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ALMOST PERIODIC SEQUENCES AND FUNCTIONS WITH GIVEN VALUES

Michal Veselý

Abstract. We present a method for constructing almost periodic sequences and functions with values in a metric space. Applying this method, we find almost periodic sequences and functions with prescribed values. Especially, for any totally bounded countable set X in a metric space, it is proved the existence of an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ such that $\{\psi_k;\ k \in \mathbb{Z}\} = X$ and $\psi_k = \psi_{k+lq(k)}$, $l \in \mathbb{Z}$ for all k and some $q(k) \in \mathbb{N}$ which depends on k.

1. Introduction

The aim of this paper is to construct almost periodic sequences and functions which attain values in a metric space. More precisely, our aim is to find almost periodic sequences and functions whose ranges contain or consist of arbitrarily given subsets of the metric space satisfying certain conditions. We are motivated by the paper [3] where a similar problem is investigated for real-valued sequences. In that paper, using an explicit construction, it is shown that, for any bounded countable set of real numbers, there exists an almost periodic sequence whose range is this set and which attains each value in this set periodically. We will extend this result to sequences attaining values in any metric space.

Concerning almost periodic sequences with indices $k \in \mathbb{N}$ (or asymptotically almost periodic sequences), we refer to [4] where it is proved that, for any precompact sequence $\{x_k\}_{k\in\mathbb{N}}$ in a metric space $\mathcal{X}$, there exists a permutation P of the set of positive integers such that the sequence $\{x_{P(k)}\}_{k\in\mathbb{N}}$ is almost periodic. Let us point out that the definition of the asymptotic almost periodicity in [4] is based on the Bochner concept; i.e., a bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathcal{X}$ is called almost periodic if the set of sequences $\{x_{k+p}\}_{k\in\mathbb{N}}$, $p \in \mathbb{N}$, is precompact in the space of all bounded sequences in $\mathcal{X}$. It is known that, for sequences and functions with values in complete metric spaces, the Bochner definition is equivalent with the Bohr definition which is used in this paper. Moreover, these definitions remain also equivalent in an arbitrary metric space if one replaces the convergence in the Bochner definition by the Cauchy property (see [8], [9]). But, it is seen that the

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result of [4] for the almost periodicity on $\mathbb{N}$ cannot be true for the almost periodicity on $\mathbb{Z}$ or $\mathbb{R}$.

For almost periodic functions, we prove a theorem corresponding to the above mentioned one for sequences. In addition, we need that the given set is the range of a uniformly continuous function φ for which the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite. We also use the result for sequences to construct an almost periodic function whose range contains an arbitrarily given totally bounded sequence if one requires the local connection by arcs of the space of values.

In a Banach space, an other important necessary and sufficient condition for a function to be almost periodic is that it has the approximation property; i.e., a function is almost periodic if and only if there exists a sequence of trigonometric polynomials which converges uniformly to the function on the whole real line in the norm topology (see [2, Theorems 6.8, 6.15]). There exist generalizations of this result. For example, it is proved in [1] that an almost periodic function with fuzzy real numbers as values can be uniformly approximated by a sequence of generalized trigonometric polynomials. We add that fuzzy real numbers form a complete metric space. One shows that the approximation theorem remains generally unvalid if one does not require the completeness of the space of values. Thus, we cannot use this idea in our constructions for general metric spaces.

The paper is organized as follows. First of all, in Section 2, we define the notion of the almost periodicity in metric spaces. The definition is similar to the classical one of H. Bohr, only the modulus being replaced by the distance. Then, in Theorems 1 and 2, we present a process which facilitates to construct almost periodic sequences and functions having certain properties. In Sections 3 and 4, we prove the above mentioned main theorems for sequences and functions, respectively.

2. Almost periodic sequences and functions in metric spaces

Let $\mathcal{X}$ be an arbitrary metric space with a metric ϱ. For given $\varepsilon > 0$ and $x \in \mathcal{X}$, the *ε-neighbourhood* of x in $\mathcal{X}$ will be denoted by $\mathcal{O}_\varepsilon(x)$ and, as usual, $\mathbb{R}_0^+$ will denote the set of all nonnegative reals.

First we recall the definition of the Bohr almost periodicity in metric spaces (see, e.g., [8], [9]).

Definition 1. *A sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ is* almost periodic *if for every $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that any set consisting of $p(\varepsilon)$ consecutive integers contains at least one integer l for which*

$$\varrho\,(\psi_{k+l}, \psi_k) < \varepsilon\,, \quad k \in \mathbb{Z}\,.$$

*The number l is called an ε-*translation number *of $\{\psi_k\}$.*

Definition 2. *A* continuous *function $\psi\colon \mathbb{R} \to \mathcal{X}$ is* almost periodic *if for every $\varepsilon > 0$, there exists a number $p(\varepsilon) > 0$ with the property that any interval of length $p(\varepsilon)$ of the real line contains at least one point s for which*

$$\varrho\big(\psi(t+s), \psi(t)\big) < \varepsilon\,, \quad -\infty < t < +\infty\,.$$

*Similarly as in Definition 1, s is called an ε-*translation number*.*

Remark 1. It follows directly from Definition 1 that the set $\{\psi_k;\, k \in \mathbb{Z}\}$ is totally bounded in $\mathcal{X}$ if $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ is almost periodic. Analogously, the set of values of an almost periodic function with values in $\mathcal{X}$ is totally bounded in $\mathcal{X}$. One can prove it by a trivial generalization of the proof of [2, Theorem 6.5]. This result is well-known for Banach spaces, where the totally bounded (precompact) sets coincide with the relatively compact sets.

Now we mention the method of constructions of almost periodic sequences and functions in $\mathcal{X}$ which we will use later. Another methods of generating almost periodic sequences and functions with prescribed properties are also presented in [6, Section 4] and [5], respectively.

Theorem 1. *Let $\psi_0 \in \mathcal{X}$ and $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ be arbitrarily given so that*

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty \tag{1}$$

holds. Then, every sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ for which it is true

$$\begin{aligned}
&\psi_k \in \mathcal{O}_{\varepsilon_1}(\psi_{k-2^0}), && k \in \{1\} = \{2-1\},\\
&\psi_k \in \mathcal{O}_{\varepsilon_2}(\psi_{k+2^1}), && k \in \{-2,-1\},\\
&\psi_k \in \mathcal{O}_{\varepsilon_3}(\psi_{k-2^2}), && k \in \{2,\dots,2+2^2-1\},\\
&\psi_k \in \mathcal{O}_{\varepsilon_4}(\psi_{k+2^3}), && k \in \{-2^3-2,\dots,-2-1\},\\
&\psi_k \in \mathcal{O}_{\varepsilon_5}(\psi_{k-2^4}), && k \in \{2+2^2,\dots,2+2^2+2^4-1\},\\
&\quad\vdots
\end{aligned}$$

$$\psi_k \in \mathcal{O}_{\varepsilon_{2i}}(\psi_{k+2^{2i-1}}),\ k \in \{-2^{2i-1}-\dots-2^3-2,\dots,-2^{2i-3}-\dots-2^3-2-1\},$$

$$\psi_k \in \mathcal{O}_{\varepsilon_{2i+1}}(\psi_{k-2^{2i}}),\ k \in \{2+2^2+\dots+2^{2i-2},\dots,2+2^2+\dots+2^{2i-2}+2^{2i}-1\},$$

$$\vdots$$

is almost periodic.

Proof. The theorem follows from [9, Theorem 3.5] where one puts $m = 0$, $j = 1$. □

Theorem 2. *Let $M > 0$ and $x_0 \in \mathcal{X}$ be given and let $\varphi\colon [0,M] \to \mathcal{X}$ be such that*

$$\varphi(0) = \varphi(M) = x_0 .$$

If $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ satisfies (1), then any continuous function $\psi\colon \mathbb{R} \to \mathcal{X}$, $\psi|_{[0,M]} \equiv \varphi$ for which

$$\begin{aligned}\psi(t) = x_0, \quad t \in \{2M,-2M\} &\cup \{(2+2^2+\dots+2^{2(i-1)}+2^{2i})M;\, i \in \mathbb{N}\}\\ &\cup \{-(2+2^3+\dots+2^{2i-1}+2^{2i+1})M;\, i \in \mathbb{N}\}\end{aligned} \tag{2}$$

and, at the same time, for which it is valid

$$
\begin{aligned}
&\psi(t) \in \mathcal{O}_{\varepsilon_1}\left(\psi(t-M)\right), && t \in (M, 2M),\\
&\psi(t) \in \mathcal{O}_{\varepsilon_2}\left(\psi(t+2M)\right), && t \in (-2M, 0),\\
&\psi(t) \in \mathcal{O}_{\varepsilon_3}\left(\psi(t-2^2M)\right), && t \in (2M, (2+2^2)M),\\
&\psi(t) \in \mathcal{O}_{\varepsilon_4}\left(\psi(t+2^3M)\right), && t \in (-(2^3+2)M, -2M),\\
&\psi(t) \in \mathcal{O}_{\varepsilon_5}\left(\psi(t-2^4M)\right), && t \in ((2+2^2)M, (2+2^2+2^4)M),\\
&\quad\vdots\\
&\psi(t) \in \mathcal{O}_{\varepsilon_{2i}}\left(\psi(t+2^{2i-1}M)\right), && t \in (-(2^{2i-1}+\cdots+2)M, -(2^{2i-3}+\cdots+2)M),\\
&\psi(t) \in \mathcal{O}_{\varepsilon_{2i+1}}\left(\psi(t-2^{2i}M)\right), && t \in ((2+2^2+\cdots+2^{2i-2})M, (2+2^2+\cdots+2^{2i})M),\\
&\quad\vdots
\end{aligned}
$$

is almost periodic.

Proof. See [8, Theorem 3.2] for $j = 1$. □

3. Almost periodic sequences with given values

In this section, we prove that, for a countable subset of $\mathcal{X}$, there exists an almost periodic sequence whose range is exactly this set. Since the range of any almost periodic sequence is totally bounded (see Remark 1), this requirement on the set is necessary. Now we prove that the condition is sufficient as well.

Theorem 3. *Let any countable and totally bounded set $X \subseteq \mathcal{X}$ be given. There exists an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying*

(3) $$\{\psi_k;\, k \in \mathbb{Z}\} = X$$

with the property that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ such that

(4) $$\psi_l = \psi_{l+jq(l)}, \quad j \in \mathbb{Z}.$$

Proof. Let us put

$$X = \{\varphi_k;\, k \in \mathbb{N}\}.$$

Without loss of the generality we can assume that the set $\{\varphi_k;\, k \in \mathbb{N}\}$ is infinite because, for only finitely many different φ_k, we can define $\{\psi_k\}$ as periodic. Since $\{\varphi_k;\, k \in \mathbb{N}\}$ is totally bounded, for any $\varepsilon > 0$, it can be imbedded into a finite number of spheres of radius ε. Let us denote by $x_1^i, \dots, x_{m(i)}^i$ the centres of the spheres of radius 2^{-i} which cover the set for all $i \in \mathbb{N}$. Evidently, we can also assume that

(5) $$x_1^i, \dots, x_{m(i)}^i \in \{\varphi_k\,;\, k \in \mathbb{N}\}, \ i \in \mathbb{N},$$

and that

(6) $$x_1^i = \varphi_i, \quad i \in \mathbb{N}.$$

We will construct $\{\psi_k\}$ applying Theorem 1. We choose arbitrary $n(1) \in \mathbb{N}$ for which $2^{2n(1)} > m(1)$. We put

$$\begin{aligned}&\psi_0 := x_1^1,\ \psi_1 := x_2^1,\ \dots,\ \psi_{m(1)-1} := x_{m(1)}^1\,,\\ &\psi_k := x_1^1,\ k \in \{-2^{2n(1)-1} - \dots - 2^3 - 2, \dots, -1\}\\ &\qquad\qquad \cup \{m(1), \dots, 2 + 2^2 + \dots + 2^{2n(1)} - 1\}\end{aligned}$$

and

$$\varepsilon_k := L, \quad k \in \{1, \dots, 2n(1) + 1\}\,, \tag{7}$$

where

$$L := \max_{i,j \in \{1,\dots,m(1)\}} \varrho(x_i^1, x_j^1) + 1\,.$$

In the second step, we choose $n(2) > n(1) + m(2)$ $(n(2) \in \mathbb{N})$. We define

$$\begin{aligned}&\psi_k := \psi_{k+2^{2n(1)+1}}\,, k \in \{-2^{2n(1)+1} - \dots - 2^3 - 2, \dots, -2^{2n(1)-1} - \dots - 2^3 - 2 - 1\}\,,\\ &\psi_k := \psi_{k-2^{2n(1)+2}}\,, k \in \{2 + 2^2 + \dots + 2^{2n(1)}, \dots, 2 + 2^2 + \dots + 2^{2n(1)+2} - 1\}\,,\\ &\quad\vdots\\ &\psi_k := \psi_{k+2^{2n(2)-1}}\,, k \in \{-2^{2n(2)-1} - \dots - 2^3 - 2, \dots, -2^{2n(2)-3} - \dots - 2^3 - 2 - 1\}\end{aligned}$$

and we put

$$\varepsilon_k := 0\,, \quad k \in \{2n(1) + 2, \dots, 2n(2)\}\,, \quad \varepsilon_{2n(2)+1} := 2^{-1}\,. \tag{8}$$

Since $n(2) > n(1) + m(2)$, from the above definition of ψ_k, it follows that, for each $j \in \{1, \dots, m(1)\}$, there exist at least $2m(2) + 2$ integers

$$l \in \{-2^{2n(2)-1} - \dots - 2^3 - 2, \dots, 2^{2n(2)-2} + \dots + 2^2 + 2 - 1\}$$

such that $\psi_l = x_j^1$. Thus, we can define

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(2)+1}}\left(\psi_{k-2^{2n(2)}}\right)\,,\ k \in \{2 + 2^2 + \dots + 2^{2n(2)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(2)} - 1\}$$

with the property that

$$\begin{aligned}\{\psi_k\,;\ k \in &\{2 + 2^2 + \dots + 2^{2n(2)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(2)-2} + 2^{2n(2)} - 1\}\}\\ =&\{x_1^1, \dots, x_{m(1)}^1, x_1^2, \dots, x_{m(2)}^2\}\,.\end{aligned}$$

In addition, we can put

$$\psi_{2^{2n(2)}} := \psi_0 = x_1^1 \tag{9}$$

and we can assume that

$$\psi_k = x_1^1 \quad \text{for some} \quad k \in \{2 + \dots + 2^{2n(2)-2}, \dots, 2 + \dots + 2^{2n(2)} - 1\} \setminus \{2^{2n(2)}\}\,.$$

In the third step, we choose $n(3) > n(2) + m(3)$ $(n(3) \in \mathbb{N})$ and we proceed analogously. We construct $\{\psi_k\}$ for

$$k \in \{-2^{2n(2)+1} - \cdots - 2^3 - 2, \ldots, -2^{2n(2)-1} - \cdots - 2^3 - 2 - 1\},$$
$$\vdots$$
$$k \in \{-2^{2n(3)-1} - \cdots - 2^3 - 2, \ldots, -2^{2n(3)-3} - \cdots - 2^3 - 2 - 1\}$$

as in the $2(n(2)+1)$-th, …, $2n(3)$-th steps of the process (mentioned in Theorem 1) for

$$\varepsilon_k := 0, \quad k \in \{2n(2) + 2, \ldots, 2n(3)\}. \tag{10}$$

Especially, we have

$$\begin{aligned} &\psi_k = x_1^1, \quad k \in J_0^3, \\ &J_0^3 := \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\} \cap \{-2^{2n(3)-1} - \cdots - 2, \ldots, 2 + \cdots + 2^{2n(3)-2} - 1\}. \end{aligned} \tag{11}$$

As in the second step, for all $j(1) \in \{1, 2\}$ and $j(2) \in \{1, \ldots, m(j(1))\}$, there exist at least $2m(3) + 2$ integers

$$l \in \{-2^{2n(3)-1} - \cdots - 2^3 - 2, \ldots, 2 + 2^2 + \cdots + 2^{2n(3)-2} - 1\} \setminus \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\}$$

such that $\psi_l = x_{j(2)}^{j(1)}$. It is seen that, to get

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(3)+1}}\left(\psi_{k-2^{2n(3)}}\right),\ k \in \{2+2^2+\cdots+2^{2n(3)-2}, \ldots, 2+2^2+\cdots+2^{2n(3)}-1\},$$

where

$$\varepsilon_{2n(3)+1} := 2^{-2}, \tag{12}$$

satisfying

$$\begin{aligned} \{\psi_k;\, k \in &\{2 + 2^2 + \cdots + 2^{2n(3)-2}, \ldots, 2 + 2^2 + \cdots + 2^{2n(3)-2} + 2^{2n(3)} - 1\}\} \\ &= \{x_1^1, \ldots, x_{m(1)}^1, \ldots, x_1^3, \ldots, x_{m(3)}^3\}, \end{aligned}$$

we need less than (or equal to) $m(3) + 1$ such integers l. Thus, we can define these ψ_k so that

$$\begin{aligned} &\psi_k = x_1^1, \quad k \in I_0^3, \\ &I_0^3 := \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(3)-2}, \ldots, 2 + \cdots + 2^{2n(3)} - 1\}, \end{aligned} \tag{13}$$

$$\psi_{2^{2n(3)}+1} = \psi_1 = x_2^1, \quad \psi_{2^{2n(3)}-1} = \psi_{-1} = x_1^1, \tag{14}$$

$$\begin{aligned} \psi_k = \psi_1 &\quad \text{for some } k \in \{2 + \cdots + 2^{2n(3)-2}, \ldots, 2 + \cdots + 2^{2n(3)} - 1\} \setminus \{2^{2n(3)} + 1\}, \\ \psi_k = \psi_{-1} &\quad \text{for some } k \in \{2 + \cdots + 2^{2n(3)-2}, \ldots, 2 + \cdots + 2^{2n(3)} - 1\} \setminus \{2^{2n(3)} - 1\}. \end{aligned}$$

We proceed further in the same way. In the i-th step, we have $n(i) > n(i-1)+m(i)$ $(n(i) \in \mathbb{N})$ and

$$\psi_k := \psi_{k+2^{2n(i-1)+1}}\,, \quad k \in \{-2^{2n(i-1)+1} - \cdots - 2, \dots, -2^{2n(i-1)-1} - \cdots - 2 - 1\}\,,$$
$$\vdots$$
$$\psi_k := \psi_{k+2^{2n(i)-1}}\,, \qquad k \in \{-2^{2n(i)-1} - \cdots - 2, \dots, -2^{2n(i)-3} - \cdots - 2 - 1\}$$

and we denote

$$\varepsilon_k := 0\,, \quad k \in \{2n(i-1)+2, \dots, 2n(i)\}\,, \quad \varepsilon_{2n(i)+1} := 2^{-i+1}\,. \tag{15}$$

We have also

$$\begin{aligned} &\psi_k = \psi_0, \quad k \in J_0^i\,, \\ &J_0^i := \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\} \cap \{-2^{2n(i)-1} - \cdots - 2, \dots, 2 + \cdots + 2^{2n(i)-2} - 1\}\,, \end{aligned} \tag{16}$$

$$\begin{aligned} &\psi_k = \psi_1, \quad k \in J_1^i\,, \\ &J_1^i := \{1 + j\, 2^{2n(3)}; j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}\,, \end{aligned} \tag{17}$$

$$\begin{aligned} &\psi_k = \psi_{-1}, \quad k \in J_{-1}^i\,, \\ &J_{-1}^i := \{-1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}\,, \end{aligned} \tag{18}$$

$$\vdots$$

$$\begin{aligned} &\psi_k = \psi_{i-3}, \quad k \in J_{i-3}^i\,, \\ &J_{i-3}^i := \{i - 3 + j\, 2^{2n(i-1)};\, j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}\,, \end{aligned}$$

$$\begin{aligned} &\psi_k = \psi_{-i+3}\,, \quad k \in J_{-i+3}^i\,, \\ &J_{-i+3}^i := \{-i + 3 + j\, 2^{2n(i-1)};\, j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\} \end{aligned}$$

if $i - 3 < 2^{2n(2)}$. If $2^{2n(2)} \le i - 3 < 2^{2n(2)+1}$, we have

$$\vdots$$

$$\begin{aligned} &\psi_k = \psi_{-2^{2n(2)}+1}\,, \quad k \in J_{-2^{2n(2)}+1}^i\,, \\ &J_{-2^{2n(2)}+1}^i := \{-2^{2n(2)} + 1 + j\, 2^{2n(2^{2n(2)}+1)};\, j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}\,, \end{aligned}$$

$$\begin{aligned} \psi_k &= \psi_{2^{2n(2)}+1}\,, \quad k \in J^i_{2^{2n(2)}+1}\,, \\ J^i_{2^{2n(2)}+1} &:= \{2^{2n(2)} + 1 + j\, 2^{2n(2^{2n(2)}+2)};\ j \in \mathbb{Z}\} \\ &\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \ldots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}\,, \\ &\vdots \end{aligned}$$

If $2^{2n(2)+1} \leq i - 3$, then we omit the values $\psi_{j\, 2^{2n(2)}}$, $\psi_{1+j\, 2^{2n(3)}}$, $\psi_{-1+j\, 2^{2n(3)}}$, … For simplicity, let $i - 2 < 2^{2n(2)}$.

Considering the construction, for all $j(1) \in \{1, \ldots, i-1\}$, $j(2) \in \{1, \ldots, m(j(1))\}$, there exist at least $2m(i) + 2$ integers

$$l \in \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \ldots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\} \setminus (J^i_0 \cup \cdots \cup J^i_{-i+3})$$

such that $\psi_l = x^{j(1)}_{j(2)}$. Evidently (similarly as in the third step), we can obtain

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(i)+1}}\left(\psi_{k-2^{2n(i)}}\right), \quad k \in \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\}$$

for which

$$\begin{aligned} &\{\psi_k;\ k \in \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)-2} + 2^{2n(i)} - 1\}\} \\ &\qquad = \{x^1_1, \ldots, x^1_{m(1)}, \ldots, x^i_1, \ldots, x^i_{m(i)}\}\,, \end{aligned} \tag{19}$$

and, in addition, we have

$$\begin{aligned} \psi_k &= \psi_0\,, \quad k \in I^i_0\,, \\ I^i_0 &:= \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\}, \end{aligned} \tag{20}$$

$$\begin{aligned} \psi_k &= \psi_1\,, \quad k \in I^i_1\,, \\ I^i_1 &:= \{1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\}\,, \end{aligned} \tag{21}$$

$$\begin{aligned} \psi_k &= \psi_{-1}\,, \quad k \in I^i_{-1}\,, \\ I^i_{-1} &:= \{-1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\}\,, \end{aligned} \tag{22}$$

$$\vdots$$

$$\begin{aligned} \psi_k &= \psi_{i-3}\,, \quad k \in I^i_{i-3}\,, \\ I^i_{i-3} &:= \{i-3+j\, 2^{2n(i-1)};\ j \in \mathbb{Z}\} \cap \{2+\cdots+2^{2n(i)-2}, \ldots, 2+\cdots+2^{2n(i)} - 1\}\,, \end{aligned}$$

$$\begin{aligned} \psi_k &= \psi_{-i+3}\,, \quad k \in I^i_{-i+3}\,, \\ I^i_{-i+3} &:= \{-i+3+j\, 2^{2n(i-1)};\ j \in \mathbb{Z}\} \cap \{2+\cdots+2^{2n(i)-2}, \ldots, 2+\cdots+2^{2n(i)} - 1\}\,, \end{aligned}$$

and

$$\psi_k = \psi_{i-2}\,, \quad k = 2^{2n(i)} + i - 2\,, \quad \psi_k = \psi_{-i+2}\,, \quad k = 2^{2n(i)} - i + 2\,,$$

$$\psi_k = \psi_{i-2}$$
$$\text{for some } k \in \{2+\dots+ 2^{2n(i)-2}, \dots, 2+\dots+2^{2n(i)} - 1\} \setminus \{2^{2n(i)} + i - 2\}\,,$$

$$\psi_k = \psi_{-i+2}$$
$$\text{for some } k \in \{2+\dots+2^{2n(i)-2}, \dots, 2+\dots+2^{2n(i)} - 1\} \setminus \{2^{2n(i)} - i + 2\}\,.$$

Using this construction, we get the sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ with the property that (see (9), (11), (13), (16), (20))

$$\psi_k = \psi_0, \quad k \in \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\}$$

and that (see (14), (17), (18), (21), (22))

$$\psi_k = \psi_1\,, \qquad k \in \{1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\}\,,$$
$$\psi_k = \psi_{-1}\,, \quad k \in \{-1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\}\,,$$

and so on; i.e., for any $l \in \mathbb{Z}$, there exists $i(l) \in \mathbb{N}$ satisfying

$$\psi_k = \psi_l\,, \quad k \in \{l + j\, 2^{2n(i(l))};\, j \in \mathbb{Z}\}\,. \tag{23}$$

Now it suffices to show that the sequence $\{\psi_k\}$ is almost periodic. Indeed, (3) follows from the process, (5), (6), and (19); (4) follows from (23) for $q(l) = 2^{2n(i(l))}$. Since we construct $\{\psi_k\}$ using Theorem 1, $\{\psi_k\}$ is almost periodic if (1) is satisfied. Immediately, see (7), (8), (10), (12), (15), we have

$$\sum_{i=1}^{\infty} \varepsilon_i = L\,(2n(1) + 1) + 1 \tag{24}$$

which completes the proof. □

4. Almost periodic functions with given values

Concerning a continuous counterpart of Theorem 3, the given set of values has to be the totally bounded graph of a continuous function (see Definition 2 and Remark 1). In addition, any almost periodic function is uniformly continuous (see, e.g., [8, Lemma 2.2]). Considering these facts, we formulate the continuous version of Theorem 3.

Theorem 4. *Let $\varphi\colon \mathbb{R} \to \mathcal{X}$ be any uniformly continuous function such that the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite and the set $\{\varphi(t);\, t \in \mathbb{R}\}$ is totally bounded. There exists an almost periodic function ψ with the property that*

$$\{\psi(k);\, k \in \mathbb{Z}\} = \{\varphi(k);\, k \in \mathbb{Z}\}\,, \quad \{\psi(t);\, t \in \mathbb{R}\} = \{\varphi(t);\, t \in \mathbb{R}\} \tag{25}$$

and that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ for which

$$\psi(l + s) = \psi\big(l + s + jq(l)\big)\,, \quad j \in \mathbb{Z}\,, \quad s \in [0, 1)\,. \tag{26}$$

Proof. We will construct $\psi\colon \mathbb{R} \to \mathcal{X}$ using Theorem 2 similarly as $\{\psi_k\}$ applying Theorem 1 in the proof of Theorem 3. Considering that the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite, let sufficiently large $M, N \in \mathbb{Z}$ have the property that $\varphi(M) = \varphi(N)$ and that, for any $l \in \mathbb{Z}$, there exists $j(l) \in \{N, N+1, \dots, M-1\}$ for which

$$\varphi(l) = \varphi\big(j(l)\big)\,, \quad \varphi(l+1) = \varphi\big(j(l)+1\big)\,. \tag{27}$$

Without loss of the generality, we can assume that $N = 0$ because, if $N < 0$, then we can redefine finitely many the below given ε_i and put $\psi \equiv \varphi$ on a sufficiently large interval.

Since φ is uniformly continuous with totally bounded range (see also (27)), for arbitrarily small $\varepsilon > 0$, there exist $l_1(\varepsilon), \dots, l_{m(\varepsilon)}(\varepsilon) \in \mathbb{Z}$ such that, for any $l \in \mathbb{Z}$, we have

$$\varrho\big(\varphi(l+s), \varphi(l_i+s)\big) < \varepsilon\,, \quad s \in [0,1]$$

for at least one integer $l_i \in \{l_1(\varepsilon), \dots, l_{m(\varepsilon)}(\varepsilon)\}$. We put $\varepsilon_i := 2^{-i}$, $i \in \mathbb{N}$, i.e.,

$$l_1^i := l_1(2^{-i}), \dots, l_{m(i)}^i := l_{m(2^{-i})}(2^{-i})\,, \quad i \in \mathbb{N}\,.$$

In addition, we will assume that

$$\{l_j^i;\, j \in \{1, \dots, m(i)\}, i \in \mathbb{N}\} = \mathbb{Z}\,. \tag{28}$$

First we define

$$\psi(t) := \varphi(t)\,, \quad t \in [0, M]\,. \tag{29}$$

We choose arbitrary $n(1) \in \mathbb{N}$ for which $2^{2n(1)}M > m(1)$. There exist (see (27))

$$j_1^1, j_2^1, \dots, j_{m(1)}^1 \in \{0, 1, \dots, M-1\}$$

such that

$$\begin{aligned}
\varphi(l_1^1) &= \psi(j_1^1)\,, \quad \varphi(l_1^1+1) = \psi(j_1^1+1)\,,\\
\varphi(l_2^1) &= \psi(j_2^1)\,, \quad \varphi(l_2^1+1) = \psi(j_2^1+1)\,,\\
&\;\;\vdots\\
\varphi(l_{m(1)}^1) &= \psi(j_{m(1)}^1), \quad \varphi(l_{m(1)}^1+1) = \psi(j_{m(1)}^1+1)\,.
\end{aligned}$$

We define

$$\begin{aligned}
\psi(s+M+j_1^1) &:= \varphi(s+l_1^1)\,, && s \in [0,1]\,,\\
\psi(t) &:= \psi(t-M)\,, && t \in (M, 2M] \setminus [M+j_1^1, M+j_1^1+1]\,,\\
\psi(s+2M+j_2^1) &:= \varphi(s+l_2^1)\,, && s \in [0,1]\,,\\
\psi(t) &:= \psi(t-2M)\,, && t \in (2M, 3M] \setminus [2M+j_2^1, 2M+j_2^1+1]\,,\\
&\;\;\vdots\\
\psi(s+m(1)M+j_{m(1)}^1) &:= \varphi(s+l_{m(1)}^1)\,, && s \in [0,1]\,,\\
\psi(t) &:= \psi(t-m(1)M)\,, && t \in (m(1)M, (m(1)+1)M]\\
& && \quad \setminus [m(1)M+j_{m(1)}^1, m(1)M+j_{m(1)}^1+1]
\end{aligned}$$

and we define ψ as periodic with period M on

$$[-(2^{2n(1)-1}+\cdots+2^3+2)M,(2+2^2+\cdots+2^{2n(1)})M]\setminus(M,(m(1)+1)M).$$

It is easily to see that we construct ψ as in Theorem 2 for

$$\varepsilon_i := L, \quad i\in\{1,\dots,2n(1)+1\} \tag{30}$$

if $L>0$ is sufficiently large.

In the second step, we choose $n(2)>n(1)+m(2)$ ($n(2)\in\mathbb{N}$) and we put

$$\psi(t) := \psi(t+2^{2n(1)+1}M)\,,\ t\in[-(2^{2n(1)+1}+\cdots+2)M,\dots,-(2^{2n(1)-1}+\cdots+2)M)\,,$$
$$\psi(t) := \psi(t-2^{2n(1)+2}M)\,,\ t\in((2+\cdots+2^{2n(1)})M,\dots,(2+\cdots+2^{2n(1)+2})M]\,,$$
$$\vdots$$
$$\psi(t) := \psi(t+2^{2n(2)-1}M)\,,\ t\in[-(2^{2n(2)-1}+\cdots+2)M,\dots,-(2^{2n(2)-3}+\cdots+2)M)$$

and

$$\varepsilon_i := 0,\ i\in\{2n(1)+2,\dots,2n(2)\},\quad \varepsilon_{2n(2)+1} := 2^{-1}. \tag{31}$$

From $n(2)>n(1)+m(2)$ and the above construction, we see that, for each integer j, $1\le j\le m(1)$, there exist at least $2m(2)+2$ intervals of the form

$$[a,a+1]\subset[-(2^{2n(2)-1}+\cdots+2)M,\dots,(2^{2n(2)-2}+\cdots+2)M]$$

such that $a\in\mathbb{Z}$ and

$$\psi|_{[a,a+1]}\equiv\varphi|_{[l^1_j,l^1_j+1]},\quad \text{i.e.,}\quad \psi(s+a)=\varphi(s+l^1_j),\quad s\in[0,1].$$

It implies that we can define continuous

$$\psi(t)\in\mathcal{O}_{\varepsilon_{2n(2)+1}}\big(\psi(t-2^{2n(2)}M)\big),\ t\in((2+\cdots+2^{2n(2)-2})M,\dots,(2+\cdots+2^{2n(2)})M]$$

for which

$$\psi|_{[2^{2n(2)}M,2^{2n(2)}M+1]}\equiv\psi|_{[0,1]}\,,$$
$$\psi|_{[k,k+1]}\equiv\psi|_{[0,1]}\quad\text{for some } k\,,$$
$$k\in\{(2+\cdots+2^{2n(2)-2})M,\dots,(2+\cdots+2^{2n(2)})M-1\}\setminus\{2^{2n(2)}M\}$$

and

$$\psi|_{[l,l+1]}\equiv\varphi|_{[j(l),j(l)+1]},\ l\in\{(2+\cdots+2^{2n(2)-2})M,\dots,(2+\cdots+2^{2n(2)})M-1\}\,,$$
$$\text{some } j(l)\in\{0,\dots,M-1,l^1_1,\dots,l^1_{m(1)},l^2_1,\dots,l^2_{m(2)}\}\,,$$
$$\{\varphi(t);\ t\in[l^1_1,l^1_1+1]\cup\cdots\cup[l^1_{m(1)},l^1_{m(1)}+1]\cup[l^2_1,l^2_1+1]\cup\cdots\cup[l^2_{m(2)},l^2_{m(2)}+1]\}$$
$$\subseteq\{\psi(t);\ t\in[(2+\cdots+2^{2n(2)-2})M,\dots,(2+\cdots+2^{2n(2)})M]\}\,.$$

In the third step, we choose $n(3)>n(2)+m(3)$ ($n(3)\in\mathbb{N}$) and we construct ψ for

$$\varepsilon_i := 0,\quad i\in\{2n(2)+2,\dots,2n(3)\},\quad \varepsilon_{2n(3)+1} := 2^{-2}. \tag{32}$$

We have continuous

$$\psi(t)\in\mathcal{O}_{\varepsilon_{2n(3)+1}}\big(\psi(t-2^{2n(3)}M)\big),\ t\in((2+\cdots+2^{2n(3)-2})M,\dots,(2+\cdots+2^{2n(3)})M]$$

satisfying

$$\psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]}\,,\ l \in \{(2+\cdots+2^{2n(3)-2})M,\dots,(2+\cdots+2^{2n(3)})M-1\}\,,$$
$$\text{at least one } j(l) \in \{0,\dots,M-1,l_1^1,l_2^1,\dots,l_{m(3)}^3\}\,,$$
$$\{\varphi(t);\ t\in[l_1^1,l_1^1+1]\cup[l_2^1,l_2^1+1]\cup\cdots\cup[l_{m(3)}^3,l_{m(3)}^3+1]\}$$
$$\subseteq\{\psi(t);\ t\in[(2+\cdots+2^{2n(3)-2})M,\dots,(2+\cdots+2^{2n(3)})M]\}\,.$$

In addition, we have

$$\psi|_{[l,l+1]}\equiv\psi|_{[0,1]}\,,\quad l\in\{j\,2^{2n(2)}M;\ j\in\mathbb{Z}\}$$
$$\cap\,\{-(2^{2n(3)-1}+\cdots+2)M,\dots,(2+\cdots+2^{2n(3)})M-1\}\,,$$
$$\psi|_{\left[2^{2n(3)}M+1,2^{2n(3)}M+2\right]}\equiv\psi|_{[1,2]}\,,\qquad \psi|_{\left[2^{2n(3)}M-1,2^{2n(3)}M\right]}\equiv\psi|_{[-1,0]}\,,$$

$$\psi|_{[k,k+1]}\equiv\psi|_{[1,2]}\quad\text{for some } k\,,$$
$$k\in\{(2+\cdots+2^{2n(3)-2})M,\dots,(2+\cdots+2^{2n(3)})M-1\}\setminus\{2^{2n(3)}M+1\}\,,$$

$$\psi|_{[k,k+1]}\equiv\psi|_{[-1,0]}\quad\text{for some } k\,,$$
$$k\in\{(2+\cdots+2^{2n(3)-2})M,\dots,(2+\cdots+2^{2n(3)})M-1\}\setminus\{2^{2n(3)}M-1\}\,.$$

Continuing in the same manner, in the i-th step, we choose $n(i)>n(i-1)+m(i)$ ($n(i)\in\mathbb{N}$) and we construct ψ for

$$\varepsilon_k:=0\,,\quad k\in\{2n(i-1)+2,\dots,2n(i)\}\,,\quad \varepsilon_{2n(i)+1}:=2^{-i+1}\,. \tag{33}$$

For simplicity, let $i-2<2^{2n(2)}M$ (see also the proof of Theorem 3 for $j\,2^{2n(2)}$ replaced by $[j\,2^{2n(2)}M,j\,2^{2n(2)}M+1]$, $1+j\,2^{2n(3)}$ by $[1+j\,2^{2n(3)}M,1+j\,2^{2n(3)}M+1]$, and so on). Again, for each $j(1)\in\{1,\dots,i-1\}$, $j(2)\in\{1,\dots,m(j(1))\}$, there exist at least $2m(i)+2$ integers

$$l\in\{-(2^{2n(i)-1}+\cdots+2)M,\dots,(2+\cdots+2^{2n(i)-2})M-1\}$$
$$\setminus\big(\{j\,2^{2n(2)}M;\ j\in\mathbb{Z}\}\cup\{1+j\,2^{2n(3)}M;\ j\in\mathbb{Z}\}\cup\{-1+j\,2^{2n(3)}M;\ j\in\mathbb{Z}\}$$
$$\cup\cdots\cup\{i-3+j\,2^{2n(i-1)}M;\ j\in\mathbb{Z}\}\cup\{3-i+j\,2^{2n(i-1)}M;\ j\in\mathbb{Z}\}\big)$$

such that

$$\psi|_{[l,l+1]}\equiv\varphi|_{\left[l_{j(2)}^{j(1)},l_{j(2)}^{j(1)}+1\right]}\,.$$

Thus, we can define continuous

$$\psi(t)\in\mathcal{O}_{\varepsilon_{2n(i)+1}}\big(\psi(t-2^{2n(i)}M)\big),\ t\in((2+\cdots+2^{2n(i)-2})M,\dots,(2+\cdots+2^{2n(i)})M]$$

satisfying

$$\psi|_{[l,l+1]}\equiv\varphi|_{[j(l),j(l)+1]}\,,\ l\in\{(2+\cdots+2^{2n(i)-2})M,\dots,(2+\cdots+2^{2n(i)})M-1\}\,,$$
$$\text{at least one } j(l)\in\{0,\dots,M-1,l_1^1,l_2^1,\dots,l_{m(i)}^i\},$$

$$\begin{aligned}&\{\varphi(t);\ t\in[l_1^1,l_1^1+1]\cup[l_2^1,l_2^1+1]\cup\cdots\cup[l_{m(i)}^i,l_{m(i)}^i+1]\}\\&\qquad\subseteq\{\psi(t);\ t\in[(2+\cdots+2^{2n(i)-2})M,\dots,(2+\cdots+2^{2n(i)})M]\}\,.\end{aligned} \tag{34}$$

In addition, we can define ψ so that

$$\begin{aligned}
\psi|_{[l,l+1]} \equiv \psi|_{[0,1]},\ & l \in \{j\, 2^{2n(2)} M;\ j \in \mathbb{Z}\} \\
& \cap \{-(2^{2n(i)-1} + \dots + 2)M, \dots, (2 + \dots + 2^{2n(i)})M - 1\}, \\
\psi|_{[l,l+1]} \equiv \psi|_{[1,2]},\ & l \in \{1 + j\, 2^{2n(3)} M;\ j \in \mathbb{Z}\} \\
& \cap \{-(2^{2n(i)-1} + \dots + 2)M, \dots, (2 + \dots + 2^{2n(i)})M - 1\}, \\
\psi|_{[l,l+1]} \equiv \psi|_{[-1,0]},\ & l \in \{-1 + j\, 2^{2n(3)} M;\ j \in \mathbb{Z}\} \\
& \cap \{-(2^{2n(i)-1} + \dots + 2)M, \dots, (2 + \dots + 2^{2n(i)})M - 1\}, \\
& \vdots \\
\psi|_{[l,l+1]} \equiv \psi|_{[i-3,i-2]},\ & l \in \{i - 3 + j\, 2^{2n(i-1)} M;\ j \in \mathbb{Z}\} \\
& \cap \{-(2^{2n(i)-1} + \dots + 2)M, \dots, (2 + \dots + 2^{2n(i)})M - 1\}, \\
\psi|_{[l,l+1]} \equiv \psi|_{[3-i,4-i]},\ & l \in \{3 - i + j\, 2^{2n(i-1)} M;\ j \in \mathbb{Z}\} \\
& \cap \{-(2^{2n(i)-1} + \dots + 2)M, \dots, (2 + \dots + 2^{2n(i)})M - 1\},
\end{aligned}$$

$$\psi|_{\left[2^{2n(i)}M+i-2,\,2^{2n(i)}M+i-1\right]} \equiv \psi|_{[i-2,i-1]}, \quad \psi|_{\left[2^{2n(i)}M+2-i,\,2^{2n(i)}M+3-i\right]} \equiv \psi|_{[2-i,3-i]},$$

$$\begin{aligned}
&\psi|_{[k,k+1]} \equiv \psi|_{[i-2,i-1]} \quad \text{for some } k, \\
&\qquad k \in \{(2 + \dots + 2^{2n(i)-2})M, \dots, (2 + \dots + 2^{2n(i)})M - 1\} \setminus \{2^{2n(i)}M + i - 2\}, \\
&\psi|_{[k,k+1]} \equiv \psi|_{[2-i,3-i]} \quad \text{for some } k, \\
&\qquad k \in \{(2 + \dots + 2^{2n(i)-2})M, \dots, (2 + \dots + 2^{2n(i)})M - 1\} \setminus \{2^{2n(i)}M + 2 - i\}.
\end{aligned}$$

For $\{\varphi(k)\}_{k\in\mathbb{Z}}$ which is not constant, it is valid

$$\min_{\varphi(i)\neq\varphi(j)} \varrho(\varphi(i), \varphi(j)) \geq 2^{-K}$$

for some $K \in \mathbb{N}$. If we begin the construction by

$$l_1^K := l_1(2^{-K}), \dots, l_{m(K)}^K := l_{m(2^{-K})}(2^{-K}),$$

then we have to obtain

$$\psi(k) = \psi(k + M), \quad k \in \mathbb{Z}.$$

Hence, we can construct the above ψ in order that the sequence $\{\psi(k)\}_{k\in\mathbb{Z}}$ is periodic with period M which gives (2) and the continuity of ψ. We construct ψ using the process from Theorem 2 for all $i \in \mathbb{N}$ and we obtain an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$. Indeed, we have (29) and, summarizing (30), (31), (32), ..., (33), ..., we get (24). For periodic $\{\psi(k)\}_{k\in\mathbb{Z}}$, the first identity in (25) follows from (27) and (29) and the second one from the construction, (28), and (34). As in the proof of Theorem 3, we see that, for any $l \in \mathbb{Z}$, there exists $i(l) \in \mathbb{N}$ satisfying

$$\psi|_{[k,k+1]} \equiv \psi|_{[l,l+1]}, \quad k \in \{l + j\, 2^{2n(i(l))} M;\ j \in \mathbb{Z}\}.$$

It gives (26) for $q(l) = 2^{2n(i(l))}M$. The theorem is proved. □

As an example which illustrates the previous theorem, we mention the following statement:

Corollary 1. *For any continuous function* $F\colon [0,1] \to \mathcal{X}$, *there exists an almost periodic function* ψ *with the property that*

$$\{\psi(t);\, t \in \mathbb{R}\} = \{F(t);\, t \in (0,1)\}\,.$$

Proof. It suffices to show that there exists a uniformly continuous function $\varphi\colon \mathbb{R} \to \mathcal{X}$ for which $\{\varphi(k);\, k \in \mathbb{Z}\} = \{F(1/2)\}$ and $\{\varphi(t);\, t \in \mathbb{R}\} = \{F(t);\, t \in (0,1)\}$, and to apply Theorem 4. For example, one can put

$$\begin{aligned}
\varphi(k+s) &:= F\Big(\frac{1}{2} + s\Big)\,, && k \in \mathbb{N},\ s \in \Big[0, \frac{k}{2k+1}\Big)\,,\\
\varphi(k+s) &:= F\Big(\frac{1}{2} + \frac{k}{2k+1}\Big)\,, && k \in \mathbb{N},\ s \in \Big[\frac{k}{2k+1}, 1 - \frac{k}{2k+1}\Big)\,,\\
\varphi(k+s) &:= F\Big(\frac{1}{2} + 1 - s\Big)\,, && k \in \mathbb{N},\ s \in \Big[1 - \frac{k}{2k+1}, 1\Big)\,;\\
\varphi(k+s) &:= F\Big(\frac{1}{2} - s\Big)\,, && k \in \mathbb{Z}\setminus\mathbb{N},\ s \in \Big[0, \frac{k}{2k-1}\Big)\,,\\
\varphi(k+s) &:= F\Big(\frac{1}{2} - \frac{k}{2k-1}\Big)\,, && k \in \mathbb{Z}\setminus\mathbb{N},\ s \in \Big[\frac{k}{2k-1}, 1 - \frac{k}{2k-1}\Big)\,,\\
\varphi(k+s) &:= F\Big(\frac{1}{2} + s - 1\Big)\,, && k \in \mathbb{Z}\setminus\mathbb{N},\ s \in \Big[1 - \frac{k}{2k-1}, 1\Big)\,.
\end{aligned}$$

□

In Theorem 4, we have constructed an almost periodic function ψ for which the set $\{\psi(k);\, k \in \mathbb{Z}\}$ has to be finite. Now we use Theorem 3 to obtain an almost periodic function with infinitely many given values on $\mathbb{Z}$. In Banach spaces, for any almost periodic sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$, there exists an almost periodic function ψ for which $\psi(k) = \varphi_k$, $k \in \mathbb{Z}$ (consider a natural generalization of [2, Theorem 1.27]). Since this statement does not need to be true in a metric space, we have to require a condition about the local connection by arcs of given values.

Theorem 5. *Let any countable and totally bounded set* $X \subseteq \mathcal{X}$ *be given. If all* $x, y \in X$ *can be connected in* $\mathcal{X}$ *by continuous curves which depend uniformly continuously on* x *and* y, *then there exists an almost periodic function* $\psi : \mathbb{R} \to \mathcal{X}$ *such that*

$$\{\psi(k);\, k \in \mathbb{Z}\} = X \tag{35}$$

and that, for any $l \in \mathbb{Z}$, *there exists* $q(l) \in \mathbb{N}$ *for which*

$$\psi(l+s) = \psi(l+s+jq(l))\,, \quad j \in \mathbb{Z},\ s \in [0,1)\,.$$

Proof. Using Theorem 3, we get an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying (3). Let continuous functions $f_k : [0,1] \to \mathcal{X}$, $k \in \mathbb{Z}$ for which $f_k(0) = \psi_k$, $f_k(1) = \psi_{k+1}$ be from the statement of the theorem. Obviously, the function

$$\psi(k+s) := f_k(s)\,, \quad k \in \mathbb{Z},\ s \in [0,1)$$

defined on $\mathbb{R}$ is continuous and (35) is satisfied. From the proof of Theorem 3, it follows (see (23)) that, for any $l \in \mathbb{Z}$, there exist $r(l,1), r(l,2) \in \mathbb{N}$ with the property that

$$\psi_l = \psi_{l+j\,2^{r(l,1)}}, \quad \psi_{l+1} = \psi_{l+1+j\,2^{r(l,2)}}, \quad j \in \mathbb{Z}.$$

Thus, for any $l \in \mathbb{Z}$, there exists $r(l) \in \mathbb{N}$ such that

$$\psi(l) = \psi(l + j\,2^{r(l)}), \quad \psi(l+1) = \psi(l+1+j\,2^{r(l)}), \quad j \in \mathbb{Z}$$

which implies

$$\psi(l+s) = \psi(l+s+j\,2^{r(l)}), \quad j \in \mathbb{Z},\ s \in [0,1].$$

It remains to show that ψ is almost periodic. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be the number corresponding to ε from the definition of the uniform continuity of the connections of the values φ_i, $i \in \mathbb{N}$. Let $l \in \mathbb{Z}$ be a δ-translation number of $\{\psi_k\}$, i.e., let

$$\varrho(\psi_{k+l}, \psi_k) < \delta, \quad k \in \mathbb{Z}. \tag{36}$$

By the definition of the function ψ, we have

$$\varrho(\psi(t+l), \psi(t)) < \varepsilon, \quad t \in \mathbb{R}.$$

Indeed, it suffices to consider (36) for k and $k+1$. Since any δ-translation number of $\{\psi_k\}$ is an ε-translation number of $\psi(t)$, $t \in \mathbb{R}$ and since the set of all δ-translation numbers of almost periodic $\{\psi_k\}$ is relative dense in $\mathbb{Z}$, function ψ is almost periodic as well. □

We remark that the first interesting generalization of the approximation theorem (mentioned in Introduction) for a complete metric space is due to H. Tornehave and it can be found in [7]. It is required there that, for every compact subset S, there exists a positive number d such that any points x, $y \in S$ with distance less than d can be connected by a continuous curve which depends continuously on x and y and which reduces to x for $x = y$. This requirement motivates the main condition of the above given Theorem 5.

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Department of Mathematics and Statistics,
Masaryk University,
Kotlářská 2, CZ-611 37 Brno, Czech Republic
E-mail: michal.vesely@mail.muni.cz