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A REMARK ON THE MORITA THEOREM FOR OPERADS

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Abstract. We extend a result of M. M. Kapranov and Y. Manin concerning the Morita theory for linear operads. We also give a cyclic operad version of their result.

1. Introduction

A "fragment of the Morita theory for operads" was constructed by M. Kapranov and Y. Manin in [2]. They have proved the operadic analogue of the following algebra result:

Let R be a ring and d a positive integer. Then the categories of (left, say) R-modules and $\mathrm{End}_R(R^{(d)})$*-modules are equivalent, where $R^{(d)}$ is the R-module of d-tuples $(x_1,\dots,x_d)$, $x_i\in R$.*

Precisely, the authors proved that given an operad P in the category of vector spaces and a positive integer d, there is an operad $\mathrm{Mat}(d,P)$ such that the categories of (operadic) P-algebras and $\mathrm{Mat}(d,P)$-algebras are equivalent [2, Theorem 1.9.1]. At the basis of this operadic analogue lies the observation that an operad can be viewed as a monoid in a certain monoidal category. It is implicit in [2] that the categories of left (right) P-modules and $\mathrm{Mat}(d,P)$-modules are equivalent too.

The goal of this paper is to prove an operadic analogue of the following algebra result:

Let R be a ring and let M be a right R-module which is finitely generated projective and a generator in the category of right R-modules. Then the categories of (left, say) R-modules and $\mathrm{End}_R(M)$*-modules are equivalent.*

This operadic analogue is a consequence of Proposition 4.4. The aforementioned result from [2] is then obtained as a corollary. We also prove (Theorem 5.1) a cyclic operad version of [2, Theorem 1.9.1].

The paper is this long in order to make the presentation reasonably self-contained. The full categorical algebra approach that we adopt was inspired by [8] and [10], and it makes the proofs minimal.

Notation. k is a commutative ring. Mod_k is the category of k-modules and Alg_k the category of (associative and unital) k-algebras. For $V\in\mathrm{Mod}_k$, $V^\vee=\mathrm{Hom}_k(V,k)$.

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When there is no danger of confusion we write $\otimes$ for the tensor product over k. For $n \geq 0$, S_n is the symmetric group on n letters, where S_0 and S_1 both denote the trivial group. If $(\mathcal{C}, \otimes, I)$ is a monoidal category with unit I and X is an object of $\mathcal{C}$, we agree that $X^{\otimes 0} = I$. We write Σ for the category whose objects are integers $n \geq 0$ and with morphisms $\Sigma(m, n) = \emptyset$ if $m \neq n$ and $\Sigma(n, n) = \mathrm{S}_n$.

2. Monoidal structures on the category of collections

In this section we recall some facts from [3], [1], [2], [9]. The category of **collections in** Mod_k is the functor category $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$. Its objects will be written $X = \{X(n)\}_{n \geq 0}$.

2.1. The category $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ has a **levelwise** monoidal structure. For $X, Y \in \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$, we put $(X \otimes Y)(n) = X(n) \otimes_k Y(n)$ and $\mathrm{Com}\,(n) = k$. Then $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \otimes, \mathrm{Com})$ is a closed category with unit Com, the internal hom being constructed levelwise. One has

$$Y^X = \{\mathrm{Hom}_k(X(n), Y(n))\}_{n \geq 0}$$

where $(f\sigma)(x) = f(x\sigma^{-1})\sigma$ for $\sigma \in \mathrm{S}_n$.

2.2. Since Σ is symmetric monoidal category, $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ has a **convolution product**

$$X \boxtimes Y = \int^{m,n} k\Sigma(m+n, -) \otimes_k X(m) \otimes_k Y(n)\,.$$

Explicitly,

$$X \boxtimes Y(n) = \bigoplus_{i+j=n} Ind_{\mathrm{S}_i \times \mathrm{S}_j}^{\mathrm{S}_n}(X(i) \otimes_k Y(j))$$

with unit the collection $\mathbf{1}(n) = k$ if $n = 0$, $\mathbf{1}(n) = 0$ otherwise. The internal hom is

$$\underline{\mathrm{Hom}}(X, Y)(r) = \int_n \mathrm{Hom}_k\big(X(n), Y(n+r)\big)$$

so that

$$\underline{\mathrm{Hom}}(X, Y) = \{\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(X, Y[n])\}_{n \geq 0}$$

where $Y[n] = \{Y(n+r)\}_{r \geq 0}$ and S_r acts on the last r entries of $n + r$.

There is an adjoint pair

$$(\_)\{0\} \colon \mathrm{Mod}_k \rightleftarrows \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}} : \Gamma_0$$

where

$$V\{0\}(n) = \begin{cases} V\,, & \text{if } n = 0 \\ 0\,, & \text{otherwise} \end{cases}$$

and $\Gamma_0(X) = \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(\mathbf{1}, X)$. The functor $(\_)\{0\}$ is a (full and faithful) symmetric strong monoidal functor. Therefore $(\mathrm{Mod}_k, \otimes, k)$ acts as a monoidal category on $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ by $V * X = V\{0\} \boxtimes X$.

2.3. $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ has also a **composition product**

$$X \circ Y = \int^n X(n)\{0\} \boxtimes Y^{\boxtimes n} = \bigoplus_{n \geq 0} X(n) \otimes_{k\mathrm{S}_n} Y^{\boxtimes n}$$

with unit the collection $I(n) = k$ if $n = 1$, $I(n) = 0$ otherwise. The monoidal category $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$ is right closed in the sense that $- \circ X$ has a right adjoint $[X, -]$, for every collection X. One has

$$[X, Y] = \{\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(X^{\boxtimes n}, Y)\}_{n \geq 0}\,.$$

A general argument implies that $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ is a $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$-category, that is, a category enriched over $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$. In particular, $[X, X]$ is a monoid with respect to $- \circ -$ for every collection X. There is an adjoint pair

$$(\_)\{1\} \colon \mathrm{Mod}_k \rightleftarrows \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}} : \Gamma_1$$

where

$$V\{1\}(n) = \begin{cases} V\,, & \text{if } n = 1 \\ 0\,, & \text{otherwise} \end{cases}$$

and $\Gamma_1(X) = \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(I, X)$. The functor $(\_)\{1\}$ is a (full and faithful) strong monoidal functor. Notice that $V\{0\} \circ X \cong V\{0\}$ for every collection X.

2.4. We list some relations between the three monoidal structures.

(a) We have natural isomorphisms $(X \circ Z) \boxtimes (Y \circ Z) \cong (X \boxtimes Y) \circ Z$ and $\mathbf{1} \circ X \cong \mathbf{1}$, which make $- \circ X$, for any collection X, into a strong symmetric monoidal functor with respect to the $\boxtimes$ monoidal structure. Therefore $(V * X) \circ Y \cong V * (X \circ Y)$, cf. 2.3.

(b) There is a natural diagonal map $(X \otimes Y)^{\boxtimes p} \to X^{\boxtimes p} \otimes Y^{\boxtimes p}$ which induces a natural twist map

$$(X_1 \otimes X_2) \circ (Y_1 \otimes Y_2) \longrightarrow (X_1 \circ Y_1) \otimes (X_2 \circ Y_2)$$

This implies that the category of monoids in $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$ is closed under $- \otimes -$.

2.5. We have an isomorphism

$$\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(V * X, Y) \cong \mathrm{Hom}_k(V, \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(X, Y))$$

If V and W are finitely generated and projective k-modules, we obtain an isomorphism

$$\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(V * X, W * Y) \cong \mathrm{Hom}_k(V, W) \otimes_k \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}(X, Y)\,.$$

This gives

$$[V * X, W * Y] \cong [V\{0\}, W\{0\}] \otimes [X, Y]$$

hence in particular

$$[V * X, V * X] \cong [V\{0\}, V\{0\}] \otimes [X, X]\,.$$

This last isomorphism is an isomorphism of monoids with respect to $- \circ -$.

2.6. Let $\mathbf{M}$ be an arbitary cocomplete monoidal category and let $\mathcal{C}$ be a small category. We endow the functor category $\mathbf{M}^{\mathcal{C}}$ with the point-wise monoidal structure. When $\mathcal{C}$ is sifted (which means $\mathcal{C}$ is nonempty and the diagonal functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is final), the colimit functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}$ is strong monoidal. Therefore, for a sifted category $\mathcal{C}$, the functor $- \circ -$ preserves colimits indexed over $\mathcal{C}$ in the second variable. In particular, $- \circ -$ preserves reflexive coequalisers in the second variable.

3. Categorical algebra in $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$

A k-**linear operad**, or an **operad in** Mod_k, is a monoid in $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$. We denote the resulting category by $\mathrm{Op}(\mathrm{Mod}_k)$. The adjunction $((\_)\{1\}, \Gamma_1)$ from 2.3 induces an adjunction

$$(\_)\{1\}\colon\ \mathrm{Alg}_k \rightleftarrows \mathrm{Op}(\mathrm{Mod}_k)\colon \Gamma_1\,.$$

A collection $A = \{A(n)\}_{n\geq 0}$ is an operad in Mod_k if and only if there are **structure maps**

$$\mu_A\colon A(n) \otimes A(k_1) \otimes \cdots \otimes A(k_n) \to A(k_1 + \cdots + k_n)$$

and a **unit map** $\eta_A\colon k \to A(1)$ satisfying some associativity, equivariance and unit axioms. The object Com from 2.1 is an operad. Another example of operad is $\mathrm{As} = \{k\mathrm{S}_n\}_{n\geq 0}$ [5, Example 3]. The canonical map $\mathrm{As} \to \mathrm{Com}$ is a morphism of operads.

3.1. Let A be an operad. We denote by Mod_A the category of right A-modules (in the sense of categorical algebra). The hom in this category between two objects M and N is denoted by $\mathrm{Hom}_A(M, N)$. An object of Mod_A is often denoted by M_A, N_A,...A right A-module M comes equipped with a **structure map** $\mu_M\colon M \circ A \to M$ satisfying associativity and unit axioms. The forgetful functor $\mathrm{Mod}_A \to \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ has a left adjoint $- \circ A$ and a right adjoint $[A, -]$. The category of left A-modules is denoted by ${}_A\,\mathrm{Mod}$; to give a left A-module structure on a collection X is to give a morphism of operads $A \to [X, X]$. If B is another operad, the category of (A, B)-bimodules is denoted by ${}_A\,\mathrm{Mod}_B$. The hom in this category between two objects M and N is denoted by $\mathrm{Hom}_{A,B}(M, N)$.

3.2. The convolution product $\boxtimes$ (2.2) extends to a closed category structure on Mod_A which we again denote by $\boxtimes$, having the same unit $\mathbf{1}$. For $M, N \in \mathrm{Mod}_A$, the right A-module structure on $M \boxtimes N$ is induced by the natural transformation

$$(M \boxtimes N) \circ A \cong (M \circ A) \boxtimes (N \circ A) \to M \boxtimes N$$

The internal hom of M and N is

$$\underline{\mathrm{Hom}}_A(M, N) = \{\mathrm{Hom}_A(M, N[n])\}_{n\geq 0}$$

The category $(\mathrm{Mod}_k, \otimes, k)$ acts as a monoidal category on Mod_A by $V * M = V\{0\} \boxtimes M$, cf. 2.4($a$).

3.3. Let A be an operad. We have a functor

$$- \circ - \colon \operatorname{Mod}_k^{\Sigma^{\mathrm{op}}} \times Mod_A \longrightarrow \operatorname{Mod}_A$$

which is the composition product and a functor

$$[-,-]_A \colon (\operatorname{Mod}_A)^{\mathrm{op}} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_k^{\Sigma^{\mathrm{op}}}$$

where $[M,N]_A$ is the equaliser

$$[M,N]_A \longrightarrow [M,N] \overset{u}{\underset{v}{\rightrightarrows}} [M \circ A, N]$$

For $f \in [M,N](n)$, the map u is given by $u(f) = \mu_N(f \circ A)$ and the map v by $v(f) = f\mu_M^{\boxtimes n}$, so that

$$[M,N]_A = \{\operatorname{Hom}_A(M^{\boxtimes n}, N)\}_{n \geq 0}$$

We have a natural isomorphism

$$\operatorname{Hom}_A(X \circ M, N) \cong \operatorname{Mod}_k^{\Sigma^{\mathrm{op}}}\left(X, [M,N]_A\right). \tag{1}$$

The functor $- \circ -$ is an action of $(\operatorname{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$ on Mod_A, therefore a general argument shows that Mod_A is a category enriched over $(\operatorname{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$. In particular, for M_A, N_A one has that $[M,M]_A$ is an operad, M is a $([M,M]_A, A)$-bimodule, $[M,N]_A$ is a $([N,N]_A, [M,M]_A)$-bimodule, $[A,A]_A \cong A$ as operads and as (A,A)-bimodules, and $[A,M]_A \cong M$ as right A-modules.

3.4. Let A, B, C be three operads. Observe that if ${}_BM_C$, ${}_AN_C$ then $[M,N]_C \in_A \operatorname{Mod}_B$. This can be seen using (1). Therefore we have a functor

$$[-,-]_C \colon ({}_B\operatorname{Mod}_C)^{\mathrm{op}} \times_A \operatorname{Mod}_C \longrightarrow_A \operatorname{Mod}_B$$

We fix ${}_BN_C$, and we consider the functor $[N,-]_C \colon {}_A\operatorname{Mod}_C \to_A \operatorname{Mod}_B$. This functor has a left adjoint $- \circ_B N \colon {}_A\operatorname{Mod}_B \to_A \operatorname{Mod}_C$, where for ${}_AM_B$, $M \circ_B N$ is the (reflexive) coequaliser of the pair $M \circ B \circ N \rightrightarrows M \circ N$. It is clear that $M \circ_B N \in \operatorname{Mod}_C$. To see that $M \circ_B N \in_A \operatorname{Mod}$, we notice that for any collection X, we have by 2.6 an isomorphism

$$(X \circ M) \circ_B N \cong X \circ (M \circ_B N). \tag{2}$$

In particular, for $X = A$ this isomorphism gives $M \circ_B N$ the structure of a left A-module, which is compatible with the right C-module action. The adjunction property follows from the definitions.

We obtain a functor

$$- \circ_B - \colon {}_A\operatorname{Mod}_B \times {}_B\operatorname{Mod}_C \longrightarrow_A \operatorname{Mod}_C$$

and a natural isomorphism

$$\operatorname{Hom}_{A,C}(M \circ_B N, P) \cong \operatorname{Hom}_{A,B}\left(M, [N,P]_C\right). \tag{3}$$

3.5. Let A, B be two operads. Consider the situation $M_{A},{}_{A}N_{B},{}_{B}P$. There is an associativity isomorphism

$$(M \circ_A N) \circ_B P \cong M \circ_A (N \circ_B P) \tag{4}$$

and we shall briefly indicate how to obtain it. Consider the following commutative diagram with the last column on the right and all rows coequalisers:

$$\begin{tikzcd}
M \circ A \circ N \circ B \circ P \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M \circ N \circ B \circ P \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & (M \circ_A N) \circ B \circ P \arrow[d, shift left] \arrow[d, shift right] \\
M \circ A \circ N \circ P \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & M \circ N \circ P \arrow[r] \arrow[dd] & (M \circ_A N) \circ P \arrow[d] \\
 & & (M \circ_A N) \circ_B P \arrow[d, equal] \\
M \circ A \circ (N \circ_B P) \arrow[r, shift left] \arrow[r, shift right] & M \circ (N \circ_B P) \arrow[r] & M \circ_A (N \circ_B P)
\end{tikzcd}$$

By 2.6 the first two columns are also coequalisers, and since colimits commute with colimits we obtain the desired isomorphism (4). If ${}_{C}M_{A},{}_{A}N_{B},{}_{B}P_{D}$, where C, D are operads, the isomorphism (4) is an isomorphism of (C, D)-bimodules.

3.6. Let A be an operad. An A-**algebra** is a left A-module of the form $V\{0\}$, $V \in \mathrm{Mod}_k$. The resulting category is denoted by ${}_{A}\,\mathrm{Alg}$. A k-module V is an A-algebra if and only if there are **structure maps** $\mu\colon A(n) \otimes V^{\otimes n} \to V$ satisfying associativity, equivariance and unit axioms. Given an A-algebra V and a right A-module M, the collection $M \circ_A V\{0\}$ is of the form $W\{0\}$, for some k-module W. If M is an (A, B)-bimodule, then $M \circ_B V\{0\} \in_A \mathrm{Alg}$.

3.7. For V and W finitely generated projective k-modules, tensoring with Hom_k $(V^{\otimes n}, W)$ preserves equalisers in Mod_k, hence from 2.5 we obtain the formula

$$[V * M, W * N]_A \cong [V\{0\}, W\{0\}] \otimes [M, N]_A \,.$$

In particular, $[V * M, V * M]_A \cong [V\{0\}, V\{0\}] \otimes [M, M]_A$ as operads.

4. A Morita type theorem for operads

A **Morita context in** $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$ is a $(\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}, \circ, I)$-category whose set of objects has two elements. In matrix representation we write a Morita context as

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

so that A, B are operads, $M \in_A \mathrm{Mod}_B$ and $N \in_B \mathrm{Mod}_A$.

Example 4.1. Let A be an operad. Every $P \in \mathrm{Mod}_A$ gives rise to a Morita context

$$\begin{pmatrix} A & P^* \\ P & [P, P]_A \end{pmatrix}$$

where $P^* = [P, A]_A$. Indeed, the evaluation map $[P, A]_A \circ P \to A$ induces a morphism of (A, A)-bimodules

$$P^* \circ_{[P,P]_A} P \to A \tag{5}$$

and the composition $P \circ [P, A]_A \cong [A, P]_A \circ [P, A]_A \to [P, P]_A$ induces a morphism of $([P, P]_A, [P, P]_A)$-bimodules

$$P \circ_A [P, A]_A \to [P, P]_A \tag{6}$$

Let us now fix an operad A. Let $P \in \mathrm{Mod}_A$. We have pairs of functors

$$\mathrm{Mod}_{[P,P]_A} \underset{-\circ_A P^*}{\overset{-\circ_{[P,P]_A} P}{\rightleftarrows}} \mathrm{Mod}_A$$

and

$${}_A\,\mathrm{Mod} \underset{P^*\circ_{[P,P]_A}-}{\overset{P\circ_A -}{\rightleftarrows}} {}_{[P,P]_A}\,\mathrm{Mod}$$

The last pair of functors restricts by 3.6 to the corresponding categories of algebras. The pairs $(- \circ_{[P,P]_A} P, - \circ_A P^*)$ and $(P \circ_A -, P^* \circ_{[P,P]_A} -)$ become inverse equivalences if and only if the maps (5) and (6) are isomorphisms in the respective categories. In Proposition 4.4 we give sufficient conditions for (5) and (6) to be isomorphisms of bimodules.

The unit I of the composition product is a projective and small object in $\mathrm{Mod}_k^{\Sigma^{\mathrm{op}}}$, hence using the adjunction $- \circ A\colon \mathrm{Mod}_k^{\Sigma^{\mathrm{op}}} \rightleftarrows \mathrm{Mod}_A\colon U$, where U is the forgetful functor, it follows that A is a projective and small object in Mod_A.

Lemma 4.2. (a) *If P_A, Q_A are projective then so is $P \boxtimes Q$.*

(b) *P_A is projective if and only if $[P, -]_A$ is a right exact functor.*

(c) *If P_A, Q_A are small then so is $P \boxtimes Q$.*

Proof. (a) By adjunction it suffices to show that if P_A is projective then $\underline{\mathrm{Hom}}_A(P, -)$ preserves epimorphisms. This is the case since P is projective and by the construction of the internal hom in Mod_A (3.2).

(b) The implication $\Rightarrow$ follows from (a) and the construction of $[-, -]_A$ (3.3). The converse is clear.

(c) By adjunction it suffices to show that if P_A is small then $\underline{\mathrm{Hom}}_A(P, -)$ preserves coproducts. This is the case from the construction of the internal hom in Mod_A (3.2). □

A right A-module is **relative projective** if it is a direct summand of $A^{(\Lambda)} (= \bigoplus_{\Lambda} A)$, for some set Λ. A relative projective A-module is **of finite rank** if the set Λ is finite. Any relative projective module is projective, but the converse is not true.

Example 4.3. Let A be a k-algebra and consider the category $\mathrm{Mod}_{A\{1\}}$. Then $As \circ A\{1\}$ is projective (since As is) but not relative projective. Otherwise

$$(As \circ A\{1\})(n) = \begin{cases} 0\,, & \text{if } n \neq 1 \\ \text{some projective } A\text{-module}\,, & n = 1 \end{cases}$$

and one can see that $(As \circ A\{1\})(n) = A^{\otimes n} \otimes k\mathrm{S}_n$.

A right A-module P is an A-**generator** if there is an epimorphism of right A-modules $P^{(\Lambda)} \twoheadrightarrow A$, for some set Λ. Any generator of Mod_A is an A-generator, but the case $P = A\{1\}$ for A a k-algebra shows that the converse is false.

Proposition 4.4. *If P_A is small, relative projective and an A-generator, then the natural morphisms* (5) *and* (6) *are isomorphisms of bimodules.*

Proof. Suppose that $P \oplus Q \cong A^{(\Lambda)}$ and there is an epimorphism $P^{(\Lambda')} \twoheadrightarrow A$. We have commutative diagrams

$$\begin{tikzcd}
P \circ_A P^* \arrow[r] \arrow[d] & {[P,P]_A} \arrow[d] \\
A^{(\Lambda)} \circ_A P^* \arrow[r] \arrow[d] & {[P,A^{(\Lambda)}]_A} \arrow[d] \\
P \circ_A P^* \arrow[r] & {[P,P]_A}
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[P,A]_A \circ_{[P,P]_A} P} \arrow[r] \arrow[d] & A \arrow[d] \\
{[P,P^{(\Lambda')}]_A \circ_{[P,P]_A} P} \arrow[r] \arrow[d] & P^{(\Lambda')} \arrow[d] \\
{[P,A]_A \circ_{[P,P]_A} P} \arrow[r] & A
\end{tikzcd}$$

in which all composites of vertical arrows are the identity. Lemma 4.2(c) and the construction of $[-,-]_A$ (3.3) imply that the middle horizontal arrows in the two diagrams are isomorphisms. $\square$

Since A_A is small, any right A-module which is relative projective and of finite rank is small. In particular

Corollary 4.5 ([2, Theorem 1.9.1])**.** *For $d \geq 1$ a fixed integer, the categories ${}_A\mathrm{Alg}$ and ${}_{[A^{(d)},A^{(d)}]_A}\mathrm{Alg}$ are equivalent.*

Example 4.6. Take $k = \mathbb{Z}$ and consider the operad $A\{1\}$, where A is a ring. Let P be a right A-module and let M be the collection given by

$$M(n) = \begin{cases} 0\,, & \text{if } n \neq 1 \\ P\,, & n = 1 \end{cases}$$

It is a right $A\{1\}$-module. In this case the canonical morphisms (5) and (6) become the well-known maps

$$P^* \otimes_{\mathrm{End}_A(P)} P \to A$$
$$P \otimes P^* \to \mathrm{End}_A(P)$$

Let now M be small, relative projective and an $A\{1\}$-generator. Then M is of the above form, with P small, projective right A-module and a generator.

We end this section with the following observation. Let $\mathrm{End}^{\boxtimes}(\mathrm{Id}_{\mathrm{Mod}_A})$ be the set of $\boxtimes$-monoidal natural transformations of the identity functor on Mod_A. We have

Lemma 4.7. $\mathrm{End}^{\boxtimes}(\mathrm{Id}_{\mathrm{Mod}_A})$ *is a sub-k-algebra of the centre of* $A(1)$.

Proof. Let $M \in \mathrm{Mod}_A$. Since $[A, M]_A \cong M$ as A-modules, we obtain from 3.3 that to give a right A-linear map $A^{\boxtimes n} \to M$ is to give an element of $M(n)$ and that $\mathrm{End}_A(A) \cong A(1)$ as k-algebras. There is a k-algebras homomorphism

$$\mathrm{End}^{\boxtimes}(\mathrm{Id}_{\mathrm{Mod}_A}) \xrightarrow{\mathrm{ev}_A} \mathrm{End}_A(A)$$

$\mathrm{ev}_A(\alpha) = \alpha_A$. If $f\colon A^{\boxtimes n} \to M$ is right A-linear then $\alpha_M f = f\alpha_{A^{\boxtimes n}}$. But $\alpha_{A^{\boxtimes n}} = \alpha_A^{\boxtimes n}$, therefore ev_A is injective. $\square$

5. A Morita type theorem for cyclic operads

In this section we give a cyclic version of [2, Theorem 1.9.1]. We first recall from [5] the relevant notions. For us, “cyclic operad” means (unital) cyclic operad in the sense of [5, Proposition 42].

We fix a cyclic operad A. We shall denote by A the underlying operad of A. Then $A(1)$ is a k-algebra with involution. Let V, T be k-modules. A k-linear map $b\colon V \otimes V \to T$ is called **bilinear form on V with values in T**. Such a b gives rise to a map

$$\phi\colon\ \mathrm{Hom}_k(V^{\otimes n}, V) \to \mathrm{Hom}_k(V^{\otimes (n+1)}, T)$$

$f \mapsto b(V \otimes f)$. If, moreover, V is an A-algebra, b is said to be **invariant** if the composite map

$$A(n) \to \mathrm{Hom}_k(V^{\otimes n}, V) \xrightarrow{\phi} \mathrm{Hom}_k(V^{\otimes (n+1)}, T)$$

is $k\mathrm{S}_{n+1}$-linear. On elements, invariance reads: for any $\sigma \in \mathrm{S}_{n+1}$,

$$b\big(v_{\sigma^{-1}(0)} \otimes a(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})\big) = b\big(v_0 \otimes a\sigma(v_1, \ldots, v_n)\big)\,.$$

When $n = 1$ and $a = 1 \in A(1)$, this implies that b is symmetric. A bilinear form b on V with values in k is simply called **bilinear form on** V. A bilinear form b on V is **nondegenerate** if the adjoint transpose $\bar{b} : V \to V^{\vee}$ is an isomorphism. In this case V is self dual with (V, b) as dual. A **cyclic** A-algebra is a pair (V, b), where V is an A-algebra and b is an invariant nondegenerate bilinear form on V. We write $\mathrm{Cyc}({}_A\mathrm{Alg})$ for the resulting category, with the obvious notion of arrow. If B is another cyclic operad and (W, b') is a cyclic B-algebra, then $(V \otimes W, b \otimes b')$ is a cyclic $A \otimes B$-algebra.

Let $d \geq 1$ be a fixed integer. Since $A^{(d)} = k^{(d)} * A$ in Mod_A, we have by 3.7 that $[A^{(d)}, A^{(d)}]_A$ is a cyclic operad.

Theorem 5.1. *The categories* $\mathrm{Cyc}({}_A\mathrm{Alg})$ *and* $\mathrm{Cyc}({}_{[A^{(d)}, A^{(d)}]_A}\mathrm{Alg})$ *are equivalent.*

Proof. The proof will be divided into several steps.

Step 1. The functor

$$A^{(d)} \circ_A (\_)\{0\} \colon {}_A\operatorname{Alg} \longrightarrow_{[A^{(d)},A^{(d)}]_A} \operatorname{Alg}$$

is naturally isomorphic to $(\_)^{(d)}$, and so we have a functor

$$\operatorname{Cyc}({}_A\operatorname{Alg}) \xrightarrow{(\_)^{(d)}} \operatorname{Cyc}({}_{[A^{(d)},A^{(d)}]_A}\operatorname{Alg}) .$$

Step 2. Put $Q = k^{(d)}$, $e_i = (0, \dots, 1, \dots, 0)$ (where 1 is on the i^{th}-place) and let $p_j \in Q^\vee$ be the projection on the j^{th} coordinate ($i, j \in \{1, \dots, d\}$). There is a natural $k\mathrm{S}_n$-linear isomorphism

$$\phi \colon Q \otimes Q^{\vee\otimes n} \longrightarrow \operatorname{Hom}_k(Q^{\otimes n}, Q) ;$$

$$\phi(v \otimes f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = \prod_{i=1}^{n} f_i(v_i) v .$$

For $n = 1$, $E_{i,j} := \phi(e_i \otimes p_j)$ is the matrix unit in the matrix ring $M_d(k)$. Put $E^n_{i,j} = \phi(e_i \otimes p_j^{\otimes n})$; we have the relations

$$\phi(e_1 \otimes p_{j_1} \otimes \cdots \otimes p_{j_n}) = E^n_{1,1}(E_{1,j_1}, \dots, E_{1,j_n})$$

and

$$E^n_{1,1}(E^{i_1}_{1,1}, \dots, E^{i_n}_{1,1}) = E^{i_1+\cdots+i_n}_{1,1}$$

in the operad $[Q\{0\}, Q\{0\}]$.

There is a natural morphism $\chi \colon A \longrightarrow [Q\{0\}, Q\{0\}] \otimes A$, $A(n) \ni a \mapsto E^n_{1,1} \otimes a$, which is multiplicative with respect to $- \circ -$, that is, the diagram

$$\begin{tikzcd}
A \arrow[r, "\chi"] & {[Q\{0\}, Q\{0\}] \otimes A} \\
A \circ A \arrow[u] \arrow[r, "\chi\circ\chi"'] & ([Q\{0\}, Q\{0\}] \otimes A) \circ ([Q\{0\}, Q\{0\}] \otimes A) \arrow[u]
\end{tikzcd}$$

commutes.

Step 3. Let V be a $[Q\{0\}, Q\{0\}] \otimes A$-algebra with structure map μ. We define $E_{1,1}V$ as

$$\operatorname{Im}(E_{1,1} \otimes 1 \otimes V \xrightarrow{\mu_1} V)$$

The associativity of μ implies that the composite

$$A \circ (E_{1,1}V)\{0\} \xrightarrow{\chi\circ\mathrm{id}} ([Q\{0\}, Q\{0\}] \otimes A) \circ (E_{1,1}V)[0] \xrightarrow{\mu} V\{0\}$$

has image in $(E_{1,1}V)\{0\}$, and so by the previous step we have an associative multiplication $A \circ (E_{1,1}V)\{0\} \to (E_{1,1}V)\{0\}$. To show that $E_{1,1}V$ is an A-algebra, it remains to show that this morphism satisfies the unit axiom. This is a consequence of the associativity of μ. Since all the constructions involved are natural, we have defined a functor

$$E_{1,1}(\_)\colon {}_{[Q\{0\},Q\{0\}]\otimes A}\,\mathrm{Alg} \longrightarrow {}_A\,\mathrm{Alg}\ .$$

Step 4. We show that $E_{1,1}(\_)$ preserves cyclic algebras. Let $(V,b) \in \mathrm{Cyc}({}_{[A^{(d)},A^{(d)}]_A}\mathrm{Alg})$. Since $E_{1,1} \equiv E_{1,1} \otimes 1 \in \mathrm{End}_k(Q) \otimes A(1)$ is an idempotent, we have

$$V = E_{1,1}V \oplus (1 - E_{1,1}) \cdot V \tag{7}$$

as k-modules, therefore $E_{1,1}V$ is finitely generated and projective. Define $\alpha\colon E_{1,1}V \to (E_{1,1}V)^{\vee}$ as

$$E_{1,1}v \mapsto \big(E_{1,1}w \mapsto b(v \otimes E_{1,1}w)\big)\ .$$

Because b is invariant and $E_{1,1}$ is a projector (that is, a self-adjoint idempotent), $b(v \otimes E_{1,1}w) = b(E_{1,1}v \otimes E_{1,1}w)$. We claim that α is an isomorphism. Injectivity is clear. For surjectivity, notice that any $f \in (E_{1,1}V)^{\vee}$ can be extended by (7) to an element $f' \in V^{\vee}$. Then one uses the fact that the adjoint $\bar{b}$ of b is an isomorphism.

Now, the invariance of $b|_{E_{1,1}V}$ is a consequence of the invariance of b. Summing up, $(E_{1,1}V, b|_{E_{1,1}V})$ is a cyclic A-algebra.

Step 5. We show that $\mathrm{Id}_{\mathrm{Cyc}({}_A\mathrm{Alg})} \xrightarrow{\cong} E_{1,1}(\_) \circ (\_)^{(d)}$.

If $(V,b) \in \mathrm{Cyc}({}_A\mathrm{Alg})$, it is immediate that the k-linear isomorphism $\eta\colon V \to E_{1,1}(V^{(d)})$, $v \mapsto (v, 0, \ldots, 0)$, is a morphism in $\mathrm{Cyc}({}_A\mathrm{Alg})$, where $V^{(d)}$ is endowed with the bilinear form $\bigoplus_{1\le i\le d} b$. Naturality of η is clear.

Step 6. We show that $\mathrm{Id}_{\mathrm{Cyc}({}_{[A^{(d)},A^{(d)}]_A}\mathrm{Alg})} \xrightarrow{\cong} (\_)^{(d)} \circ E_{1,1}(\_)$. Fix $(V,b) \in \mathrm{Cyc}({}_{[A^{(d)},A^{(d)}]_A}\mathrm{Alg})$ with structure map μ. We define $\epsilon_V\colon V \to (E_{1,1}V)^{(d)}$ by

$$v \mapsto \big(E_{1,1}v, \mu_1(E_{1,2} \otimes 1 \otimes v), \ldots, \mu_1(E_{1,d} \otimes 1 \otimes v)\big)\ .$$

Since $E_{1,i} = E_{1,1}E_{1,i}$ $(i \in \{1,\ldots,d\})$ and $V \in_{\mathrm{End}_k(Q)\otimes A(1)} \mathrm{Mod}$ via μ_1, ϵ_V is well-defined. To show that ϵ_V is a k-linear isomorphism one proceeds in exactly the same way as for the proof of the standard fact that the categories ${}_{A(1)}\,\mathrm{Mod}$ and ${}_{\mathrm{End}_{A(1)}(A(1)^{(d)})}\,\mathrm{Mod}$ are equivalent, see for example [4, §17B].

Next, we show that ϵ_V is a morphism of $[Q\{0\},Q\{0\}] \otimes A$-algebras.

$$\begin{aligned} \epsilon_V\mu(f \otimes a \otimes v_1 \otimes \cdots \otimes v_n) &= \Sigma_{i=1}^{d} e_i \otimes \mu_1\big(E_{1,i} \otimes 1 \otimes \mu_n(f \otimes a \otimes v_1 \otimes \cdots \otimes v_n)\big) \\ &= \Sigma_{i=1}^{d} e_i \otimes \mu_n\big((E_{1,i} \circ f) \otimes a \otimes v_1 \otimes \cdots \otimes v_n\big) \end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\mu_{Q\otimes E_{1,1}V}(\mathrm{id}\otimes\epsilon_V^{\otimes n})(f\otimes a\otimes v_1\otimes\cdots\otimes v_n)\\
&\qquad=\mu_{Q\otimes E_{1,1}V}\big(f\otimes a\otimes(\Sigma_{j_1=1}^d e_{j_1}\otimes E_{1,j_1}v_1)\otimes\cdots\otimes(\Sigma_{j_n=1}^d e_{j_n}\otimes E_{1,j_n}v_n)\big)\\
&\qquad=\Sigma_{j_1,\dots,j_n=1}^d\mu_{Q\otimes E_{1,1}V}\big(f\otimes a\otimes(e_{j_1}\otimes E_{1,j_1}v_1)\otimes\cdots\otimes(e_{j_n}\otimes E_{1,j_n}v_n)\big)\\
&\qquad=\Sigma_{j_1,\dots,j_n=1}^d f(e_{j_1}\otimes\cdots\otimes e_{j_n})\otimes\mu(E_{1,1}^n\otimes a\otimes E_{1,j_1}v_1\otimes\cdots\otimes E_{1,j_n}v_n)\\
&\qquad=\Sigma_{j_1,\dots,j_n=1}^d f(e_{j_1}\otimes\cdots\otimes e_{j_n})\otimes\mu(E_{1,1}^n(E_{1,j_1},\dots,E_{1,j_n})\\
&\qquad\qquad\otimes a\otimes v_1\otimes\cdots\otimes v_n) \qquad (9)
\end{aligned}$$

The expressions (8) and (9) are equal if and only if they are equal for $f=\phi(e_s\otimes p_{k_1}\otimes\cdots\otimes p_{k_n})$ ($1\le k_n\le d$, $s\in\{1,\dots,d\}$), where ϕ is the isomorphism from Step 2. For such f, $E_{1,i}\circ f$ in (7) above is $p_i(e_s)E_{1,1}^n(E_{1,k_1},\dots,E_{1,k_n})$, hence we obtain equality. Naturality of ϵ is easy to check. Finally

$$\begin{aligned}
\big(\oplus_d\,(b|_{E_{1,1}V})\big)(\epsilon_V\otimes\epsilon_V)(v\otimes w)&=\Sigma_{i=1}^d b(E_{1,i}v\otimes E_{1,i}w)\\
=\Sigma_{i=1}^d b(E_{1,i}v\otimes\overline{E_{1,i}}w)=\Sigma_{i=1}^d b(w\otimes E_{i,1}E_{1,i}v)&=b(w\otimes v)=b(v\otimes w)
\end{aligned}$$

where the second equality holds because the involution on $\mathrm{End}_k(Q)$ induced by b is the same as the involution on the matrix ring $M_d(k)$, which is the transpose. $\square$

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