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HILBERT INEQUALITY FOR VECTOR VALUED FUNCTIONS

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Abstract. In this paper we consider a class of Hankel operators with operator valued symbols on the Hardy space $\mathcal{H}^2_{\Xi}(\mathbb{T})$ where Ξ is a separable infinite dimensional Hilbert space and showed that these operators are unitarily equivalent to a class of integral operators in $L^2(0,\infty)\otimes\Xi$. We then obtained a generalization of Hilbert inequality for vector valued functions. In the continuous case the corresponding integral operator has matrix valued kernels and in the discrete case the sum involves inner product of vectors in the Hilbert space Ξ.

1. Introduction

If $a_m,\, b_n \ge 0$ satisfy $0<\sum_{m=1}^{\infty} a_m^2<\infty$ and $0<\sum_{n=1}^{\infty} b_n^2<\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\Big(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\Big)^{\frac{1}{2}} \tag{1.1}$$

where the constant factor π is the best possible.

The integral version of the inequality (1.1) is as follows:

If $f,\, g\ge 0$ and $f,\, g\in L^2(0,\infty)$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy<\pi\Big(\int_0^{\infty}f^2(x)\,dx\int_0^{\infty}g^2(x)\,dx\Big)^{\frac{1}{2}} \tag{1.2}$$

where the constant factor π is the best possible.

The inequalities (1.1) and (1.2) are the well-known Hilbert's inequality (see Hardy et. al [4], Ch-9). Hardy and Riesz [3] gave the following generalizations of (1.1) and (1.2) for conjugate parameters.

Let $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$, $a_m,\, b_n\ge 0$ satisfy $0<\sum_{m=1}^{\infty}a_m^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\Big(\sum_{m=1}^{\infty}a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}b_n^q\Big)^{\frac{1}{q}} \tag{1.3}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible.

The integral version of the inequality (1.3) is as follows:

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Let $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$, $f\in L^p(0,\infty)$, $g\in L^q(0,\infty)$. Then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy<\frac{\pi}{\sin(\pi/p)}\Big(\int_0^\infty f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^q(x)dx\Big)^{\frac{1}{q}} \tag{1.4}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible.

The inequalities (1.3) and (1.4) are well-known as Hardy-Hilbert's inequality. These inequalities are important in analysis and its applications (see [6, Ch-5], [4, Ch-9]).

Let $L^2(\mathbb{T})$ denote the Hilbert space of square integrable, Lebesgue measurable complex valued functions on the unit circle $\mathbb{T}=\{z\in\mathbb{C}\mid |z|=1\}$, with pointwise algebraic operations and inner product

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta\,.$$

Let $L^\infty(\mathbb{T})$ denote the Banach space of essentially bounded, Lebesgue measurable, complex valued functions on $\mathbb{T}$ with pointwise algebraic operations and essential supremum norm

$$\|f\|_\infty=\operatorname*{ess\,sup}_{|z|=1}|f(z)|\,.$$

For $p=2$ or ∞, let $\mathcal{H}^p(\mathbb{T})$ be the closed subspace $\{f\in L^p(\mathbb{T}):\widehat{f}(n)=0$ for $n<0\}$ of $L^p(\mathbb{T})$, with the restriction of the norm of L^p. Here $\widehat{f}(n)$ denote the nth Fourier coefficient of f. The spaces $\mathcal{H}^2(\mathbb{T})$ and $\mathcal{H}^\infty(\mathbb{T})$ are called Hardy spaces. The space $\mathcal{H}^2(\mathbb{T})$ is a Hilbert space and $\mathcal{H}^\infty(\mathbb{T})$ is a Banach space. Clearly, $\mathcal{H}^\infty(\mathbb{T})\subset\mathcal{H}^2(\mathbb{T})$.

For $\varphi\in L^\infty(\mathbb{T})$, the Hankel operator S_φ with symbol φ, from $\mathcal{H}^2(\mathbb{T})$ into itself is defined by $S_\varphi f=PJ(\varphi f)$ where P is the orthogonal projection from $L^2(\mathbb{T})$ onto $\mathcal{H}^2(\mathbb{T})$ and $J\colon L^2(\mathbb{T})\to L^2(\mathbb{T})$ is defined by $Jf(e^{it})=f(e^{-it})$. There are some useful unitary equivalences between Hankel operators and Hankel integral operators as we discuss in the following examples.

Example 1.1. Consider the function

$$\varphi(e^{i\theta})=-i(\pi-\theta)\,,\quad 0\le\theta<2\pi\,.$$

Then $\varphi\in L^\infty(\mathbb{T})$ and if

$$\varphi(e^{i\theta})=\sum_{n=-\infty}^{\infty}a_ne^{in\theta}$$

then

$$a_n=\begin{cases}0 & \text{if}\quad n=0\,;\\ -\frac{1}{n} & \text{if}\quad n\neq 0\,.\end{cases}$$

Hence the matrix of the Hankel operator $S_{z\varphi}$ with respect to the standard orthonormal basis of $\mathcal{H}^2(\mathbb{T})$ is the Hilbert's matrix

$$\Gamma = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is not difficult to see that the Hilbert's matrix Γ as an operator on $l^2(\mathbb{Z}_+)$ is unitarily equivalent to the integral operator

$$\left(K_{\widetilde{h}} f\right)(x) = \int_0^\infty \widetilde{h}(x+y) f(y) dy,\ f \in L^2(0,\infty)\,, \tag{1.5}$$

where $\widetilde{h}(x) = \frac{e^{-x}}{x}$. This integral operator is known as Hankel integral operator. For more details see [8].

Example 1.2. Consider the classical singular integral operator, the Carleman's operator defined on $L^2(0,\infty)$ by

$$\left(K_h f\right)(x) = \int_0^\infty h(x+y) f(y) dy, \tag{1.6}$$

where the kernel function is $h(x) = \frac{1}{x}$. It is easy to verify (see [8]) that the Carleman's operator K_h is unitarily equivalent to the Hankel operator defined on $\mathcal{H}^2(\mathbb{T})$ whose matrix with respect to the standard orthonormal basis is

$$S = 2 \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots & \cdots \\ \frac{1}{3} & 0 & \frac{1}{5} & \cdots & \cdots & \cdots \\ 0 & \frac{1}{5} & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{5} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{1.7}$$

Such integral operators are widely studied in the literature (see [7, 8]). It is not difficult to see that $\|S_{z\varphi}\| = \|K_{\widetilde{h}}\| = \|K_h\| = \pi$.

Example 1.3. Let $\psi(z) = \chi\left(i\frac{1-z}{1+z}\right)$, $|z| = 1$, $z \neq -1$ where

$$\chi(t) = \begin{cases} 1, & t \in [-1,1] \\ 0, & t \notin [-1,1]\,. \end{cases}$$

Then ψ is the characteristic function of the set $\{z \in \mathbb{C} : |z| = 1, \mathrm{Re}\, z \geq 0\}$. In the Hilbert space $l^2(\mathbb{Z}_+)$ introduce the Hankel operator Γ_ψ with symbol ψ defined by

$$(1.8) \qquad (\Gamma_\psi x)_n = \sum_{k=0}^{\infty} c_{n+k+1} x_k \,,$$

for $x = (x_0, x_1, x_2, \cdots) \in l^2(\mathbb{Z}_+)$, where c_k are the Fourier coefficients of the function ψ,

$$c_k(\psi) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} \psi(e^{i\theta})\, d\theta = \frac{2}{\pi k} \sin\left(\frac{\pi k}{2}\right), \quad k \in \mathbb{N}\,.$$

The operator Γ_ψ defined in (1.8) is unitarily equivalent to the Hankel integral operator $K_{\frac{1}{2}}$ defined on $L^2(0,\infty)$ as

$$\big(K_{\frac{1}{2}} f\big)(x) = \int_0^{\infty} \frac{2}{\pi} \, \frac{\sin(x+y)}{x+y} f(y)\, dy\,.$$

For details see [5].

In this paper we observe that certain Hankel operators on $\mathcal{H}^2_{\Xi}(\mathbb{T})$ are unitarily equivalent to a class of integral operators on $L^2_{\Xi}(0,\infty)$, where Ξ is an infinite dimensional separable Hilbert space and using the unitary equivalence of these operators generalize the Hilbert inequality for vector valued functions. In §22, we deal with the Hankel integral operator $K_{\widetilde{h}}$ defined in (1.5). We show that the norm of $K_{\widetilde{h}}$ as an operator on $L^p(0,\infty)$ is equal to $\pi/\sin(\pi/p)$, $1 < p < \infty$ and derive the associated integral inequality. As a consequence of this we have obtained an integral inequality involving the kernel $[\cosh(t-s)]^{-1}$ in $L^2(-\infty,\infty)$. In §33, we obtain the discrete and integral version of Hilbert's inequality for $\mathbb{C}^n-$valued functions and show that in the continuous case the corresponding integral operator has matrix valued kernel and in the discrete case the sum involves the inner product of vectors in the Hilbert space $\mathbb{C}^n$. In §44, we consider the case of $\mathcal{H}^2_{\Xi}(\mathbb{T})$ and generalize the results of §33. Further, we also generalize the discrete version of Hilbert inequality for sequences in a Hilbert space $\mathbb{H}$.

2. Norm of the Hankel integral operator $K_{\widetilde{h}}$ and the associated inequalities

In this section we find the norm of $K_{\widetilde{h}}$ as an operator from $L^p(0,\infty)$ into itself, $1 < p < \infty$. But we establish first the discrete version of a Hilbert type inequality.

Theorem 2.1. *If a_m, $b_n \in \mathbb{C}$ satisfy $0 < \sum_{m=0}^{\infty} |a_m|^2 < \infty$ and $0 < \sum_{n=0}^{\infty} |b_n|^2 < \infty$, then*

$$(2.1) \qquad \Big| \sum_{\substack{m,\, n=0 \\ m+n \ \text{even}}}^{\infty} \frac{a_m \bar{b}_n}{m+n+1} \Big| \leq \frac{\pi}{2} \Big(\sum_{m=0}^{\infty} |a_m|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} |b_n|^2 \Big)^{\frac{1}{2}}$$

and the constant factor $\frac{\pi}{2}$ is the best possible.

Proof. Let $\psi(z) = \sum_{n=0}^{\infty} \frac{2}{2n+1} z^{-(2n+1)}$. Then

$$\hat{\psi}(-k) = \begin{cases} \frac{2}{k} & \text{if } k \text{ is odd, } k > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence for $m, n = 0, 1, 2, \cdots,$

$$\begin{aligned} c_{mn} &= \langle S_{z\psi} z^m, z^n \rangle = \langle PJ(z\psi z^m), z^n \rangle = \langle J(\psi z^{m+1}), z^n \rangle \\ &= \langle \psi z^{m+1}, z^{-n} \rangle = \langle \psi, z^{-(m+n+1)} \rangle \\ &= \begin{cases} \frac{2}{m+n+1} & \text{if } m+n+1 \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the matrix S given in (1.7) is the matrix of $S_{z\psi}$. Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and suppose $f, g \in \mathcal{H}^2(\mathbb{T})$. Now as we have mentioned in example 1.2 that $\|S_{z\psi}\| = \|K_h\|$. It follows from the Hilbert integral inequality (1.2) that $\|K_h\| = \pi$. Hence $\|S_{z\psi}\| = \pi$. It follows from Cauchy-Schwarz inequality that

$$|\langle S_{z\psi} f, g \rangle| \le \|S_{z\psi}\| \, \|f\| \, \|g\| \le \pi \|f\| \, \|g\| \,.$$

But

$$\begin{aligned} |\langle S_{z\psi} f, g \rangle| &= \Big| \Big\langle S_{z\psi} \Big(\sum_{m=0}^{\infty} a_m z^m \Big), \Big(\sum_{n=0}^{\infty} b_n z^n \Big) \Big\rangle \Big| \\ &= \Big| \sum_{m,n=0}^{\infty} a_m \bar{b}_n \langle S_{z\psi} z^m, z^n \rangle \Big| \\ &= \Big| \, 2 \sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} \frac{a_m \bar{b}_n}{m+n+1} \Big| \,. \end{aligned}$$

Thus

$$\Big| \sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} \frac{a_m \bar{b}_n}{m+n+1} \Big| \le \frac{\pi}{2} \Big\{ \sum_{m=0}^{\infty} |a_m|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{n=0}^{\infty} |b_n|^2 \Big\}^{\frac{1}{2}} \,.$$

□

We now proceed to show that the norm of the operator $K_{\widetilde{h}}$ as an operator from $L^p(0, \infty)$ into itself is equal to $\frac{\pi}{\sin(\pi/p)}$, if $1 < p < \infty$. It also gives us the following Hardy-Hilbert type integral inequality.

Theorem 2.2. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$, $f \in L^p(0, \infty)$, $g \in L^q(0, \infty)$. Then*

$$\int_0^{\infty} \int_0^{\infty} \frac{e^{-(x+y)}}{x+y} f(x) g(y) \, dx \, dy \le \frac{\pi}{\sin \frac{\pi}{p}} \Big(\int_0^{\infty} f^p(x) \, dx \Big)^{\frac{1}{p}} \Big(\int_0^{\infty} g^q(y) \, dy \Big)^{\frac{1}{q}}$$

and the constant factor $\frac{\pi}{\sin \frac{\pi}{p}}$ is the best possible.

Proof. It follows from Hardy-Hilbert's integral inequality (1.4), that

$$\int_0^\infty \int_0^\infty \frac{e^{-(x+y)}}{x+y} f(x)g(y)\,dx\,dy \le \frac{\pi}{\sin\frac{\pi}{p}} \Big(\int_0^\infty e^{-px} f^p(x)\,dx\Big)^{\frac{1}{p}} \Big(\int_0^\infty e^{-qy} g^q(y)\,dy\Big)^{\frac{1}{q}}$$
$$\le \frac{\pi}{\sin\frac{\pi}{p}} \Big(\int_0^\infty f^p(x)\,dx\Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(y)\,dy\Big)^{\frac{1}{q}}$$

as $e^{-pt} \le 1$ for $t \in (0,\infty)$.

It remains to show that the constant factor 1 in the inequality

$$\int_0^\infty e^{-px} f^p(x)\,dx \le \int_0^\infty f^p(x)\,dx \tag{2.2}$$

is the best possible.

Suppose there exists a constant $k,\ 0 < k < 1$ such that

$$\int_0^\infty e^{-px} f^p(x)\,dx < k \int_0^\infty f^p(x)\,dx \tag{2.3}$$

for all $f \in L^p(0,\infty)$.
Setting

$$\widetilde{f}(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{p}\log\frac{1}{k} \\ 0, & x > \frac{1}{p}\log\frac{1}{k}\,. \end{cases}$$

we have

$$\int_0^\infty \widetilde{f}^p(x)\,dx = \int_0^{\frac{1}{p}\log\frac{1}{k}} dx = \frac{1}{p}\log\frac{1}{k}\,;$$

hence $\widetilde{f} \in L^p(0,\infty)$. Now

$$\int_0^\infty \left(e^{-px} - k\right) \widetilde{f}^p(x)\,dx = \frac{1}{p} + \frac{k}{p}\log\left(\frac{k}{e}\right)\,. \tag{2.4}$$

Consider the function $g(t) = -e^{-pt} + 1 - kpt,\ t \in [0,\infty)$. Then $g'(t) = pe^{-pt} - kp = 0$ for $t = \frac{1}{p}\log\frac{1}{k}$ and $g''(t) = -p^2 e^{-pt} < 0$ for $t = \frac{1}{p}\log\frac{1}{k}$. Hence $g(t) > g(0)$ for $t = \frac{1}{p}\log\frac{1}{k}$. Therefore $1 + k\log\left(\frac{k}{e}\right) > 0$. Now from (2.4) we get

$$\int_0^\infty \left(e^{-px} - k\right) \widetilde{f}^p(x)\,dx > 0\,.$$

This is a contradiction to the assumption (2.3), which shows that the constant factor 1 in the inequality (2.2) is the best possible. Again the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible in the Hardy-Hilbert's integral inequality (1.4). Hence the result follows. □

Remark 2.3. It follows from Theorem 2.2 that

$$\|K_h\| = \|K_{\widetilde{h}}\| = \frac{\pi}{\sin(\pi/p)}\,.$$

As a consequence of Theorem 2.2, we obtain the following integral inequality involving the kernel $[\cosh(t-s)]^{-1}$ in $L^2(-\infty,\infty)$.

Corollary 2.4. *If $f, g \in L^2(-\infty,\infty)$ then*

$$\Big|\int_{-\infty}^{\infty}[\cosh(t-s)]^{-1}f(s)g(t)\,ds\,dt\Big| \le \pi\|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}\,.$$

Proof. Consider the map $W\colon L^2(0,\infty)\to L^2(-\infty,\infty)$ defined by

$$Wf(t)=\sqrt{2}e^t f(e^{2t})\,.$$

The operator W is a unitary operator. Let f be a continuous function with compact support in $(0,\infty)$ and $h(x+y)=\frac{1}{x+y}$, $x=e^{2t}$, $y=e^{2s}$. Then

$$\begin{aligned}(K_h f)(x) &= \int_0^{\infty}\frac{f(y)}{x+y}\,dy=\int_{-\infty}^{\infty}\frac{f(e^{2s})2e^{2s}}{e^{2t}+e^{2s}}\,ds=\frac{1}{\sqrt{2}e^t}\int_{-\infty}^{\infty}\frac{2e^se^t}{e^{2t}+e^{2s}}Wf(s)\,ds\\ &=\frac{1}{\sqrt{2}e^t}\int_{-\infty}^{\infty}[\cosh(t-s)]^{-1}Wf(s)\,ds=(W^*CWf)(x)\end{aligned}$$

since if $g\in L^2(-\infty,\infty)$ then $\frac{g(t)}{\sqrt{2}e^t}=\frac{1}{\sqrt{2x}}g(\frac12\log x)=W^*g(x)$. Thus $K_h=W^*CW$ where C is the convolution with $(\cosh t)^{-1}$. That is,

$$(Cf)(t)=\int_{-\infty}^{\infty}[\cosh(t-s)]^{-1}f(s)\,ds\,.$$

Since K_h and C are unitarily equivalent hence $\|C\|=\pi$ and

$$|\langle Cf,g\rangle|\le\pi\|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}\,.$$

Thus

$$\Big|\int_{-\infty}^{\infty}[\cosh(t-s)]^{-1}f(s)g(t)\,ds\,dt\Big| \le \pi\|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}\,.$$

□

3. Hilbert inequality for vector valued functions

In this section we generalize the discrete version of the Hilbert inequality (1.1) and here the sum involves the inner product of vectors in a Hilbert space $\mathbb{H}$. Let $\mathcal{L}(\mathbb{H})$ denote the set of all bounded linear operators from the Hilbert space $\mathbb{H}$ into itself.

Theorem 3.1. *Let (x_n) and (y_n) be two sequences in $\mathbb{H}$ such that $0<\sum_{n=1}^{\infty}\|x_n\|^2<\infty$ and $0<\sum_{n=1}^{\infty}\|y_n\|^2<\infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\langle x_m,y_n\rangle|}{m+n}<\pi\Big\{\sum_{m=1}^{\infty}\|x_m\|^2\Big\}^{\frac12}\Big\{\sum_{n=1}^{\infty}\|y_n\|^2\Big\}^{\frac12}, \tag{3.1}$$

where the constant factor π is the best possible.

Proof. Let $\mathbb{H} \neq \{0\}$ be a Hilbert space and $\mathcal{E}$ be an orthonormal basis for $\mathbb{H}$. The set $\{e \in \mathcal{E} | \ \langle z, e\rangle \neq 0$ for some $z = x_m$ or $y_n\}$ is countable, let us enumerate this set as the sequence $(e_1, e_2, e_3, \dots)$. Then every x_m and y_n can be expressed as

$$x_m = \sum_{k=1}^{\infty} a_{mk} e_k \, ; \qquad y_n = \sum_{k=1}^{\infty} b_{nk} e_k \, ,$$

where $a_{mk} = \langle x_m, e_k\rangle$, $b_{nk} = \langle y_n, e_k\rangle$. Then

$$\langle x_m, y_n\rangle = \sum_{k=1}^{\infty} a_{mk} \bar{b}_{nk} \, .$$

By Parseval relation $\|x_m\|^2 = \sum_{k=1}^{\infty} |a_{mk}|^2$, for every m and $\|y_n\|^2 = \sum_{k=1}^{\infty} |b_{nk}|^2$, for every n. So, we have $|a_{mk}| \leq \|x_m\|$ for all m and $|b_{nk}| \leq \|y_n\|$ for all n. Hence for each k, $\sum_{m=1}^{\infty} |a_{mk}|^2 < \infty$ and $\sum_{n=1}^{\infty} |b_{nk}|^2 < \infty$. Now using Hilbert's inequality, we have for each k,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_{mk}| \, |b_{nk}|}{m+n} < \pi \Big\{ \sum_{m=1}^{\infty} |a_{mk}|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{n=1}^{\infty} |b_{nk}|^2 \Big\}^{\frac{1}{2}} .$$

Taking summation over k from 1 to p, and using Cauchy-Schwartz inequality, we get

$$\begin{aligned} \sum_{k=1}^{p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_{mk}| \, |b_{nk}|}{m+n} &< \pi \Big\{ \sum_{k=1}^{p} \sum_{m=1}^{\infty} |a_{mk}|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{k=1}^{p} \sum_{n=1}^{\infty} |b_{nk}|^2 \Big\}^{\frac{1}{2}} \\ &= \pi \Big\{ \sum_{m=1}^{\infty} \sum_{k=1}^{p} |a_{mk}|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{n=1}^{\infty} \sum_{k=1}^{p} |b_{nk}|^2 \Big\}^{\frac{1}{2}} . \end{aligned}$$

It follows therefore that for every $p \geq 1$,

$$\sum_{k=1}^{p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_{mk}| \, |b_{nk}|}{m+n} < \pi \Big\{ \sum_{m=1}^{\infty} \|x_m\|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{n=1}^{\infty} \|y_n\|^2 \Big\}^{\frac{1}{2}} . \tag{3.2}$$

Notice that

$$|\langle x_m, y_n\rangle| = \Big| \sum_{k=1}^{\infty} a_{mk} \bar{b}_{nk} \Big| \leq \sum_{k=1}^{\infty} |a_{mk}| \ |b_{nk}| \, .$$

It follows from the relation $|a_{mk}| \, |b_{nk}| \leq \frac{1}{2}(|a_{mk}|^2 + |b_{nk}|^2)$ and the convergence of the series $\sum_{k=1}^{\infty} |a_{mk}|^2$ and $\sum_{k=1}^{\infty} |b_{nk}|^2$. Thus letting $p \to \infty$ in (3.2), we obtain (3.1). In particular for the Hilbert space $\mathbb{H} = \mathbb{R}$, (3.1) reduces to the Hilbert's inequality (1.1). Since the constant factor π in (1.1) is the best possible, so we conclude that the constant factor π in (3.1) is the best possible. □

We shall now present the integral version of the inequality (3.1) and derive some related inequalities using tensor products.

Let $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ denote the Hilbert space of $\mathbb{C}^n$-valued, norm-square integrable, measurable functions on $\mathbb{T}$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ the corresponding Hardy space of functions in $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ with vanishing negative Fourier coefficients. We note that $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}) =$

$L^2(\mathbb{T}) \otimes \mathbb{C}^n$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = \mathcal{H}^2(\mathbb{T}) \otimes \mathbb{C}^n$ where the Hilbert space tensor product is used. When endowed with the inner product defined by

$$\langle f, g\rangle_{\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})} = \int_{\mathbb{T}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} dz\,, \quad \text{for} \quad f, g \in \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})\,,$$

the spaces $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ become separable Hilbert spaces. Here the measures dz denotes the normalized Lebesgue measure on $\mathbb{T}$. If Φ is a bounded, measurable $M_n = M_n(\mathbb{C})$-valued function (the algebra of $n \times n$ matrices with complex entries) in $\mathcal{L}^\infty_{M_n}(\mathbb{T}) = L^\infty(\mathbb{T}) \otimes M_n$, then S_Φ denotes the Hankel operator defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ by

$$S_\Phi f = \widetilde{P}\widetilde{J}(\Phi f) \quad \text{for} \quad f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\,,$$

where $\widetilde{P}$ is the orthogonal projection of $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ onto $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ and $\widetilde{J}\colon \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}) \to \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ is defined by $\widetilde{J}F(e^{it}) = F(e^{-it})$ and $(\Phi f)(e^{it}) = \Phi(e^{it}) f(e^{it})$.

Let $\Phi \in \mathcal{L}^\infty_{\mathbb{M}_n}(\mathbb{T})$ and

$$\Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix}.$$

Then each entry ϕ_{ij} of Φ is in $L^\infty(\mathbb{T})$ and

$$\mathbf{S}_\Phi = \begin{pmatrix} S_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & S_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\phi_{nn}} \end{pmatrix}.$$

This is so as $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = \underbrace{\mathcal{H}^2(\mathbb{T}) \oplus \mathcal{H}^2(\mathbb{T}) \oplus \cdots \oplus \mathcal{H}^2(\mathbb{T})}_{n-\text{times}}$.

Let $\mathcal{L}^2_{\mathbb{C}^n}(0,\infty) = L^2(0,\infty) \otimes \mathbb{C}^n = L^2(0,\infty) \oplus L^2(0,\infty) \oplus \cdots \oplus L^2(0,\infty)$. For F, $G \in \mathcal{L}^2_{\mathbb{C}^n}(0,\infty)$, the norm is defined by

$$\|F\|_{\mathcal{L}^2_{\mathbb{C}^n}} = \Big(\int_0^\infty \|F(x)\|^2_{\mathbb{C}^n}\, dx\Big)^{\frac{1}{2}}$$

and the inner product is defined by

$$\langle F, G\rangle = \int_0^\infty \langle F(x), G(x)\rangle_{\mathbb{C}^n}\, dx\,.$$

With the above inner product $\mathcal{L}^2_{\mathbb{C}^n}(0,\infty)$ is a Hilbert space. For detail see [1]. Let

$$H(x+y) = \begin{pmatrix} \frac{e^{-(x+y)}}{x+y} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{e^{-(x+y)}}{x+y} \end{pmatrix}_{n\times n}.$$

Define $B_H\colon \mathcal{L}^2_{\mathbb{C}^n}(0,\infty) \to \mathcal{L}^2_{\mathbb{C}^n}(0,\infty)$ by

$$(B_H F)(x) = \int_0^\infty H(x+y)F(y)\,dy\,.$$

The map B_H is well-defined, linear and for $G \in \mathcal{L}^2_{\mathbb{C}^n}(0,\infty)$,

$$\langle B_H F, G\rangle = \int_0^\infty \int_0^\infty G^*(x)H(x+y)F(y)\,dy\,dx\,,$$

where $G^*(x)$ denotes the adjoint of $G(x)$. Notice that

$$B_H = \begin{pmatrix} K_{\widetilde{h}_{11}} & 0 & \cdots & 0 \\ 0 & K_{\widetilde{h}_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{\widetilde{h}_{nn}} \end{pmatrix}$$

where $\widetilde{h}_{ij}(x) = \frac{e^{-x}}{x}$ for all $i,\ j = 1,2,\ldots,n$.

Lemma 3.2. *The operator $B_H\colon \mathcal{L}^2_{\mathbb{C}^n}(0,\infty) \to \mathcal{L}^2_{\mathbb{C}^n}(0,\infty)$ is a bounded operator and $\|B_H\| = \pi$.*

Proof. Let $F = (f_1, f_2, \ldots, f_n)^T$, where $f_i \in L^2(0,\infty)$ for all $i = 1,2,\ldots,n$. Then $G = B_H F = (g_1, g_2, \ldots, g_n)^T$ and $g_i \in L^2(0,\infty)$ for all $i = 1,2,\ldots,n$.

Now

$$\begin{aligned}
\|B_H F\|^2 &= \int_0^\infty \|(B_H F)(x)\|^2_{\mathbb{C}^n}\,dx = \int_0^\infty \|G(x)\|^2_{\mathbb{C}^n}\,dx \\
&= \int_0^\infty \Big(\sum_{j=1}^n |g_j(x)|^2\Big)\,dx = \sum_{j=1}^n \int_0^\infty |g_j(x)|^2\,dx \\
&= \sum_{j=1}^n \int_0^\infty \big|\big(K_{\widetilde{h}_{jj}} f_j\big)(x)\big|^2\,dx \\
&= \sum_{j=1}^n \|K_{\widetilde{h}_{jj}} f_j\|^2 \le \sum_{j=1}^n \|K_{\widetilde{h}_{jj}}\|^2 \|f_j\|^2 \le \sum_{j=1}^n \pi^2 \|f_j\|^2 \\
&= \pi^2 \sum_{j=1}^n \int_0^\infty |f_j(x)|^2\,dx = \pi^2 \int_0^\infty \Big(\sum_{j=1}^n |f_j(x)|^2\Big)\,dx \\
&= \pi^2 \int_0^\infty \|F(x)\|^2_{\mathbb{C}^n}\,dx = \pi^2 \|F\|^2\,.
\end{aligned}$$

Thus $\|B_H\| \le \pi$.

Now it remains to show that that $\|B_H\| \ge \pi$.

Let $f \in L^2(0,\infty)$ and $F = (f, 0, \cdots, 0)^T$. Then $\|F\| = \|f\|$. So,

$$|\langle K_{\widetilde{h}_{11}} f, f\rangle| = |\langle B_H F, F\rangle| \le \|B_H\| \|F\|^2 = \|B_H\| \|f\|^2$$

gives $\pi = \|K_{\widetilde{h}_{11}}\| \le \|B_H\|$ as $K_{\widetilde{h}_{11}}$ is self-adjoint. Hence $\|B_H\| = \pi$. $\square$

Now we generalize the Theorem 2.2, for the case $p = q = 2$, to vector-valued functions.

Theorem 3.3. *If $F, G \in \mathcal{L}^2_{\mathbb{C}^n}(0,\infty)$, then*

$$\Big|\int_0^\infty\int_0^\infty G^*(x)\,H(x+y)\,F(y)\,dx\,dy\Big| \le \pi\Big(\int_0^\infty \|F(x)\|^2\,dx\Big)^{\frac12}\Big(\int_0^\infty \|G(y)\|^2\,dy\Big)^{\frac12},$$

where the constant factor π is the best possible.

Proof. Since $\|B_H\| = \pi$, so, the result follows from the fact that

$$|\langle B_H F, G\rangle| \le \pi\|F\|_{\mathcal{L}^2_{\mathbb{C}^n}}\|G\|_{\mathcal{L}^2_{\mathbb{C}^n}}\,, \quad \text{for all} \quad F, G \in \mathcal{L}^2_{\mathbb{C}^n}(0,\infty)\,.$$

□

Now let $\phi_{lj}(e^{i\theta}) = -i(\pi-\theta)e^{i\theta}$, $0 \le \theta < 2\pi$, $1 \le l,\ j \le n$ and

$$\Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0\\ 0 & \phi_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \phi_{nn}\end{pmatrix}.$$

It is not difficult to see that

$$\mathbf{S}_\Phi = \begin{pmatrix} S_{\phi_{11}} & 0 & \cdots & 0\\ 0 & S_{\phi_{22}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & S_{\phi_{nn}}\end{pmatrix}$$

is unitarily equivalent to

$$B_H = \begin{pmatrix} K_{\widetilde{h}_{11}} & 0 & \cdots & 0\\ 0 & K_{\widetilde{h}_{22}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & K_{\widetilde{h}_{nn}}\end{pmatrix},$$

where $\widetilde{h}_{ij}(x) = \frac{e^{-x}}{x}$, $1 \le i,\ j \le n$. Hence $\|\mathbf{S}_\Phi\| = \pi$.

Let $e_k = (0,0,\ldots,0,1,0,\ldots,0)$ with 1 in the k^{th} place and $\gamma_{kl} = e^{ilt}\otimes e_k$, $k = 1,2,\ldots,n$, $l = 0,1,2,\ldots$. Then $\{e_k\}_{k=1}^n$ form an orthonormal basis for $\mathbb{C}^n$ and $\{\gamma_{kl}\}_{k=1,2,\ldots,n;l=0,1,\ldots,\infty}$ form an orthonormal basis for $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = \mathcal{H}^2(\mathbb{T})\otimes\mathbb{C}^n$.

Theorem 3.4. *Let $\widetilde{F} = f\otimes x \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ and $\widetilde{G} = g\otimes y \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Then*

$$\Big|\sum_{l,\ l'=0}^\infty\sum_{k=1}^n \frac{\langle f\otimes x, e^{ilt}\otimes e_k\rangle\overline{\langle g\otimes y, e^{il't}\otimes e_k\rangle}}{l+l'+1}\Big| \le \pi\|f\otimes x\|\,\|g\otimes y\|\,.$$

Proof. Notice that

$$\langle\widetilde{F},\gamma_{kl}\rangle = \langle f\otimes x, e^{ilt}\otimes e_k\rangle = \langle f, e^{ilt}\rangle\langle x, e_k\rangle$$

and

$$\langle\widetilde{G},\gamma_{ml'}\rangle = \langle g\otimes y, e^{il't}\otimes e_m\rangle = \langle g, e^{il't}\rangle\langle y, e_m\rangle\,.$$

Hence

$$
\begin{aligned}
\langle S_\Phi \widetilde{F}, \widetilde{G}\rangle &= \sum_{k,\, m=1}^{n} \sum_{l,\, l'=0}^{\infty} \langle \widetilde{F}, \gamma_{kl}\rangle \overline{\langle \widetilde{G}, \gamma_{ml'}\rangle} \langle S_\Phi(\gamma_{kl}), \gamma_{ml'}\rangle \\
&= \sum_{k,\, m=1}^{n} \sum_{l,\, l'=0}^{\infty} \langle \widetilde{F}, \gamma_{kl}\rangle \overline{\langle \widetilde{G}, \gamma_{ml'}\rangle} \langle (S_\phi \otimes I_{\mathbb{C}^n})(e^{ilt} \otimes e_k), e^{il't} \otimes e_m\rangle \\
&= \sum_{k,\, m=1}^{n} \sum_{l,\, l'=0}^{\infty} \langle f, e^{ilt}\rangle \langle x, e_k\rangle \overline{\langle g, e^{il't}\rangle}\ \overline{\langle y, e_m\rangle}\ \langle S_\phi e^{ilt} \otimes e_k, e^{il't} \otimes e_m\rangle \\
&= \sum_{k,\, m=1}^{n} \sum_{l,\, l'=0}^{\infty} \langle f, e^{ilt}\rangle \langle x, e_k\rangle \overline{\langle g, e^{il't}\rangle}\ \overline{\langle y, e_m\rangle}\ \langle S_\phi e^{ilt}, e^{il't}\rangle \langle e_k, e_m\rangle \\
&= \sum_{k=1}^{n} \sum_{l,\, l'=0}^{\infty} \langle f, e^{ilt}\rangle \langle x, e_k\rangle \overline{\langle g, e^{il't}\rangle}\ \overline{\langle y, e_k\rangle}\ \langle S_\phi e^{ilt}, e^{il't}\rangle\,.
\end{aligned}
$$

Thus

$$
\big|\langle S_\Phi \widetilde{F}, \widetilde{G}\rangle\big| = \Big| \sum_{l,\, l'=0}^{\infty} \sum_{k=1}^{n} \frac{\langle f \otimes x, e^{ilt} \otimes e_k\rangle \overline{\langle g \otimes y, e^{il't} \otimes e_k\rangle}}{l + l' + 1} \Big|
$$

and since S_Φ is a bounded linear operator in $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ and $\|S_\Phi\| = \pi$, we obtain

$$
\big|\langle S_\Phi \widetilde{F}, \widetilde{G}\rangle\big| \le \pi \|\widetilde{F}\|_{\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})} \|\widetilde{G}\|_{\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})} = \pi \|f \otimes x\|\, \|g \otimes y\|\,.
$$

The result follows. □

Corollary 3.5. *If* $\sum_{k=1}^{n} \sum_{l=0}^{\infty} |a_{kl}|^2 < \infty$ *and* $\sum_{k=1}^{n} \sum_{l'=0}^{\infty} |b_{kl'}|^2 < \infty$*, then*

$$
\Big| \sum_{l,l'=0}^{\infty} \sum_{k=1}^{n} \frac{a_{kl} \overline{b}_{kl'}}{l + l' + 1} \Big| \le \pi \Big(\sum_{k=1}^{n} \sum_{l=0}^{\infty} |a_{kl}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \sum_{l'=0}^{\infty} |b_{kl'}|^2 \Big)^{\frac{1}{2}}
$$

and the constant π *is best possible.*

Proof. It is possible to find x_k, y_k, $k = 1, 2, \ldots, n$, and sequences $(c_l)_{l=0}^{\infty}$, $(c_{l'})_{l'=0}^{\infty}$ such that $a_{kl} = x_k c_l$, $b_{kl'} = y_k c_{l'}$, $\sum_{l=0}^{\infty} |c_l|^2 < \infty$ and $\sum_{l'=0}^{\infty} |c_{l'}|^2 < \infty$. Let $f(e^{it}) = \sum_{l=0}^{\infty} c_l e^{ilt}$ and $g(e^{it}) = \sum_{l'=0}^{\infty} c_{l'} e^{il't}$. Then $f,\, g \in \mathcal{H}^2(\mathbb{T})$. So, for $x = (x_k)_{k=1}^{n}$, $y = (y_k)_{k=1}^{n} \in \mathbb{C}^n$, we have $f \otimes x,\, g \otimes y \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Now

$$
\|f \otimes x\|^2 = \|f\|^2 \|x\|^2 = \sum_{l=0}^{\infty} |c_l|^2 \sum_{k=1}^{n} |x_k|^2 = \sum_{k=1}^{n} \sum_{l=0}^{\infty} |c_l|^2 |x_k|^2 = \sum_{k=1}^{n} \sum_{l=0}^{\infty} |a_{kl}|^2\,.
$$

Similarly,

$$
\|g \otimes y\|^2 = \sum_{k=1}^{n} \sum_{l'=0}^{\infty} |b_{kl'}|^2\,.
$$

On the other hand, $\langle f \otimes x, e^{ilt} \otimes e_k\rangle = \langle f, e^{ilt}\rangle \langle x, e_k\rangle = x_k c_l = a_{kl}$ and $\langle g \otimes y, e^{il't} \otimes e_k\rangle = \langle g, e^{il't}\rangle \langle y, e_k\rangle = y_k c_{l'} = b_{kl'}$. Hence the results follows from Theorem 3.4. Since $\|S_\Phi\| = \pi$, the constant π is the best possible. □

4. Hankel operators with operator valued symbols

Let Ξ be a separable infinite dimensional Hilbert space. The measure m will denote the normalised Lebesgue measure on $\mathbb{T}$. The space L^2_Ξ is defined to be the set of all (equivalence classes of) measurable, norm-square integrable, Ξ-valued functions defined on $\mathbb{T}$. When endowed with the inner product defined by the equation

$$\langle f, g\rangle = \int_{\mathbb{T}} \langle f(z), g(z)\rangle_\Xi \, dm\,, \quad f, g \in L^2_\Xi\,,$$

the space L^2_Ξ becomes a separable Hilbert space. The subspace of L^2_Ξ consisting of those functions with vanishing negative Fourier coefficients will be denoted by H^2_Ξ. Each function in H^2_Ξ admits a natural analytic continuation into $\mathbb{D}$.

A function Φ from $\mathbb{T}$ into $\mathcal{L}(\Xi)$ is called weakly measurable in case the complex-valued function $z \mapsto \langle \Phi(z)x, y\rangle$ is Lebesgue measurable for every x and y in Ξ. If Φ is weakly measurable then the real-valued function $z \to \|\Phi(z)\|$ is measurable and the space of all (equivalence classes of) weakly measurable, essentially bounded, $\mathcal{L}(\Xi)$-valued functions on $\mathbb{T}$ will be denoted by $L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$.

The space $L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$ is a C^*- algebra with the algebraic operations defined pointwise and norm defined by the equation

$$\|\Phi\|_\infty = \operatorname*{ess\,sup}_{z\in\mathbb{T}} \|\Phi(z)\|\,, \quad \Phi \in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$$

where $\|\Phi(z)\| = \sup\limits_n \sup\limits_m |\langle \Phi(z)e_n, e_m\rangle|$, $z \in \mathbb{T}$, $\{e_n\}_{n=0}^\infty$ is the orthonormal basis for Ξ and involution is defined by the equation $\Phi^*(z) = (\Phi(z))^*$. The mapping $\zeta \to \Phi(\zeta)f$, $\zeta \in \mathbb{T}$ are measurable for $f \in \Xi$. This follows from the Pettis theorem (see [1]) as Ξ is separable.

For a function $\Phi \in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$ we define the Fourier coefficients

$$C_n(\Phi) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\,int}\Phi(e^{it})\,dt\,, \quad n \in \mathbb{Z}\,.$$

The integral is understood in the strong sense, i.e.,

$$C_n(\Phi)f = \frac{1}{2\pi}\int_0^{2\pi} e^{-\,int}\Phi(e^{it})f dt, f \in \Xi.$$

We have clearly $\|C_n(\Phi)\| \le \|\Phi\|_\infty$ for all integers n. The space $H^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$ is the subspace of $L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$ consisting of those functions Φ whose Fourier coefficients $C_n(\Phi)$ vanish if $n < 0$. For $\Phi \in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$, we define the Hankel operator S_Φ from $H^2_\Xi(\mathbb{T})$ into itself as $S_\Phi f = Q(\mathfrak{J}(\Phi f))$ where Q is the orthogonal projection from $L^2_\Xi(\mathbb{T})$ onto $H^2_\Xi(\mathbb{T})$ and the symbol Φf denote the function on $\mathbb{T}$ defined by $(\Phi f)(e^{it}) = \Phi(e^{it})f(e^{it})$ and $\mathfrak{J}\colon L^2_\Xi(\mathbb{T}) \to L^2_\Xi(\mathbb{T})$ is defined by $\mathfrak{J}F(e^{it}) = F(e^{-it})$.

In the following theorem we extend Theorem 3.3 for Ξ-valued functions.

Theorem 4.1. *Let $H(x) = \frac{e^{-x}}{x} \otimes I_\Xi$ where I_Ξ is the identity operator from the Hilbert space Ξ into itself. Let $L^2_\Xi(0,\infty) = L^2(0,\infty) \otimes \Xi$ and define K_H :*

$L^2_\Xi(0,\infty) \to L^2_\Xi(0,\infty)$ by

$$(K_H F)(x) = \int_0^\infty H(x+y)F(y)\,dy\,.$$

Then for $F, G \in L^2_\Xi(0,\infty)$,

$$\Big|\int_0^\infty \big\langle (K_H F)(x), G(x)\big\rangle_\Xi\,dx\Big| \le \pi \|F\|_{L^2_\Xi(0,\infty)} \|G\|_{L^2_\Xi(0,\infty)}\,.$$

Proof. Let $\widetilde{h}(x) = \frac{e^{-x}}{x}$ and define $K_{\widetilde{h}} \in \mathcal{L}(L^2(0,\infty))$ by

$$(K_{\widetilde{h}} f)(x) = \int_0^\infty \frac{e^{-(x+y)}}{x+y} f(y)\,dy\,.$$

It is not difficult to see that the operator K_H is well-defined and since $L^2_\Xi(0,\infty) = L^2(0,\infty) \otimes \Xi$ we have $K_H = \sum_{n=0}^\infty \oplus K_{\widetilde{h}} = K_{\widetilde{h}} \otimes I_\Xi$ where $(K_{\widetilde{h}} \otimes I_\Xi)(f \otimes z) = K_{\widetilde{h}} f \otimes z$ if $f \in L^2(0,\infty)$ and $z \in \Xi$. Now $\|K_H\| = \big\|\sum_{n=0}^\infty \oplus K_{\widetilde{h}}\big\| = \|K_{\widetilde{h}}\| = \pi$. Thus by Cauchy-Schwarz inequality it follows that

$$\begin{aligned}\big|\langle K_H F, G\rangle\big| &\le \|K_H\|\,\|F\|_{L^2_\Xi(0,\infty)}\|G\|_{L^2_\Xi(0,\infty)}\\ &= \pi\|F\|_{L^2_\Xi(0,\infty)}\|G\|_{L^2_\Xi(0,\infty)}\,.\end{aligned}$$

Hence

$$\Big|\int_0^\infty \big\langle (K_H F)(x), G(x)\big\rangle_\Xi\,dx\Big| \le \pi \|F\|_{L^2_\Xi(0,\infty)} \|G\|_{L^2_\Xi(0,\infty)}\,.$$

□

Theorem 4.2. *If $\widetilde{F} = f \otimes x$, $\widetilde{G} = g \otimes y \in \mathcal{H}^2_\Xi(\mathbb{T}) = \mathcal{H}^2(\mathbb{T}) \otimes \Xi$, then*

$$\Big|\sum_{l,\,l'=0}^\infty \sum_{k=0}^\infty \frac{\langle f\otimes x, e^{ilt}\otimes e_k\rangle \overline{\langle g \otimes y, e^{il't}\otimes e_k\rangle}}{l+l'+1}\Big| \le \pi \|f\otimes x\|\,\|g\otimes y\|\,.$$

Proof. Let $\phi(e^{i\theta}) = -i(\pi-\theta)e^{i\theta}$, $0 \le \theta < 2\pi$ and $\Phi = \phi \otimes I_\Xi$. Then $\Phi \in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{T})$. Let S_Φ be the Hankel operator from $\mathcal{H}^2_\Xi(\mathbb{T})$ into itself with symbol Φ. Notice that since $\mathcal{H}^2_\Xi(\mathbb{T}) = \mathcal{H}^2(\mathbb{T}) \otimes \Xi$, we have $S_\Phi = S_\phi \otimes I_\Xi$. Thus $\|S_\Phi\| = \|S_\phi\| = \pi$. Let $\Upsilon_{kl} = e^{ilt} \otimes e_k$, $k = 0,1,2,\ldots$ and $l = 0,1,2,\ldots$. The sequence $\{\Upsilon_{kl}\}$ form an orthonormal basis for $\mathcal{H}^2_\Xi(\mathbb{T})$. Then

$$\langle S_\Phi \widetilde{F}, \widetilde{G}\rangle = \sum_{l,\,l'=0}^\infty \sum_{k=0}^\infty \frac{\langle f\otimes x, e^{ilt}\otimes e_k\rangle \overline{\langle g \otimes y, e^{il't}\otimes e_k\rangle}}{l+l'+1}\,.$$

Since

$$\big|\langle S_\Phi \widetilde{F}, \widetilde{G}\rangle\big| \le \|S_\Phi\|\,\|\widetilde{F}\|\,\|\widetilde{G}\| = \pi\|f\otimes x\|\,\|g\otimes y\|\,,$$

the result follows. □

Corollary 4.3. *Let $\widetilde{F} = f \otimes x$ and $\widetilde{G} = g \otimes y$ where f, $g \in \mathcal{H}^2(\mathbb{T})$ and x, $y \in \Xi$. Let $c_l(f)$ and $c_{l'}(g)$ denote the l^{th} and $l'^{\,th}$ Fourier coefficients of f and g respectively. Then*

$$\Big|\sum_{l,\,l'=0}^\infty \frac{\langle c_l(f)x, c_{l'}(g)y\rangle_\Xi}{l+l'+1}\Big| \le \pi \|\widetilde{F}\|_{\mathcal{H}^2_\Xi(\mathbb{T})}\|\widetilde{G}\|_{\mathcal{H}^2_\Xi(\mathbb{T})}\,.$$

Proof. Let $\Upsilon_{kl} = e^{ilt} \otimes e_k$, $k = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$. Then the sequence $\{\Upsilon_{kl}\}$ form an orthonormal basis for $\mathcal{H}^2_{\Xi}(\mathbb{T})$. Hence $\langle \widetilde{F}, \Upsilon_{kl} \rangle = c_l(f)\langle x, e_k \rangle$ and $\langle \widetilde{g}, \Upsilon_{kl'} \rangle = c_{l'}(g)\langle y, e_k \rangle$. Also

$$\sum_{l,\, l'=0}^{\infty} \sum_{k=0}^{\infty} \frac{\langle f \otimes x, e^{ilt} \otimes e_k \rangle \overline{\langle g \otimes y, e^{il't} \otimes e_k \rangle}}{l + l' + 1} = \sum_{l,\, l'=0}^{\infty} \sum_{k=0}^{\infty} \frac{\langle c_l(f)x, e_k \rangle \overline{\langle c_{l'}(g)y, e_k \rangle}}{l + l' + 1}$$

$$= \sum_{l,\, l'=0}^{\infty} \sum_{k=0}^{\infty} \frac{\langle c_l(f)x, e_k \rangle \langle e_k, c_{l'}(g)y \rangle}{l + l' + 1} = \sum_{l,\, l'=0}^{\infty} \frac{\langle c_l(f)x, c_{l'}(g)y \rangle_{\Xi}}{l + l' + 1}.$$

Now the result follows from Theorem 4.2. □

Corollary 4.4. *If* $\sum_{l,k=0}^{\infty} |a_{kl}|^2 < \infty$ *and* $\sum_{l',k=0}^{\infty} |b_{kl'}|^2 < \infty$, *then*

$$\Big| \sum_{k,\, l,\, l'=0}^{\infty} \frac{a_{kl}\bar{b}_{kl'}}{l + l' + 1} \Big| \leq \pi \Big(\sum_{k,\, l=0}^{\infty} |a_{kl}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k,\, l'=0}^{\infty} |b_{kl'}|^2 \Big)^{\frac{1}{2}}$$

and the constant π *is sharp.*

Proof. The proof is similar to the proof of Corollary 3.5. □

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