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THEORY OF RAPID VARIATION ON TIME SCALES WITH APPLICATIONS TO DYNAMIC EQUATIONS

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Abstract. In the first part of this paper we establish the theory of rapid variation on time scales, which corresponds to existing theory from continuous and discrete cases. We introduce two definitions of rapid variation on time scales. We will study their properties and then show the relation between them. In the second part of this paper, we establish necessary and sufficient conditions for all positive solutions of the second order half-linear dynamic equations on time scales to be rapidly varying. Note that these results are new even for the linear (dynamic) case and for the half-linear discrete case. In the third part of this paper we give a complete characterization of all positive solutions of linear dynamic equations and of all positive decreasing solutions of half-linear dynamic equations with respect to their regularly or rapidly varying behavior. The paper is finished by concluding comments and open problems of these themes.

1. Introduction

Recall that a measurable function $f\colon [a,\infty)\to(0,\infty)$ of a real variable is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ if it satisfies

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\begin{cases}\infty \quad \text{resp.} \quad 0 \quad \text{for } \lambda>1\,,\\ 0 \quad \text{resp.} \quad \infty \quad \text{for } 0<\lambda<1\,;\end{cases} \tag{1}$$

we write $f\in\mathcal{RPV}_{\mathbb{R}}(\infty)$, resp. $f\in\mathcal{RPV}_{\mathbb{R}}(-\infty)$. Note that it is easy to show that in relation (1) it is not necessary to include both cases $\lambda>1$ and $0<\lambda<1$, i.e.,

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\infty \ \ (\text{resp.}\ \ 0)\,,\quad \lambda>1 \ \Leftrightarrow\ \lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=0 \ \ (\text{resp.}\ \ \infty)\,,\ \lambda\in(0,1)\,.$$

For more information about the rapid variation on $\mathbb{R}$, see for example [1] and references therein. In [17], the concept of rapidly varying sequences was introduced in the following way. Let $[u]$ denote the integer part of u. A positive sequence $\{f_k\}$,

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$k \in \{a, a+1, \dots\} \subset \mathbb{Z}$ is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ if it satisfies

$$\lim_{k\to\infty} \frac{f_{[\lambda k]}}{f_k} = \begin{cases} \infty & \text{resp. } 0 \quad \text{for } \lambda > 1\,, \\ 0 & \text{resp. } \infty \quad \text{for } 0 < \lambda < 1\,; \end{cases} \tag{2}$$

we write $f \in \mathcal{RPV}_{\mathbb{Z}}(\infty)$, resp. $f \in \mathcal{RPV}_{\mathbb{Z}}(-\infty)$. Similarly, as in the previous case, one can show that

$$\lim_{k\to\infty} \frac{f_{[\lambda k]}}{f_k} = \infty \;\; (\text{resp. } 0)\,, \quad \lambda > 1 \;\;\Leftrightarrow\;\; \lim_{k\to\infty} \frac{f_{[\lambda k]}}{f_k} = 0 \;\; (\text{resp. } \infty)\,, \quad \lambda \in (0,1)\,.$$

Note that these types of definitions of rapidly varying functions (1) and rapidly varying sequences (2), which include a parameter λ, correspond to the classical Karamata type definition of regularly varying functions, see [1, 8, 11, 12, 13, 25] and references therein. In [17] it was shown that if a positive sequence $\{f_k\}$ has the property that Δf_k increases, then $f \in \mathcal{RPV}_{\mathbb{Z}}(-\infty)$ if and only if

$$\lim_{k\to\infty} \frac{k\Delta f_k}{f_k} = -\infty\,. \tag{3}$$

This result shows that under certain conditions there exists an alternative (in some cases more practical) possibility, how to define rapidly varying sequences (resp. functions). For further reading of rapid and regular variation in discrete case we refer, e.g., to [3, 4, 5, 7, 17, 15, 16, 26] and the references therein.

In this paper we extend the theory of rapid variation to time scales (i.e., considered functions are defined on nonempty closed subsets of $\mathbb{R}$, see [2, 9]). We work with two definitions of rapid variation, precisely, with a Karamata type definition and with a definition using Δ-derivative, where the latter one is motivated by (3). Our aim is to show properties of rapidly varying functions and relation between both mentioned definitions. The theory of rapid variation on time scales automatically holds for the continuous and discrete cases, moreover, at the same time, the theory works also on other time scales which may be different from the "classical" ones. Finally, note that the theory of rapid variation on time scales naturally extends and completes our knowledge concerning the theory of regular variation on time scales, which was earlier studied in [20, 24, 23].

As an application, we study asymptotic properties of solutions of the second order half-linear dynamic equation

$$[\Phi(x^\Delta)]^\Delta - p(t)\Phi(x^\sigma) = 0 \tag{4}$$

on a time scale, where $p > 0$ is an rd-continuous function, and $\Phi(x) = |x|^{\alpha-1}\operatorname{sgn} x$, $\alpha > 1$. Note that this results automatically hold for the second order linear dynamic equation

$$x^{\Delta\Delta} - p(t)x^\sigma = 0 \tag{5}$$

as a special case of equation (4) (when $\alpha = 2$).

In this paper, the time scale $\mathbb{T}$ is assumed to be unbounded above, $\min \mathbb{T} = a$ (with $a > 0$) and the graininess satisfies $\mu(t) = o(t)$. This condition will be discussed at the end of the paper. As we show, if we want to obtain a reasonable theory, we cannot omit this additional requirement on the graininess.

2. Preliminaries

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}$, σ, ρ, f^σ, μ, f^Δ, and $\int_a^b f^\Delta(s)\,\Delta s$ stand for the time scale, forward jump operator, backward jump operator, $f\circ\sigma$, graininess, Δ-derivative of f, and Δ-integral of f from a to b. See [9], which is the initiating paper of the time scale theory, and [2] containing a lot of information on time scale calculus.

In [20], the concept of regular variation on $\mathbb{T}$ was introduced in the following way. A measurable function $f\colon \mathbb{T}\to(0,\infty)$ is said to be *regularly varying of index* ϑ, $\vartheta\in\mathbb{R}$, if there exists a positive rd-continuously Δ-differentiable function g satisfying

$$f(t)\sim Cg(t) \quad\text{and}\quad \lim_{t\to\infty}\frac{tg^\Delta(t)}{g(t)}=\vartheta\,, \tag{6}$$

C being a positive constant; we write $f\in\mathcal{RV}_\mathbb{T}(\vartheta)$. If $\vartheta=0$, then f is said to be *slowly varying*; we write $f\in\mathcal{SV}_\mathbb{T}$. Moreover, the function g is said to be *normalized regularly varying of index* ϑ; we write $g\in\mathcal{NRV}_\mathbb{T}(\vartheta)$. If $\vartheta=0$, then g is said to be *normalized slowly varying*; we write $g\in\mathcal{NSV}_\mathbb{T}$. In [24], we introduced a Karamata type definition of regularly varying function on time scales and developed and enriched the existing theory with new statements (the embedding theorem, a relation between previous and Karamata type definition, etc.). Here is the Karamata type definition. Let $f\colon\mathbb{T}\to(0,\infty)$ be a measurable function satisfying

$$\lim_{t\to\infty}\frac{f(\tau(\lambda t))}{f(t)}=\lambda^\vartheta \tag{7}$$

uniformly on each compact λ-set in $(0,\infty)$, where $\tau\colon\mathbb{R}\to\mathbb{T}$ is defined as $\tau(t)=\max\{s\in\mathbb{T}\colon s\le t\}$. Then f is said to be *regularly varying of index* ϑ ($\vartheta\in\mathbb{R}$) *in the sense of Karamata*; we write $f\in\mathcal{KRV}_\mathbb{T}(\vartheta)$. If $\vartheta=0$, then f is said to be *slowly varying in the sense of Karamata*; we write $f\in\mathcal{KSV}_\mathbb{T}$. For further information about theory of regular variation on $\mathbb{T}$ see, e.g., [20, 22, 24, 23].

3. Theory of rapid variation on time scales

In this section we establish the theory of rapid variation on time scales. Recall that throughout the paper, $\mathbb{T}$ is assumed to be unbounded above, $\min\mathbb{T}=a$ (with $a>0$) and $\mu(t)=o(t)$.

Definition 1. Let c, d be the real constants such that $0<c\le d$ and $\vartheta\in\mathbb{R}$. A measurable function $f\colon\mathbb{T}\to(0,\infty)$ is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ if there exist function $\varphi\colon\mathbb{T}\to(0,\infty)$ satisfying $\varphi\in\mathcal{RV}_\mathbb{T}(\vartheta)$ or $c\le\varphi(t)\le d$ for large t and a positive rd-continuously Δ-differentiable function ω such that $f(t)=\varphi(t)\omega(t)$ and

$$\lim_{t\to\infty}\frac{t\omega^\Delta(t)}{\omega(t)}=\infty\,,\quad\text{resp.}\quad \lim_{t\to\infty}\frac{t\omega^\Delta(t)}{\omega(t)}=-\infty\,; \tag{8}$$

we write $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$, resp. $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. Moreover, the function ω is said to be *normalized rapidly varying of index* ∞, resp. *normalized rapidly varying of index* $-\infty$; we write $\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$, resp. $\omega \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$.

Proposition 1.

(i) *It holds* $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ *if and only if* $1/f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$.

(ii) *Let* $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. *Then for every* $\vartheta \in [0,\infty)$ *the function* $f(t)/t^{\vartheta}$ *is increasing for large* t *and* $\lim_{t\to\infty} f(t)/t^{\vartheta} = \infty$.

(iii) *Let* $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. *Then for every* $\vartheta \in [0,\infty)$ *the function* $f(t)t^{\vartheta}$ *is decreasing for large* t *and* $\lim_{t\to\infty} f(t)t^{\vartheta} = 0$.

(iv) $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$ *implies* $f^{\Delta}(t) > 0$ *for large* t *and* $f(t)$ *is increasing for large* t, *moreover* f *and* f^{Δ} *are tending to* ∞.

(v) $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$ *implies* $f^{\Delta}(t) < 0$ *for large* t *and* $f(t)$ *is decreasing for large* t, *moreover* f *is tending to* 0. *If* f *is convex for large* t *or if there exists* $h > 0$ *such that* $\mu(t) > h$ *for large* t, *then* f^{Δ} *is tending to* 0.

Proof. (i) Let $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$, $f = \varphi\,\omega$. First, we show that $\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty) \Leftrightarrow 1/\omega \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. Due to (8), $\omega^{\Delta}(t) > 0$ for large t. Therefore,

$$\begin{aligned}
\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty) \;&\Leftrightarrow\; \lim_{t\to\infty} \frac{\omega(t)}{t\omega^{\Delta}(t)} = 0 \;\Leftrightarrow\; \lim_{t\to\infty} \frac{\omega^{\sigma}(t) - \mu(t)\omega^{\Delta}(t)}{t\omega^{\Delta}(t)} = 0\\
&\Leftrightarrow\; \lim_{t\to\infty} \left(\frac{\omega^{\sigma}(t)}{t\omega^{\Delta}(t)} - \frac{\mu(t)}{t}\right) = 0 \;\Leftrightarrow\; \lim_{t\to\infty} \frac{\omega^{\sigma}(t)}{t\omega^{\Delta}(t)} = 0\\
&\Leftrightarrow\; \lim_{t\to\infty} \frac{t\omega^{\Delta}(t)}{\omega^{\sigma}(t)} = \infty \;\Leftrightarrow\; \lim_{t\to\infty} \left(\frac{t}{1/\omega(t)} \cdot \frac{-\omega^{\Delta}(t)}{\omega(t)\omega^{\sigma}(t)}\right) = -\infty\\
&\Leftrightarrow\; \lim_{t\to\infty} \frac{t(1/\omega(t))^{\Delta}}{1/\omega(t)} = -\infty \;\Leftrightarrow\; \frac{1}{\omega} \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)\,.
\end{aligned}$$

Now, since $1/\varphi \in \mathcal{RV}_{\mathbb{T}}(-\vartheta)$, see, [24, part (iv) of Proposition 1], or $0 < 1/d \le 1/\varphi(t) \le 1/c$ for large t, we have $1/f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. Similarly, $1/f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ implies $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$.

(ii) Let $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$ and $\vartheta \in [0,\infty)$. Then there exists a function $\xi(t)$, $t \le \xi(t) \le \sigma(t)$, such that

$$\left(\frac{f(t)}{t^{\vartheta}}\right)^{\Delta} = \frac{f^{\Delta}(t)t^{\vartheta} - f(t)(t^{\vartheta})^{\Delta}}{t^{\vartheta}(\sigma(t))^{\vartheta}} = \frac{f^{\Delta}(t)t^{\vartheta} - \vartheta f(t)(\xi(t))^{\vartheta-1}}{t^{\vartheta}(\sigma(t))^{\vartheta}}\,. \tag{9}$$

In view of

$$\frac{tf^{\Delta}(t)}{f(t)} > \vartheta\left(\frac{\xi(t)}{t}\right)^{\vartheta-1} \quad \text{for large } t$$

(indeed, $tf^{\Delta}(t)/f(t) \to \infty$ and $\xi(t)/t \to 1$ as $t \to \infty$), which is equivalent to

$$f^{\Delta}(t)t^{\vartheta} > \vartheta f(t)(\xi(t))^{\vartheta-1} \quad \text{for large } t\,,$$

(9) is positive for large t and hence $f(t)/t^{\vartheta}$ is increasing for large t. By a contradiction, suppose that $\lim_{t\to\infty} f(t)/t^{\vartheta} = L$, $L \in (0,\infty)$ (note that the limit of

this function exists for all $\vartheta \geq 0$ as a finite or infinite number, because the function $f(t)/t^{\vartheta}$ is increasing). Then $f(t) \sim Lt^{\vartheta}$ and hence $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$, which is a contradiction with $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. Therefore, $\lim_{t\to\infty} f(t)/t^{\vartheta} = \infty$.

(iii) It follows from (i) and (ii).

(iv) Let $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. If we take $\vartheta = 0$ in (ii), we get $f(t)$ is increasing (thus $f^{\Delta}(t) > 0$) for large t and $\lim_{t\to\infty} f(t) = \infty$. To prove that $\lim_{t\to\infty} f^{\Delta}(t) = \infty$, it is enough to show that $\liminf_{t\to\infty} f^{\Delta}(t) = \infty$. We know that $f^{\Delta}(t) > 0$. Assume that $\liminf_{t\to\infty} f^{\Delta}(t) = c$, $c > 0$. Then, in view of $\lim_{t\to\infty} t/f(t) = 0$ (which follows from (ii)), $\liminf_{t\to\infty} tf^{\Delta}(t)/f(t) = 0$, a contradiction with (8). So $\liminf_{t\to\infty} f^{\Delta}(t) = \infty$ and hence $\lim_{t\to\infty} f^{\Delta}(t) = \infty$.

(v) Analogously as in case (iv), we get (by using (iii) for $\vartheta = 0$) that $f(t)$ is decreasing (thus $f^{\Delta}(t) < 0$) for large t and $\lim_{t\to\infty} f(t) = 0$. Let f be convex for large t. Then $f^{\Delta}(t)$ increases for large t and $\lim_{t\to\infty} f^{\Delta}(t)$ exists as a nonpositive number. By a contradiction, assume that $\lim_{t\to\infty} f^{\Delta}(t) = k < 0$. Hence, $f^{\Delta}(t) \leq k$ for large t. By integration of the last inequality from t_0 to t (where $t_0 \in \mathbb{T}$ is sufficiently large), we get $f(t) \leq kt + q$ ($q = kt_0 - f(t_0)$) for large t. Hence, $f(t) < 0$ for large t, a contradiction. Let (for large t) $\mu(t)$ be bounded from below by a positive constant h. Then in view of that $f(t)$ is decreasing for large t,

$$0 > f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)} > \frac{f^{\sigma}(t) - f(t)}{h} \quad \text{for large } t. \tag{10}$$

If $t \to \infty$ in (10), we get (by using $\lim_{t\to\infty} f(t) = 0$) $\lim_{t\to\infty}((f^{\sigma}(t) - f(t))/h) = 0$, hence $\lim_{t\to\infty} f^{\Delta}(t) = 0$. □

Remark 1. (i) From the above proposition it is easy to see that the function $f(t) = a^t$ with $a > 1$ is a typical representative of the class $\mathcal{RPV}_{\mathbb{T}}(\infty)$, while the function $f(t) = a^t$ with $a \in (0,1)$ is a typical representative of the class $\mathcal{RPV}_{\mathbb{T}}(-\infty)$. Of course, as we can see also from Definition 1, these classes are much wider. The rapidly varying function can be understood like a product of an exponential function and a function, which is regularly varying or bounded. However, the exact representation is not known for now. We conjecture that it could be somewhere near to this one: for $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ resp. $\mathcal{RPV}_{\mathbb{T}}(-\infty)$,

$$f(t) = \varphi(t)a^{g(t)}\,, \qquad a > 1 \text{ resp. } a \in (0,1)\,,$$

where φ is defined as in Definition 1 and $g(t) \geq h(t)$, $h \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ with $\vartheta > 0$. Observe that this "representation" is sufficiently wide and includes many various rapidly varying functions, e.g., $(\sin(t) + b)a^t$, $\ln(t)a^t$, $t^{\gamma}a^t$, $a^{t^{\vartheta}}$ and a^{b^t} with $a \in (0,1) \cup (1,\infty)$, $b > 1$, $\gamma \in (-\infty,\infty)$ and $\vartheta > 0$.

(ii) Case (ii), resp. (iii) (and of course (iv), resp. (v)) of Proposition 1 does not hold generally for $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$, resp. $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. It is enough to take, e.g., a function $f(t) = a^{t-2\sin t}$ with $a > 1$, resp. $f(t) = a^{t-2\sin t}$ with $a < 1$. Note that $f(t) \in \mathcal{RPV}_{\mathbb{T}}(\pm\infty)$ in view of $a^{t-2\sin t} = a^{-2\sin t}a^t$ with bounded $a^{-2\sin t}$.

(iii) The assumption of convexity or existence of $h > 0$ in Proposition 1 in part (v) (unlike part (iv)) is important, because without this condition only

$\limsup_{t\to\infty} f^\Delta(t) = 0$ holds, as can we see in the following example. Let $n \in \mathbb{N}$ and consider function f defined on the discrete time scale $\mathbb{T} = \mathbb{N} \cup \{n + (1/2)^{n+2}\}$ such that

$$f(t) = \begin{cases} \left(\frac{1}{2}\right)^t & \text{for } t = n\,, \\ \frac{3}{4}\left(\frac{1}{2}\right)^t & \text{for } t = n + \left(\frac{1}{2}\right)^{n+2}. \end{cases}$$

Then $f(t) \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$, $\liminf_{t\to\infty} f^\Delta(t) = -1$ and $\limsup_{t\to\infty} f^\Delta(t) = 0$.

Now we introduce a Karamata type definition, see (1), (2) and (7).

Definition 2 (Karamata type definition)**.** Let $\tau\colon \mathbb{R} \to \mathbb{T}$ be defined as $\tau(t) = \max\{s \in \mathbb{T} : s \le t\}$. A measurable function $f\colon \mathbb{T} \to (0,\infty)$ satisfying

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \begin{cases} \infty \text{ resp. } 0 & \text{for } \lambda > 1\,, \\ 0 \text{ resp. } \infty & \text{for } 0 < \lambda < 1 \end{cases} \tag{11}$$

is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ *in the sense of Karamata.* We write $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$, resp. $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$.

Note that the classes $\mathcal{KRPV}_{\mathbb{T}}(\infty)$ and $\mathcal{KRPV}_{\mathbb{T}}(-\infty)$ can be described similarly as the classes $\mathcal{RPV}_{\mathbb{T}}(\infty)$ and $\mathcal{RPV}_{\mathbb{T}}(-\infty)$, see part (i) of Remark 1. Now we prove some properties of rapidly varying functions in the sense of Karamata.

Proposition 2.

(I) $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$ *if and only if* $1/f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$.

(II) *Let* $f\colon \mathbb{T} \to (0,\infty)$ *be a measurable function, monotone for large* t*. Then*

(i) $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$ *implies* f *is increasing for large* t *and* $\lim_{t\to\infty} f(t) = \infty$.

(ii) $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$ *implies* f *is decreasing for large* t *and* $\lim_{t\to\infty} f(t) = 0$.

(iii) $\displaystyle\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty \quad (\lambda > 1)$ *implies* $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$.

(iv) $\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = 0 \quad (\lambda > 1)$ *implies* $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$.

Proof. (I) We have

$$f \in \mathcal{KRPV}_{\mathbb{T}}(\infty) \;\Leftrightarrow\; \lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \begin{cases} \infty & \text{for } \lambda > 1, \\ 0 & \text{for } 0 < \lambda < 1 \end{cases} \;\Leftrightarrow$$

$$\Leftrightarrow \lim_{t\to\infty} \frac{\frac{1}{f(\tau(\lambda t))}}{\frac{1}{f(t)}} = \begin{cases} 0 & \text{for } \lambda > 1, \\ \infty & \text{for } 0 < \lambda < 1 \end{cases} \;\Leftrightarrow\; \frac{1}{f} \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)\,.$$

(II) (i) Let $\lambda > 1$ and $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = \infty$ hold. Suppose that $f(t)$ is nonincreasing for large t. Then $\limsup_{t\to\infty} f(\tau(\lambda t))/f(t) \le 1$, a contradiction. Next, let $\lim_{t\to\infty} f(t) = c < \infty$. Then, $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = 1$, a contradiction.

(ii) This part we can prove analogically as the part (i).

(iii) Let $\lambda > 1$ and $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = \infty$ hold. From (i) we know that $f(t)$ is increasing for large t. Therefore,

$$\infty = \lim_{t\to\infty} \frac{f(\tau(\lambda\tau(\frac{t}{\lambda})))}{f(\tau(\frac{t}{\lambda}))} \le \lim_{t\to\infty} \frac{f(t)}{f(\tau(\frac{1}{\lambda}t))}$$

(due to $f(\tau(\lambda\tau(\frac{t}{\lambda}))) \le f(t)$). Hence, $\lim_{t\to\infty} f(t)/f(\tau(\lambda t)) = \infty$ for $0 < \lambda < 1$ and thus $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = 0$ for $0 < \lambda < 1$. Therefore, $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$.

(iv) This part we can prove analogically as the part (iii). □

Now, we show that the Karamata type definition is under certain conditions equivalent to Definition 1.

Lemma 1. *Let f be a positive rd-continuously differentiable function and let $f^{\Delta}(t)$ be increasing for large t. Then*

(i) $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$ *iff* $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ *iff* $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$.

(ii) $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$ *iff* $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ *iff* $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$.

Moreover, $f^{\Delta}(t)$ be increasing for large t is not to be assumed in all if parts.

Proof. (i) We will proceed in the following way:

$$f \in \mathcal{KRPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{NRPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{RPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)\,.$$

Let $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$. First, observe that $f(t)$ is monotone for large t. Indeed, $f(t)$ is convex, so there exists t_0 such that $f(t)$ is monotone for $t > t_0$. Hence, $f(t)$ is increasing for large t due to Proposition 2. Now, for all $\lambda < 1$, we have

$$\begin{aligned} f(t) - f(\tau(\lambda t)) &= \int_{\tau(\lambda t)}^{t} f^{\Delta}(s)\Delta s \le f^{\Delta}(t)[t - \tau(\lambda t)] \le f^{\Delta}(t)[t - (\lambda t - \mu(\tau(\lambda t)))] \\ &= f^{\Delta}(t)[t(1-\lambda) + \mu(\tau(\lambda t))]\,. \end{aligned}$$

Hence,

$$\frac{f^{\Delta}(t)[t(1-\lambda) + \mu(\tau(\lambda t))]}{f(t)} \ge \frac{f(t) - f(\tau(\lambda t))}{f(t)}\,. \tag{12}$$

Note that $\mu(\tau(\lambda t))/f(t) \to 0$ as $t \to \infty$. Really, $f(t)$ is convex and increasing, so there exists $t_0 \in \mathbb{T}$ such that $f(t) > t$ for $t > t_0$ and hence,

$$0 = \lim_{t\to\infty} \frac{\mu(\tau(\lambda t))}{t} \ge \lim_{t\to\infty} \frac{\mu(\tau(\lambda t))}{f(t)} \ge 0\,.$$

Since $\lambda < 1$ is independent of t and can be chosen arbitrarily close to 1, in view of $\mu(\tau(\lambda t))/f(t) \to 0$ as $t \to \infty$ and $f(\tau(\lambda t))/f(t) \to 0$ as $t \to \infty$, from inequality (12) we have

$$\liminf_{t\to\infty} \frac{t f^{\Delta}(t)}{f(t)} \ge \sup_{\lambda<1} \frac{1}{1-\lambda} = \infty \tag{13}$$

and thus $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. The part $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ holds trivially. Let $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ and take $\lambda > 1$. Then, by Definition 1

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \lim_{t\to\infty} \frac{\varphi(\lambda t)}{\varphi(t)} \cdot \frac{\omega(\lambda t)}{\omega(t)} = \lim_{t\to\infty} h_{\lambda}(t) \frac{\omega(\lambda t)}{\omega(t)}\,. \tag{14}$$

Let $\varphi \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. Hence, $\varphi \in \mathcal{KRV}_{\mathbb{T}}(\vartheta)$ by [24, Theorem 2], which implies that $h(t) \to \lambda^{\vartheta}$ as $t \to \infty$. Let φ is bounded, i.e., $0 < c \le \varphi(t) \le d$ for large t. Then,

$$\frac{c}{d} \le \liminf_{t\to\infty} h_{\lambda}(t) \le h_{\lambda}(t) \le \limsup_{t\to\infty} h_{\lambda}(t) \le \frac{d}{c}\,.$$

Together, $h_\lambda(t)$ is bounded both above and below for large t by the positive constants. Due to $\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$, $\omega(t)$ is increasing for large t (by Proposition 1). Now, for all $\lambda > 1$, we have

$$\begin{aligned}\omega(\tau(\lambda t)) \geq \omega(\tau(\lambda t)) - \omega(t) = \int_t^{\tau(\lambda t)} \omega^\Delta(s)\Delta s \geq \omega^\Delta(t)[\tau(\lambda t) - t] \\ \geq \omega^\Delta(t)[\lambda t - \mu(\tau(\lambda t)) - t] = \omega^\Delta(t)\big[t(\lambda - 1) - \mu(\tau(\lambda t))\big]\,.\end{aligned}$$

Hence,

$$\frac{\omega(\tau(\lambda t))}{\omega(t)} \geq \frac{\omega^\Delta(t)[t(\lambda-1) - \mu(\tau(\lambda t))]}{\omega(t)}\,. \tag{15}$$

Since $\lambda > 1$, in view of $\mu(\tau(\lambda t))/\omega(t) \to 0$ as $t \to \infty$ (a similar reasoning as in the first implication of part (i) of this proof), from (14) and (15) we have

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} \geq \lim_{t\to\infty} h_\lambda(t) \frac{t\omega^\Delta(t)(\lambda - 1)}{\omega(t)} = \infty \qquad (\lambda > 1)\,,$$

and thus (by Proposition 2) $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$.

(ii) We will proceed analogically as in case (i). Let $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$. Similarly as in part (i), we get $f(t)$ is decreasing for large t due to Proposition 2. Now, for all $\lambda > 1$, we have

$$-f(\tau(\lambda t)) + f(t) = \int_t^{\tau(\lambda t)} (-f^\Delta(s))\Delta s \leq -f^\Delta(t)(\tau(\lambda t) - t)) \leq -f^\Delta(t)(\lambda - 1)\,t\,.$$

Hence,

$$-\frac{tf^\Delta(t)}{f(t)} \geq \frac{1}{\lambda - 1} \cdot \frac{-f(\tau(\lambda t)) + f(t)}{f(t)} = \frac{1}{\lambda - 1}\Big(1 - \frac{f(\tau(\lambda t))}{f(t)}\Big)\,.$$

Since $\lambda > 1$ is independent of t and can be chosen arbitrarily close to 1, in view of $f(\tau(\lambda t))/f(t) \to 0$ as $t \to \infty$, from the above inequality we have

$$\liminf_{t\to\infty} -\frac{tf^\Delta(t)}{f(t)} \geq \sup_{\lambda>1} \frac{1}{\lambda - 1} = \infty$$

and thus $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. The part $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty) \Rightarrow f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ holds trivially. Let $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. By using Proposition 1, part (i) of this lemma and Proposition 2 we can successively write:

$$f \in \mathcal{RPV}_{\mathbb{T}}(-\infty) \Rightarrow \frac{1}{f} \in \mathcal{RPV}_{\mathbb{T}}(\infty) \Rightarrow \frac{1}{f} \in \mathcal{KRPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)\,.$$

□

Remark 2 (Important)**.** (i) Note that the concept of normalized rapid variation is not known in the literature concerning the continuous (resp. discrete) theory and it seems that there is no reason to distinguish the two cases of rapidly varying behavior in this situation. We conjecture that in this case, every positive differentiable function f (resp. every positive sequence), which is rapidly varying, is automatically normalized rapidly varying. Hence, there is no point to consider both definitions (specially, when we study asymptotic properties of differential or difference equations and deal with functions which are differentiable). However, the situation is different in the general time scale case, where the previous assertion is not true in general (only if f is convex and Δ-differentiable, then, in view of Lemma 1, these two definitions are equivalent). Indeed, take, e.g., $\mathbb{T} = \mathbb{N} \cup \{n+2^{-n}\}$, $n \in \mathbb{N}$, and f, φ, $\omega\colon \mathbb{T} \to \mathbb{R}$ satisfying the assumptions of Definition 1 such that

$$\varphi(t) = \begin{cases} 1 + 2^{-t} & \text{for } t = n\,, \\ 1 - 2^{-t} & \text{for } t = n + 2^{-n} \end{cases} \qquad \text{and} \quad \omega(t) = 2^t \quad (t \in \mathbb{T})\,.$$

Then $\varphi(t) \to 1$ as $t \to \infty$ and $\omega(t) \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. Moreover,

$$f(t) = \varphi(t)\omega(t) = \begin{cases} 2^t + 1 & \text{for } t = n\,, \\ 2^t - 1 & \text{for } t = n + 2^{-n} \end{cases}$$

is of the class $C^1_{\mathrm{rd}}(\mathbb{T})$. It is not difficult to verify that $f(t)$ is decreasing in each $t = n$, $n \in \mathbb{N}$. Hence, $f^{\Delta}(t)$ is negative for every $t = n$, thus $\liminf_{t\to\infty} tf^{\Delta}(t)/f(t) \le 0$ and hence $f \notin \mathcal{NRPV}_{\mathbb{T}}(\infty)$.

(ii) Looking at Definition 1 and a condition on a function φ, the reader may ask why we require the function φ just in this form. The other eventualities are, e.g., to consider φ in the following forms:

(a) $\varphi(t) \sim C$, where $C > 0$ (a less general form),

(b) $t^c \le \varphi(t) \le t^d$, where $c, d \in \mathbb{R}$, $c \le d$ (a more general form).

However, the case (a) is less general then in our definition. Moreover, observe that the function φ from the previous example from (i) satisfies condition (a). The case (b) is more general but not convenient since our theory focuses on a generalization in the sense of a "domain of definition" rather than considering "badly behaving" functions.

4. Applications to half-linear dynamic equations

As an application of the theory of rapid variation, we study asymptotic behavior of solutions of half-linear dynamic equation in the form (4). In [18] the reader can find many useful information about half-linear dynamic equations and the monograph [2] is a very good source for many results about linear dynamic equations. In view of the structure of equation (4), it is not difficult to see that every positive solution y of (4) satisfies $y^{\Delta\Delta} > 0$, i.e., y is convex and y^{Δ} is increasing.

Theorem 1. *Equation* (4) *has solutions $u \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ and $v \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ if and only if for all $\lambda > 1$*

$$\lim_{t\to\infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s)\Delta s = \infty\,. \tag{16}$$

Moreover, all positive decreasing solutions of (4) *belong to $\mathcal{NRPV}_{\mathbb{T}}(-\infty)$ and all positive increasing solutions of* (4) *belong to $\mathcal{NRPV}_{\mathbb{T}}(\infty)$.*

Proof. "If": Let u be a positive decreasing solution of (4) and let (16) hold. By integration of equation (5) from t to $\tau(\sqrt{\lambda}t)$ $(\lambda > 1)$ we get

$$\Phi\big(u^{\Delta}(\tau(\sqrt{\lambda}t))\big) - \Phi\big(u^{\Delta}(t)\big) = \int_t^{\tau(\sqrt{\lambda}t)} p(s)\Phi\big(u^{\sigma}(s)\big)\Delta s\,.$$

Since $u^{\Delta} < 0$ and u is positive decreasing with zero limit, we can write

$$\begin{aligned} -u^{\Delta}(t) &\geq \Phi^{-1}\Big(\int_t^{\tau(\sqrt{\lambda}t)} p(s)\Phi\big(u^{\sigma}(s)\big)\Delta s\Big) \\ &\geq u\big(\tau(\sqrt{\lambda}t)\big)\Phi^{-1}\Big(\int_t^{\tau(\sqrt{\lambda}t)} p(s)\Delta s\Big)\,. \end{aligned} \tag{17}$$

In the last inequality we use the fact that

$$\int_a^b f^{\sigma}(t)g(t)\Delta t \geq f(b)\int_a^b g(t)\Delta t \qquad (a,b\in\mathbb{T};\ a<b) \tag{18}$$

holds for arbitrary positive decreasing function f and positive function g. This inequality follows from the time scales version of the second mean value theorem of integral calculus, see [18, Lemma 2.5]. By integration of (17) from t to $\tau(\sqrt{\lambda}t)$ $(\lambda > 1)$ we get

$$u(t) - u\big(\tau(\sqrt{\lambda}t)\big) \geq \int_t^{\tau(\sqrt{\lambda}t)} u\big(\tau(\sqrt{\lambda}s)\big)\Phi^{-1}\Big(\int_s^{\tau(\sqrt{\lambda}s)} p(r)\Delta r\Big)\Delta s\,.$$

By using (18) with $u(\tau(\sqrt{\lambda}\rho(\tau(\sqrt{\lambda}t)))) \geq u(\tau(\lambda t))$ we get

$$u(t) \geq u\big(\tau(\lambda t)\big)\int_t^{\tau(\sqrt{\lambda}t)} \Phi^{-1}\Big(\int_s^{\tau(\sqrt{\lambda}s)} p(r)\Delta r\Big)\Delta s\,. \tag{19}$$

In view of (16) for any arbitrarily large constant $M > 0$ there exists t_0 sufficiently large such that

$$\int_t^{\tau(\sqrt{\lambda}t)} p(s)\Delta s \geq \frac{M}{t^{\alpha-1}}\,, \qquad t > t_0\,. \tag{20}$$

Since u is positive, from (19) and (20) we get

$$\begin{aligned} \frac{u(t)}{u(\tau(\lambda t))} &\geq \Phi^{-1}(M)\int_t^{\tau(\sqrt{\lambda}t)} \Phi^{-1}\Big(\frac{1}{s^{\alpha-1}}\Big)\,\Delta s = \Phi^{-1}(M)\int_t^{\tau(\sqrt{\lambda}t)} \frac{1}{s}\,\Delta s \\ &\geq \Phi^{-1}(M)\int_t^{\tau(\sqrt{\lambda}t)} \frac{1}{s}\,\mathrm{d}s = \Phi^{-1}(M)\ln\frac{\tau(\sqrt{\lambda}t)}{t} \\ &\geq \Phi^{-1}(M)\ln\frac{\sqrt{\lambda}t - \mu(\tau(\sqrt{\lambda}t))}{t} = \Phi^{-1}(M)\ln\Big(\sqrt{\lambda} - \frac{\mu(\tau(\sqrt{\lambda}t))}{t}\Big)\,. \end{aligned}$$

where the inequality $\int_t^{\tau(\sqrt{\lambda}t)}(1/s)\,\Delta s \geq \int_t^{\tau(\sqrt{\lambda}t)}(1/s)\,\mathrm{d}s$ (used also in further part of the proof) follows from [19, Lemma 1.1]. Since $\mu(\tau(\sqrt{\lambda}t))/t \to 0$ as $t \to \infty$ and since M was arbitrarily large, this implies $u(t)/u(\tau(\lambda t)) \to \infty$ as $t \to \infty$. Consequently,

$u(\tau(\lambda t))/u(t) \to 0$ as $t \to \infty$ for $\lambda > 1$, which implies (due to Proposition 2) that $u \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$ and hence, by Lemma 1, $u \in (\mathcal{N})\mathcal{RPV}_{\mathbb{T}}(-\infty)$.

Let v be a positive increasing solution of (4) and let (16) hold. By integration of equation (4) from $\tau(t/\sqrt{\lambda})$ to t ($\lambda > 1$) we get

$$\Phi(v^{\Delta}(t)) - \Phi\Big(v^{\Delta}\Big(\tau\Big(\frac{t}{\sqrt{\lambda}}\Big)\Big)\Big) = \int_{\tau\left(\frac{t}{\sqrt{\lambda}}\right)}^{t} p(s)\Phi\big(v^{\sigma}(s)\big)\Delta s\,.$$

Since $v^{\Delta} > 0$ and v is positive increasing, we get

$$\begin{aligned} v^{\Delta}(t) &\geq \Phi^{-1}\Big(\int_{\tau\left(\frac{t}{\sqrt{\lambda}}\right)}^{t} p(s)\Phi\big(v^{\sigma}(s)\big)\Delta s\Big) \\ &\geq v\Big(\tau\Big(\frac{t}{\sqrt{\lambda}}\Big)\Big)\Phi^{-1}\Big(\int_{\tau\left(\frac{t}{\sqrt{\lambda}}\right)}^{t} p(s)\Delta s\Big)\,. \end{aligned}$$

By integration of the last inequality from $\sigma(\tau(t/\sqrt{\lambda}))$ to t ($\lambda > 1$) we get

$$v(t) - v^{\sigma}\Big(\tau\Big(\frac{t}{\sqrt{\lambda}}\Big)\Big) \geq \int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^{t} v\Big(\tau\Big(\frac{s}{\sqrt{\lambda}}\Big)\Big)\Phi^{-1}\Big(\int_{\tau\left(\frac{s}{\sqrt{\lambda}}\right)}^{s} p(r)\Delta r\Big)\Delta s\,.$$

By using the same ideas as before we get

$$v(t) \geq v\Big(\tau\Big(\frac{t}{\lambda}\Big)\Big)\int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^{t} \Phi^{-1}\Big(\int_{\tau\left(\frac{s}{\sqrt{\lambda}}\right)}^{s} p(r)\Delta r\Big)\Delta s\,, \tag{21}$$

where we use

$$v\Big(\tau\Big(\frac{\sigma\big(\tau\big(\frac{t}{\sqrt{\lambda}}\big)\big)}{\sqrt{\lambda}}\Big)\Big) \geq v\Big(\tau\Big(\frac{t}{\lambda}\Big)\Big)\,.$$

Inequality (21) can be rewritten on the form

$$\frac{v(t)}{v\left(\tau\left(\frac{t}{\lambda}\right)\right)} \geq \int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^{t} \Phi^{-1}\Big(\int_{\tau\left(\frac{s}{\sqrt{\lambda}}\right)}^{s} p(r)\Delta r\Big)\Delta s\,, \tag{22}$$

In view of (16), which can be equivalently written with $\sqrt{\lambda}$ instead of λ, we have (due to $\{\tau(t/\sqrt{\lambda})\} \subseteq \mathbb{T}$ for large t)

$$\lim_{t\to\infty}\Big(\tau\Big(\frac{t}{\sqrt{\lambda}}\Big)\Big)^{\alpha-1}\int_{\tau\left(\frac{t}{\sqrt{\lambda}}\right)}^{\tau\left(\sqrt{\lambda}\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)} p(s)\Delta s = \infty\,.$$

Therefore, thanks to $\tau(t/\sqrt{\lambda}) \leq t/\sqrt{\lambda} < t$ and $\tau(\sqrt{\lambda}\tau(t/\sqrt{\lambda})) \leq t$, we get

$$\lim_{t\to\infty} t^{\alpha-1}\int_{\tau\left(\frac{t}{\sqrt{\lambda}}\right)}^{t} p(s)\Delta s = \infty\,,$$

which means that for arbitrarily large $M > 0$, there exists s_0 sufficiently large such that

$$\int_{\tau\left(\frac{s}{\sqrt{\lambda}}\right)}^{s} p(r)\Delta r \geq \frac{M}{s^{\alpha-1}}\,, \qquad s > s_0\,, \tag{23}$$

and since v is positive, then from (22) and (23), we get (by using the similar calculation as in previous case for the decreasing solution u)

$$\frac{v(t)}{v\left(\tau\left(\frac{t}{\lambda}\right)\right)} \geq \Phi^{-1}(M) \int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^{t} \frac{1}{s}\,\Delta s \geq \Phi^{-1}(M) \int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^{t} \frac{1}{s}\,\mathrm{d}s$$

$$\geq \Phi^{-1}(M) \ln \frac{t}{\frac{t}{\sqrt{\lambda}} + \mu\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)} = \Phi^{-1}(M) \ln \frac{\sqrt{\lambda}}{1 + \frac{\mu\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}{\frac{t}{\sqrt{\lambda}}}}.$$

Since $\mu(t) = o(t)$, $\mu(\tau(t/\sqrt{\lambda}))/(t/\sqrt{\lambda}) \to 0$ as $t \to \infty$ and since M was arbitrarily large, this yields $v(t)/v(\tau((t/\lambda))) \to \infty$ as $t \to \infty$ for $\lambda > 1$, i.e., $v(\tau(\lambda t))/v(t) \to 0$ as $t \to \infty$ for $\lambda < 1$, which implies, similarly as in the proof of Lemma 1, first implication of part (i) (indeed, v satisfies (12) and (13)), that $v \in (\mathcal{N})\mathcal{RPV}_{\mathbb{T}}(\infty)$.

"Only if": Let u be a positive decreasing rapidly varying solution of (4). Thanks to $u^{\Delta\Delta} > 0$ (see equation (4)), we have u^{Δ} increases and due to Lemma 1, $u \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. Hence, $u^{\Delta}(t)$ is negative with zero limit and $u(t) \to 0$ as $t \to \infty$ (by Proposition 1). Moreover, $-u^{\Delta}(t)$ decreases. For $\lambda > 1$ we have

$$\begin{aligned} -u^{\Delta}\big(\tau(\lambda t)\big)\,\tau(\lambda t)\Big(1 - \frac{t}{\tau(\lambda t)}\Big) &= -u^{\Delta}\big(\tau(\lambda t)\big)\big(\tau(\lambda t) - t\big) \\ &= -u^{\Delta}\big(\tau(\lambda t)\big) \int_t^{\tau(\lambda t)} \Delta s \\ &\leq - \int_t^{\tau(\lambda t)} u^{\Delta}(s)\Delta s = u(t) - u\big(\tau(\lambda t)\big)\,. \end{aligned} \tag{24}$$

From the fact that

$$1 - \frac{t}{\tau(\lambda t)} \geq 1 - \frac{t}{\lambda t - \mu(\tau(\lambda t))} = 1 - \frac{1}{\lambda - \frac{\mu(\tau(\lambda t))}{t}} = 1 - \frac{1}{\lambda\big(1 - \frac{\mu(\tau(\lambda t))}{\lambda t}\big)}\,,$$

we have (due to $\mu(\tau(\lambda t))/\lambda t \to 0$ as $t \to \infty$):

$$\lim_{t\to\infty} \Big(1 - \frac{t}{\tau(\lambda t)}\Big) \geq \lim_{t\to\infty} \left(1 - \frac{1}{\lambda\big(1 - \frac{\mu(\tau(\lambda t))}{\tau(\lambda t)}\big)}\right) = 1 - \frac{1}{\lambda} > 0\,.$$

Since $\lim_{t\to\infty} \big(u(t) - u\big(\tau(\lambda t)\big)\big) = 0$, inequality (24) implies

$$\lim_{t\to\infty} \tau(\lambda t) u^{\Delta}(\tau(\lambda t)) = 0\,. \tag{25}$$

Due to $u^{\Delta}(t)$ is negative increasing,

$$\frac{u^{\Delta}(\tau(\lambda t))}{u^{\Delta}(t)} \leq 1\,.$$

Now we want to show that

$$\limsup_{t\to\infty} \frac{u^{\Delta}(\tau(\lambda t))}{u^{\Delta}(t)} < 1\,, \qquad \lambda > 1\,. \tag{26}$$

By a contradiction, assume that there exist $\lambda_0 > 1$ and an unbounded sequence $\{t_k\}_{k=1}^{\infty} \subseteq \mathbb{T}$ such that

$$\lim_{t_k \to \infty} \frac{u^{\Delta}(\tau(\lambda_0^2 t_k))}{u^{\Delta}(t_k)} = 1\,. \tag{27}$$

Let y be a continuous positive decreasing function of a real variable, such that

$$y(t) = -u^{\Delta}(t) \quad \text{for all} \quad t \in \{t_k\}_{k=1}^{\infty}$$

and

$$y(t) \geq -u^{\Delta}(t) \quad \text{for all} \quad t \in \mathbb{T}\,.$$

Thanks to $\mu(t)/t \to 0$ as $t \to \infty$, we have for large t

$$\frac{\mu(\tau(\lambda_0 t))}{\lambda_0 t} \leq \frac{\mu(\tau(\lambda_0 t))}{\tau(\lambda_0 t)} \leq \lambda_0 - 1$$

and therefore, we get

$$\mu\big(\tau(\lambda_0 t)\big) \leq \lambda_0^2\, t - \lambda_0 t \leq \lambda_0^2\, t - \tau(\lambda_0 t)\,.$$

From the last inequality we have $\sigma(\tau(\lambda_0 t)) \leq \lambda_0^2\, t$ for large t and hence

$$\lambda_0 t \leq \tau(\lambda_0^2\, t)\,. \tag{28}$$

From (27), (28) and thanks to y is decreasing we have

$$1 > \frac{y(\lambda_0 t_k)}{y(t_k)} \geq \frac{y(\tau(\lambda_0^2\, t_k))}{y(t_k)} \geq \frac{u^{\Delta}(\tau(\lambda_0^2\, t_k))}{u^{\Delta}(t_k)} \to 1\,,$$

as $t_k \to \infty$. Then (see the proof of [13, Theorem 1.3]) there exists a continuous positive decreasing function z of real variable, such that $z(t) = y(t)$ for every $t \in \mathbb{T}$ sufficiently large, and $\lim_{x\to\infty}(z(\lambda_0 x)/z(x)) = 1$. Since z is monotone, $\lim_{x\to\infty}(z(\lambda x)/z(x)) = 1$ holds for every $\lambda > 0$, see [1, Proposition 1.10.1] and this implies that z is slowly varying function, see [1, Definition on page 6]. Therefore, $\lim_{x\to\infty} x z(x) = \infty$. The contradiction follows by observing that

$$z\big(\tau(\lambda t)\big) = y\big(\tau(\lambda t)\big) = -u^{\Delta}\big(\tau(\lambda t)\big)\,, \qquad t \in \{t_k\}_{k=1}^{\infty}$$

and

$$\lim_{t\to\infty} -\tau(\lambda t)\, u^{\Delta}\big(\tau(\lambda t)\big) = 0\,, \quad t \in \{t_k\}_{k=1}^{\infty}\,,$$

which holds due to (25). Hence, (26) holds. Therefore, there exists $N > 0$ such that

$$1 - \Phi\Big(\frac{u^{\Delta}(\tau(\lambda t))}{u^{\Delta}(t)}\Big) \geq N\,, \tag{29}$$

for every $\lambda > 1$ and t sufficiently large. By integration of (4) from t to $\tau(\lambda t)$ we have

$$\Phi\big(u^{\Delta}\big(\tau(\lambda t)\big)\big) - \Phi\big(u^{\Delta}(t)\big) = \int_t^{\tau(\lambda t)} p(s)\, \Phi\big(u\big(\sigma(s)\big)\big)\Delta s \leq \Phi\big(u(t)\big) \int_t^{\tau(\lambda t)} p(s)\, \Delta s\,.$$

This implies

$$-\Phi\big(u^\Delta(t)\big)\Big(1-\frac{\Phi(u^\Delta(\tau(\lambda t)))}{\Phi(u^\Delta(t))}\Big)\le \Phi\big(u(t)\big)\int_t^{\tau(\lambda t)} p(s)\,\Delta s\,.$$

From (29) and by multiplying the previous inequality by $t^{\alpha-1}$, we have

$$N\Big(\frac{-tu^\Delta(t)}{u(t)}\Big)^{\alpha-1}\le t^{\alpha-1}\int_t^{\tau(\lambda t)} p(s)\Delta s\,,$$

which (with $t\to\infty$) implies (16). □

Remark 3. Note that the previous theorem is new even for the linear case (when $\alpha=2$), where u and v form a fundamental set of solutions of (5). The sufficiency part for increasing solutions is new also for the half-linear discrete case. For more information about this case, see [17, 16]. For the continuous case, we refer to Marić's book [13] or to [14] for the corresponding results in the linear case. However, according to the best of our knowledge, the corresponding case of rapid variation for half-linear differential equations has not yet been processed in the literature. Finally note that the necessity part for increasing solutions has not been proved (even in linear case) in the differential, resp. difference or dynamic, equations setting yet.

5. Karamata functions and $\mathbb{M}$-classification

In this section we provide information about asymptotic behavior of all positive solutions of (5) and all positive decreasing solutions of (4) as $t\to\infty$. First consider the linear dynamic equation (5). Note that all nontrivial solutions of (5) are nonoscillatory (i.e., of one sign for large t) and monotone for large t. Because of linearity, without loss of generality, we may consider just positive solutions of (5); we denote this set as $\mathbb{M}$. Thanks to the monotonicity, the set $\mathbb{M}$ can be further split into the two classes $\mathbb{M}^+$ and $\mathbb{M}^-$, where

$$\begin{aligned}\mathbb{M}^+&=\{x\in\mathbb{M}:\ \exists t_x\in\mathbb{T}\text{ such that }x(t)>0,x^\Delta(t)>0\text{ for }t\ge t_x\}\,,\\ \mathbb{M}^-&=\{x\in\mathbb{M}:\ x(t)>0,x^\Delta(t)<0\}\,.\end{aligned}$$

These classes are always nonempty. To see it, the reader can follow the continuous ideas described, e.g., in [6, Chapter 4].

Now we introduce the following concept. A positive function $f\colon\mathbb{T}\to\mathbb{R}$ is said to be a *Karamata function*, if f is slowly or regularly or rapidly varying; we write $f\in\mathcal{KF}_{\mathbb{T}}$. In [24] we established necessary and sufficient conditions for all positive solutions of (5) to be regularly (resp. slowly) varying. Here we want to complete this discussion for all positive solutions to be Karamata functions. We introduce the following notation:

$$
\begin{aligned}
\mathbb{M}^-_{SV} &= \mathbb{M}^- \cap \mathcal{NSV}_{\mathbb{T}}\,,\\
\mathbb{M}^-_{RV}(\vartheta_1) &= \mathbb{M}^- \cap \mathcal{NRV}_{\mathbb{T}}(\vartheta_1), \vartheta_1 < 0\,,\\
\mathbb{M}^+_{RV}(\vartheta_2) &= \mathbb{M}^+ \cap \mathcal{NRV}_{\mathbb{T}}(\vartheta_2), \vartheta_2 = 1-\vartheta_1 > 1\,,\\
\mathbb{M}^-_{RPV}(-\infty) &= \mathbb{M}^- \cap \mathcal{NRPV}_{\mathbb{T}}(-\infty)\,,\\
\mathbb{M}^+_{RPV}(\infty) &= \mathbb{M}^+ \cap \mathcal{NRPV}_{\mathbb{T}}(\infty)\,,\\
\mathbb{M}^-_0 &= \{y \in \mathbb{M}^- : \lim_{t\to\infty} y(t) = 0\}\,,\\
\mathbb{M}^+_\infty &= \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = \infty\}\,.
\end{aligned}
$$

We distinguish three cases for the behavior of the coefficient p from equation (5):

$$\lim_{t\to\infty} t\int_t^\infty p(s)\,\Delta s = 0\,, \tag{30}$$

$$\lim_{t\to\infty} t\int_t^\infty p(s)\,\Delta s = A > 0\,, \tag{31}$$

$$\lim_{t\to\infty} t\int_t^{\tau(\lambda t)} p(s)\Delta s = \infty \quad \text{for all } \lambda > 1\,. \tag{32}$$

We claim (with the use of the results of this paper and [24]) that:

$$
\begin{aligned}
&\mathbb{M}^- = \mathbb{M}^-_{SV} && \iff (30) \iff && \mathbb{M}^+ = \mathbb{M}^+_{RV}(1) = \mathbb{M}^+_\infty\,,\\
&\mathbb{M}^- = \mathbb{M}^-_{RV}(\vartheta_1) = \mathbb{M}^-_0 && \iff (31) \iff && \mathbb{M}^+ = \mathbb{M}^+_{RV}(\vartheta_2) = \mathbb{M}^+_\infty\,,\\
&\mathbb{M}^- = \mathbb{M}^-_{RPV}(-\infty) = \mathbb{M}^-_0 && \iff (32) \implies && \mathbb{M}^+ = \mathbb{M}^+_{RPV}(\infty) = \mathbb{M}^+_\infty\,.
\end{aligned}
\tag{33}
$$

Now consider the second order half-linear dynamic equations (4). The space of all solutions is here more complicated than the space of all solution of equation (5). The reason is that we do not have property of linearity in this case. However, by the similar consideration as in linear case, we get again two classes $\mathbb{M}^+$ and $\mathbb{M}^-$, which are always nonempty. In [23] we established necessary and sufficient conditions for all positive decreasing solutions of (4) to be regularly varying. Now we complete this result in the sense of rapidly varying behavior. We distinguish three cases for behavior of coefficient $p(t)$ from equation (4):

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\Delta s = 0\,, \tag{34}$$

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\Delta s = B > 0\,, \tag{35}$$

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^{\tau(\lambda t)} p(s)\Delta s = \infty \quad \text{for all} \quad \lambda > 1\,. \tag{36}$$

With the use of the results of this paper and [23], with the same notation as in the linear case we can claim:

$$\begin{aligned} &\mathbb{M}^- = \mathbb{M}^-_{SV} && \iff (34),\\ &\mathbb{M}^- = \mathbb{M}^-_{RV}(\vartheta_1) = \mathbb{M}^-_0 && \iff (35),\\ (37)\quad &\mathbb{M}^- = \mathbb{M}^-_{RPV}(-\infty) = \mathbb{M}^-_0 && \iff (36) \implies \mathbb{M}^+ = \mathbb{M}^+_{RPV}(\infty) = \mathbb{M}^+_\infty\,. \end{aligned}$$

The reader may wonder that integral in condition (32) (resp. (36)) is from t to $\tau(\lambda(t))$, while integral in conditions (30) and (31) (resp. (34) and (35)) is from t to ∞. In [17, Example 1], it is shown that there exists function $p\colon \mathbb{T} \to \mathbb{R}$ ($\mathbb{T} = \mathbb{N}$, thus p is a sequence), which satisfies following condition

$$(38)\qquad \lim_{t\to\infty} t \int_t^\infty p(s)\,\Delta s = \infty\,, \quad \text{but} \quad \lim_{t\to\infty} t \int_t^{\tau(\lambda t)} p(s)\Delta s \neq \infty\,,$$

for some $\lambda > 1$. For simplicity, introduce the following notation

$$P = \lim_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\,\Delta s\,, \qquad P_\lambda = \lim_{t\to\infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s)\Delta s\,, \quad \lambda > 1\,.$$

In view of that example, $P = \infty$ does not imply $P_\lambda = \infty$ for all $\lambda > 1$ (only the inverse implication holds, because $\int_t^{\tau(\lambda t)} p(s)\Delta s < \int_t^\infty p(s)\,\Delta s$). But if P is finite (nonnegative) number, then P_λ is also finite (nonnegative) number for all $\lambda > 1$, and if P_λ is finite (nonnegative) number for all $\lambda > 1$, then P is also finite (nonnegative) number. A relation between P and P_λ is shown in the following theorem.

Theorem 2. *It holds*

$$P = A \geq 0 \quad \textit{if and only if} \quad P_\lambda = \frac{A(\lambda^{\alpha-1} - 1)}{\lambda^{\alpha-1}} \quad \textit{for all} \quad \lambda > 1\,.$$

Proof. In this proof we will need a special sequence of reals. Take $\lambda > 1$ and $t \in \mathbb{T}$ sufficiently large and define sequence $\{r_n\}_{n=0}^\infty$ of reals such that $\lambda^{r_n} t = \tau(\lambda^n t)$ for $n \in \mathbb{N} \cup \{0\}$. Note that $r_n = r_n(t)$. We show that r_n has the following properties:

(i) $r_n < r_{n+1}$ for all $n \in \mathbb{N}$,

(ii) $r_0 = 0 < r_1 \leq 1 < r_2 \leq 2 < \cdots < r_{n-1} \leq n-1 < r_n \leq n$ for all $n \in \mathbb{N}$,

(iii) $\tau(\lambda^{1+r_n} t) \leq \lambda^{r_{n+1}} t$ for all $n \in \mathbb{N}$,

(iv) $r_n(t) \to n$ as $t \to \infty$ for all $n \in \mathbb{N}$.

(i) Let $n \in \mathbb{N}$. First note that for $\tau(\lambda^n t)$ right-dense $\lambda^{r_n} t = \tau(\lambda^n t) = \lambda^n t < \tau(\lambda^{n+1} t) = \lambda^{r_{n+1}} t$ and (i) holds trivially. Now suppose that $\tau(\lambda^n t)$ is right-scattered. Thanks to $\mu(t) = o(t)$, $\mu(t) < (\lambda - 1)t$ for large t and we can write

$$\sigma(\tau(\lambda^n t)) \leq \lambda^n t + \mu(\tau(\lambda^n t)) < \lambda^n t + (\lambda-1)\tau(\lambda^n t) \leq \lambda^n t + (\lambda-1)\lambda^n t = \lambda^{n+1} t\,.$$

Therefore, $\tau(\lambda^n t) < \sigma(\tau(\lambda^n t)) \leq \tau(\lambda^{n+1} t)$. Hence, $\lambda^{r_n} t < \lambda^{r_{n+1}} t$ and (i) holds.

(ii) Note that $r_0 = 0$ holds trivially. Let $n \in \mathbb{N}$. By using (i) we can write $\tau(\lambda^{n-1} t) \leq \lambda^{n-1} t < \tau(\lambda^n t) = \lambda^{r_n} t \leq \lambda^n t$. Hence, $n - 1 < r_n \leq n$.

(iii) Let $n \in \mathbb{N}$. It holds

$$\tau(\lambda^{1+r_n} t) = \tau(\lambda \lambda^{r_n} t) = \tau\big(\lambda(\tau(\lambda^n t))\big) \le \tau(\lambda \lambda^n t) = \tau(\lambda^{n+1} t) = \lambda^{r_{n+1}} t\,.$$

Hence, (iii) is fulfilled.

(iv) In view of

$$1 \ge \frac{\tau(\lambda^n t)}{\lambda^n t} \ge \frac{\lambda^n t - \mu(\tau(\lambda^n t))}{\lambda^n t} = 1 - \frac{\mu(\tau(\lambda^n t))}{\lambda^n t} \to 1 \text{ as } t \to \infty\,,$$

we get $\lim_{t\to\infty} \tau(\lambda^n t)/(\lambda^n t) = 1$. Hence,

$$1 = \lim_{t\to\infty} \frac{\tau(\lambda^n t)}{\lambda^n t} = \lim_{t\to\infty} \frac{\lambda^{r_n} t}{\lambda^n t} = \lambda^{r_n - n}\,,$$

which implies $r_n(t) \to n$ as $t \to \infty$ for each $n \in \mathbb{N}$.

"If": We wish to show that if there is $\lambda > 1$ such that $P_\lambda = L$, then $P = L\lambda^{\alpha-1}/(\lambda^{\alpha-1} - 1)$. First suppose that there exist $\lambda > 1$ and $L_* > 0$ such that

$$\liminf_{t\to\infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s)\,\Delta s \ge L_*\,.$$

Let $\varepsilon > 0$ and take $t \in \mathbb{T}$ sufficiently large. Then by using the properties (i), (ii) and (iii) we get

$$(\lambda^n t)^{\alpha-1} \int_{\lambda^{r_n} t}^{\lambda^{r_{n+1}} t} p(s)\Delta s \ge (\lambda^{r_n} t)^{\alpha-1} \int_{\lambda^{r_n} t}^{\tau(\lambda^{1+r_n} t)} p(s)\Delta s \ge L_* - \varepsilon$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence,

$$t^{\alpha-1} \int_{\lambda^{r_n} t}^{\lambda^{r_{n+1}} t} p(s)\Delta s \ge \frac{L_* - \varepsilon}{(\lambda^{\alpha-1})^n} \qquad \text{for all } n \in \mathbb{N} \cup \{0\}\,.$$

Summing this inequality for n from 0 to ∞ we get

$$t^{\alpha-1} \int_t^{\infty} p(s)\,\Delta s \ge (L_* - \varepsilon) \sum_{n=0}^{\infty} \frac{1}{(\lambda^{\alpha-1})^n} = \frac{(L_* - \varepsilon)\lambda^{\alpha-1}}{\lambda^{\alpha-1} - 1}\,,$$

which implies

$$\liminf_{t\to\infty} t^{\alpha-1} \int_t^{\infty} p(s)\,\Delta s \ge \frac{L_* \lambda^{\alpha-1}}{\lambda^{\alpha-1} - 1}\,.$$

Now suppose that there exist $\lambda > 1$ and $L^* > 0$ such that

$$\limsup_{t\to\infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s)\,\Delta s \le L^*.$$

Let $\varepsilon > 0$. Take $t \in \mathbb{T}$ sufficiently large. Then

$$(\lambda^{r_n} t)^{\alpha-1} \int_{\lambda^{r_n} t}^{\tau(\lambda^{1+r_n} t)} p(s)\Delta s \le L^* + \varepsilon \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}\,.$$

Hence,

$$t^{\alpha-1}\int_{\lambda^{r_n}t}^{\tau(\lambda^{1+r_n}t)} p(s)\Delta s \le \frac{L^*+\varepsilon}{(\lambda^{\alpha-1})^{r_n}} \quad \text{for all} \quad n\in\mathbb{N}\cup\{0\}\,.$$

Summing this inequality for n from 0 to ∞ we get

$$t^{\alpha-1}\sum_{n=0}^{\infty}\int_{\lambda^{r_n}t}^{\tau(\lambda^{1+r_n}t)} p(s)\Delta s \le (L^*+\varepsilon)\sum_{n=0}^{\infty}\frac{1}{(\lambda^{\alpha-1})^{r_n}}\,. \tag{39}$$

In view of property (ii), it is clear that the series on the right-hand side of the inequality (39) can be majorized by the convergent series

$$\sum_{n=0}^{\infty}\frac{1}{(\lambda^{\alpha-1})^{n-1}}$$

for each sufficiently large t. Hence, using the property (iv), resp. $1+r_n(t)\to r_{n+1}(t)$ as $t\to\infty$ following from (iv), (39) implies

$$\limsup_{t\to\infty} t^{\alpha-1}\sum_{n=0}^{\infty}\int_{\lambda^{r_n}t}^{\lambda^{r_{n+1}}t} p(s)\Delta s \le L^*\sum_{n=0}^{\infty}\frac{1}{(\lambda^{\alpha-1})^{n}}\,,$$

i.e.,

$$\limsup_{t\to\infty} t^{\alpha-1}\int_t^{\infty} p(s)\,\Delta s \le \frac{L^*\lambda^{\alpha-1}}{\lambda^{\alpha-1}-1}\,.$$

Therefore, if $L=L_*=L^*$ part "If" follows.

"Only if": Let $P=A$ and let $\lambda>1$ be an arbitrary real number. Then

$$\begin{aligned} t^{\alpha-1}\int_t^{\infty} p(s)\,\Delta s &= t^{\alpha-1}\int_t^{\tau(\lambda t)} p(s)\,\Delta s + t^{\alpha-1}\int_{\tau(\lambda t)}^{\infty} p(s)\,\Delta s \\ &= t^{\alpha-1}\int_t^{\tau(\lambda t)} p(s)\,\Delta s + \frac{t^{\alpha-1}}{(\tau(\lambda t))^{\alpha-1}}(\tau(\lambda t))^{\alpha-1}\int_{\tau(\lambda t)}^{\infty} p(s)\,\Delta s. \end{aligned}$$

Since $(\tau(\lambda t))^{\alpha-1}\int_{\tau(\lambda t)}^{\infty} p(s)\,\Delta s \to A$ and $t/\tau(\lambda t)\to 1/\lambda$ as $t\to\infty$, we get

$$A = P_\lambda + \frac{1}{\lambda^{\alpha-1}}\,A\,, \quad \text{i.e.,} \quad P_\lambda = \frac{A(\lambda^{\alpha-1}-1)}{\lambda^{\alpha-1}}\,.$$

□

In view of the previous results, we get the following statement.

Corollary 1. *All positive solutions of* (5) *are Karamata functions if and only if for every* $\lambda>1$ *there exists the* (*finite or infinite*) *limit*

$$\lim_{t\to\infty} t\int_t^{\tau(\lambda t)} p(s)\Delta s\,. \tag{40}$$

All positive decreasing solutions of (4) *are Karamata functions if and only if for every* $\lambda > 1$ *there exists the* (*finite or infinite*) *limit*

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^{\tau(\lambda t)} p(s)\Delta s.$$

6. Concluding comments and open problems

Similarly as in the theory of regular variation on time scales, see [24], in the theory of rapid variation on $\mathbb{T}$ we distinguish three cases:

(i) $\mu(t) = o(t)$

If we want to obtain a reasonable theory, which corresponds from a certain point of view to the continuous (or discrete) theory, we need this additional requirement on the graininess of the time scale, which cannot be improved. Indeed, consider the graininess $\mu(t) = O(t)$ such that $\mu(t) \neq o(t)$ and take, e.g., the function $f(t) = (1/2)^t$ on $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$ ($\mu(t) = (q-1)t$). We expect that $f \in (\mathcal{N})\mathcal{RPV}_{\mathbb{T}}(-\infty)$. But

$$\lim_{t\to\infty} \frac{t\, f^{\Delta}(t)}{f(t)} = \frac{1}{1-q} \neq -\infty\,.$$

(ii) $\mu(t) = (q-1)t$, with $q > 1$

In [21], we established the theory of q-rapid variation, which means that the considered functions are defined as in the q-calculus, i.e., on $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$ ($\mu(t) = (q-1)t$). A function $f : q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-rapidly varying of index* ∞, resp. *of index* $-\infty$ if

$$\lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = \infty := [\infty]_q\,, \qquad \text{resp.} \qquad \lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = \frac{1}{1-q} := [-\infty]_q,$$

where $D_q f(t) = [f(qt) - f(t)]/[(q-1)t]$ is the q-derivative of a function f. The totality of q-rapidly varying functions of index $\pm\infty$ is denoted by $\mathcal{RPV}_q(\pm\infty)$. It is easy to see that the function $f(t) = a^t$ with $a > 1$, resp. $a \in (0,1)$ is a typical representative of the class $\mathcal{RPV}_q(\infty)$, resp. $\mathcal{RPV}_q(-\infty)$. Note that the theory of q-rapid variation similarly as a theory of q-regular variation, see [22], was established by using suitable modifications of the "classical" theories.

(iii) Other cases

If the graininess is eventually "very big" (or a combination of "very big" and "small"), then the theory gives no proper results. Indeed, for instance, let $\mathbb{T} = 2^{p^{\mathbb{N}_0}} = \{2^{p^k} : k \in \mathbb{N}_0\}$ with $p > 1$. Take the function $f(t) = t^{\vartheta}$ with $\vartheta > 1$. The function $f(t)$ is a typical representative of class $(\mathcal{N})\mathcal{RV}_{\mathbb{T}}(\vartheta)$, see (6). But on this time scale we can observe that if we use Definition 1, then $tf^{\Delta}(t)/f(t) = t((t^p)^{\vartheta} - t^{\vartheta})/(t^{\vartheta}(t^p - t)) = (t^{\vartheta(p-1)} - 1)/(t^{p-1} - 1) \to \infty$ as $t \to \infty$, hence $f \in (\mathcal{N})\mathcal{RPV}_{\mathbb{T}}(\infty)$. Again, let $\mathbb{T} = 2^{p^{\mathbb{N}_0}}$ with $p > 1$. Take $f(t) = a^t$, $a \neq 1$. We expect that $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$ for $a > 1$ and $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$ for $a < 1$. But for $\lambda > 1$ we get $f(\tau(\lambda t))/f(t) \to 1$ as $t \to \infty$ (really, on this time scale for each $\lambda > 1$ there exists $t_0 \in \mathbb{T}$ such that $\tau(\lambda t) = t$ for $t > t_0$) and therefore $f \notin \mathcal{KRPV}_{\mathbb{T}}(\pm\infty)$.

From the above observations, we conclude that it is advisable to consider only the cases (i) and (ii) in the theory of rapid variation on time scales.

At the end of this section we give few remarks about open problems and perspectives related to the topic of this paper.

Remark 4. In view of Proposition 2, part (II), cases (iii) and (iv), we can naturally ask, whether the following condition

$$\lim_{t\to\infty}\frac{f(\tau(\lambda t))}{f(t)}=\infty \text{ (resp. } 0) \quad \lambda>1 \quad \Leftrightarrow \quad \lim_{t\to\infty}\frac{f(\tau(\lambda t))}{f(t)}=0 \text{ (resp. } \infty) \quad \lambda\in(0,1)$$

holds as in the cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. We conjecture that if f is positive and monotone, then this equivalence holds.

Remark 5. Looking at relation (33) (resp. (37)), which can be rewritten as

$$\mathbb{M}^-=\mathbb{M}^-_{RPV}(-\infty)=\mathbb{M}^-_0 \text{ and } \mathbb{M}^+=\mathbb{M}^+_{RPV}(\infty)=\mathbb{M}^+_\infty \iff (32) \text{ (resp. (36))},$$

it is not known (even in continuous and discrete case) whether

$$M^+=\mathbb{M}^+_{RPV}(\infty)=\mathbb{M}^+_\infty \implies (32) \quad \text{(resp.(36))}\,. \tag{41}$$

If the implication (41) is true, then the theory of asymptotic behavior of all solutions of equation (5) is complete and we can claim (compare with Corollary 1) :

"*There exists a positive solution y of* (5) *such that $y\in\mathcal{KF}_\mathbb{T}$* (*resp. $y\notin\mathcal{KF}_\mathbb{T}$*) *if and only if every positive solution y of* (5) *satisfies $y\in\mathcal{KF}_\mathbb{T}$* (*resp. $y\notin\mathcal{KF}_\mathbb{T}$*) *if and only if the limit* (40) *exists* (*resp. does not exist*)*. Specially, there exists a positive decreasing solution u of* (5) *such that $u\in\mathcal{RPV}_\mathbb{T}(-\infty)$ if and only if there exists a positive increasing solution v of* (5) *such that $v\in\mathcal{RPV}_\mathbb{T}(\infty)$ if and only if the limit* (40) *is equal ∞.*"

On the other hand, thanks to the existence of a function p satisfying condition (38) we know that a positive decreasing "No-Karamata" solution $u\notin\mathcal{KF}_\mathbb{T}$ of (5) really exists. Indeed, it can be obtained as a decreasing solution of (5) with the mentioned coefficient p. However, the existence of an increasing solution v of (5) such that $v\notin\mathcal{KF}_\mathbb{T}$ has not been shown yet. From the above observations, there are three possibilities for a fundamental set of rapidly varying solutions of (5):

(i) $u\in\mathcal{RPV}_\mathbb{T}(-\infty)$, $v\in\mathcal{RPV}_\mathbb{T}(\infty)$.

(ii) $u\notin\mathcal{KF}_\mathbb{T}$ such that u is a positive decreasing, $v\in\mathcal{RPV}_\mathbb{T}(\infty)$.

(iii) $u\notin\mathcal{KF}_\mathbb{T}$ such that u is a positive decreasing, $v\notin\mathcal{KF}_\mathbb{T}$ such that v is a positive increasing.

Finally note that further possible research related to equation (4) could be the following one - to establish necessary and sufficient condition for all positive increasing solutions of (4) to be regularly varying.

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