) $\Rightarrow$ (1) Let M be a (finitely generated) right R-module and N be a submodule of M. Since $E(N)$ is an injective module, we extend the inclusion $N \subseteq E(N)$ to the map $\alpha\colon M \to E(N)$. This implies that $\alpha(N)=N$. Thus $\alpha(M)\cap N=N$. Since $N \leq_e N$, we have $N \leq_e \alpha(M)$. This implies that $\mathrm{Hom}_R\big(\alpha(M),N\big)\neq 0$. Moreover, for $K=\mathrm{Ker}(\alpha)$,

$$\mathrm{Hom}_R\big(\alpha(M),N\big)=\mathrm{Hom}_R\big(M/K,N\big)\subseteq \mathrm{Hom}_R(M,N)\,.$$

As such, $\mathrm{Hom}_R(M,N)\neq 0$.
(3) $\Rightarrow$ (2) Let N be an essential submodule of a (finitely generated) right R-module M. Then $E(N)\cong E(M)$. By (3), we can obtain that $\mathrm{Hom}_R(M,N)=0$, and so $\mathrm{Hom}_R\big(M,E(N)\big)=0$. Hence $\mathrm{Hom}_R\big(M,E(M)\big)=0$. □

By Example 1, a commutative ring need not be retractable.

Theorem 6. *Any ring that is Morita equivalent to a commutative ring is finitely mod-retractable.*

Proof. By Theorem 2, it suffices to prove the claim for a commutative ring R. Let M be a finitely generated R-module and $N \leq M$. Assume that $\mathrm{Hom}_R\big(M,E(N)\big)\neq 0$, and take $0\neq\alpha\in\mathrm{Hom}_R\big(M,E(N)\big)$. Since M is a finitely generated R-module, we can write $\alpha(M)$ as follows (where the sum is not necessarily direct): $\alpha(M)=Rm_1+Rm_2+\ldots Rm_n$ with $m_i\in E(N)$, $1\leq i\leq n$. Since N is essential in $E(N)$, thus there exists $r\in R$ such that $rm_i\in N$ for all i and $r\alpha(M)\neq 0$. Now we can define $0\neq\beta\colon\alpha(M)\to N$ such that $\beta(m_i)=rm_i$ for all $1\leq i\leq n$. Thus $0\neq\beta\alpha\in\mathrm{Hom}_R(M,N)$. This implies that $\mathrm{Hom}_R(M,N)\neq 0$. By Lemma 5, the R-module M is retractable. □

Example 7. Let R be a commutative artin ring. Assume that a ring S is Morita equivalent to R. First, note that every S-module is retractable and has a maximal submodule. We consider an S-module M. Let N be a maximal submodule of M. Hence we have a simple submodule K of N. Then there exits an S-homomorphism $f\colon M\to E(K)$, where $E(K)$ is the injective hull of K. Clearly, $f(M)$ is a finitely generated S-module. By Theorem 6, $f(M)$ is a retractable S-module and so M is a retractable S-module.

Example 7 shows that the class of right mod-retractable rings is not closed under direct sums.

Theorem 8. *The ring $\prod_{i\in\mathcal{I}}R_i$ is right mod-retractable if and only if each R_i is a right mod-retractable ring for each $i\in\mathcal{I}$, where $\mathcal{I}$ is an arbitrary finite set.*

Proof. :$\Rightarrow$ Indeed, R_i is a homomorphic image of $\prod_{i\in\mathcal{I}}R_i$. So the result follows from Theorem 4.
$\Leftarrow$: Let each e_i denote the unit element of R_i for all $i\in\mathcal{I}$. A module M over $\prod_{i\in\mathcal{I}}R_i$ can be written as set direct product $\prod_{i\in\mathcal{I}}M_i$, where ${M_i}_{R_i}=Me_i$ and external operation defined as $(r_i)_{i\in\mathcal{I}}(m_i)_{i\in\mathcal{I}}=(r_im_i)_{i\in\mathcal{I}}$. Thus retractability of M

is given by retractability of each $M_{i i\in\mathcal{I}}$. But, since each R_i is mod-retractable, this condition is satisfied. $\square$

Corollary 9. *The class of all right mod-retractable rings is closed under taking finite direct products.*

Proof. By Theorem 8. $\square$

Giving a ring R, $R[X]$ denotes the polynomial ring with X a set of commuting indeterminate over R. If $X = \{x\}$, then we use $R[x]$ in place of $R[\{x\}]$.

Theorem 10. *If $R[x]$ is a mod-retractable ring then R is a mod-retractable ring.*

Proof. Since $R \cong R[x]/R[x]x$, the result is clear from Theorem 4. $\square$

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