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TWO-MODE BIFURCATION IN SOLUTION OF A PERTURBED NONLINEAR FOURTH ORDER DIFFERENTIAL EQUATION

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Abstract. In this paper, we are interested in the study of bifurcation solutions of nonlinear wave equation of elastic beams located on elastic foundations with small perturbation by using local method of Lyapunov-Schmidt.We showed that the bifurcation equation corresponding to the elastic beams equation is given by the nonlinear system of two equations. Also, we found the parameters equation of the Discriminant set of the specified problem as well as the bifurcation diagram.

1. Introduction

It is known that many of the nonlinear problems that appear in mathematics and physics can be written in the form of operator equation,

$$F(x,\lambda,\varepsilon) = b\,, \quad x \in O \subset X\,, \quad b \in Y\,, \quad \lambda \in R^n \tag{1.1}$$

where F is a smooth Fredholm map of index zero, ε is small parameter indicate the perturbation of the equation

$$F(x,\lambda) = b\,, \quad x \in O \subset X\,, \quad b \in Y\,, \quad \lambda \in R^n\,.$$

X, Y are Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\theta(\xi,\lambda,\varepsilon) = \beta\,, \quad \xi \in \hat{M}\,, \quad \beta \in \hat{N} \tag{1.2}$$

can be used, where $\hat{M}$ and $\hat{N}$ are smooth finite dimensional manifolds.

Vainberg [11], Loginov [5] and Sapronov [6, 7] are dealing with equation (1.1) into equation (1.2) by using local method of Lyapunov-Schmidt with the conditions that, equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, etc).

Definition 1.1. Suppose that E and F are Banach spaces and $A\colon E \to F$ be a linear continuous operator. The operator A is called Fredholm operator, if
1- The kernel of A, $\mathrm{Ker}(A)$, is finite dimensional,
2- The Range of A, $\mathrm{Im}(A)$, is closed in F,

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3- The Cokernel of A, $\mathrm{Coker}(A)$, is finite dimensional.
The number $\dim(\mathrm{Ker}\, A) - \dim(\mathrm{Coker}\, A)$ is called Fredholm index of the operator A.

Definition 1.2. The discriminate set Σ of equation (1.1) is defined to be the union of all $\lambda = \bar{\lambda}$ for which the equation (1.1) has degenerate solution $\bar{x} \in O$:

$$F(\bar{x}, \bar{\lambda}, \varepsilon) = b\,, \quad \mathrm{Codim}\left(\mathrm{Im}\, \frac{\partial F}{\partial x}(\bar{x}, \bar{\lambda}, \varepsilon)\right) > 0\,.$$

The oscillations and motion of waves of the elastic beams located on elastic foundations can be described by means of the following PDE,

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + \alpha \frac{\partial^2 y}{\partial x^2} + (\beta + \varepsilon_1 x) y + \varepsilon_2 \frac{\partial y}{\partial x} + g(\lambda, \tilde{y}) = \psi\,,$$
$$\tilde{y} = (y, y_x, y_{xx}, y_{xxx}, y_{xxxx})\,.$$

where y is the deflection of beam, ε_1, ε_2 indicates the perturbation parameters, $\psi = \tilde{\varepsilon}\varphi(x)$ ($\tilde{\varepsilon}$ – small parameter) is a continuous function and $g(\lambda, \tilde{y})$ is a generic nonlinearity. It is known that, to study the oscillations of beams, equilibrium state ($w(x) = y(x, t)$) should be consider which is describe by the equation

$$\frac{d^4 w}{dx^4} + \alpha \frac{d^2 w}{dx^2} + (\beta + \varepsilon_1 x) w + \varepsilon_2 \frac{dw}{dx} + g(\lambda, \tilde{w}) = \psi\,. \tag{1.3}$$
$$\tilde{w} = (w, w', w'', w''', w'''')\,.$$

When $g(\lambda, \tilde{w}) = -kw^3$, ($k$ is a parameter) [4], $\psi = 0$ and $\varepsilon_1 = \varepsilon_2 = 0$ equation (1.3) has been studied as follows: Thompson and Stewart [10] showed numerically the existence of periodic solutions of equation (1.3) for some values of parameters. Sapronov [9] applied the local method of Lyapunov-Schmidt and found the bifurcation solutions of equation (1.3) when $g(\lambda, \tilde{w}) = w^3$, $\psi = 0$ and ε_1, $\varepsilon_2 \neq 0$ with the boundary conditions,

$$w(0) = w(\pi) = w''(0) = w'''(\pi) = 0$$

in his study he solved the bifurcation equation corresponding to the equation (1.3) and found the bifurcation diagram of a specify problem. When $g(\lambda, \tilde{w}) = w^2$, $\varepsilon_1 = \varepsilon_2 = 0$ and $\psi \neq 0$, equation (1.3) has been studied with the boundary conditions,

$$w(0) = w(1) = w''(0) = w''(1) = 0$$

by Abdul Hussain [1]. He showed that by using local method of Lyapunov-Schmidt the existence of bifurcation solutions of equation (1.3). When $g(\lambda, \tilde{w}) = w^2 + w^3$ $\psi = 0$, and $\varepsilon_1 = \varepsilon_2 = 0$ equation (1.3) was studied by Sapronov [8], in his work he found bifurcation periodic solutions of equation (1.3) by using local method of Lyapunov-Schmidt. Also, he solved the bifurcation equation corresponding to the equation (1.3) and found the bifurcation diagram of a specify problem. When $g(\lambda, \tilde{w}) = w^2 + w^3$, $\psi \neq 0$ and $\varepsilon_1, \varepsilon_2 \neq 0$ equation (1.3) was studied with the boundary conditions,

$$w(0) = w(\pi) = w''(0) = w''(\pi) = 0$$

by Abdul Hussain [2]. He found the bifurcation solutions of equation (1.3) by using local method of Lyapunov-Schmidt.

In this paper we used the local method of Lyapunov-Schmidt to find two modes bifurcation solutions of boundary value problem,

$$\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+(\beta+\varepsilon_1x)w+\varepsilon_2\frac{dw}{dx}+w^2=\psi\,, \qquad (1.4)$$
$$w(0)=w(1)=w''(0)=w''(1)=0$$

where ε_1 and ε_2 are small parameters indicates the perturbation and $\psi=\tilde{\varepsilon}\varphi(x)$ ($\tilde{\varepsilon}$ – small parameter) is a symmetric function with respect to the involution $I\colon \psi(x)\mapsto\psi(1-x)$.

2. Reduction to bifurcation equation

To the study problem (1.4) it is convenient to set the ODE in the form of operator equation, that is;

$$F(w,\lambda,\varepsilon_1,\varepsilon_2)=\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+(\beta+\varepsilon_1x)w+\varepsilon_2\frac{dw}{dx}+w^2 \qquad (2.1)$$

Where $F\colon E\to M$ is nonlinear Fredholm map of index zero from Banach space E to Banach space M, $E=C^4([0,1],R)$ is the space of all continuous functions that have derivative of order at most four, $M=C^0([0,1],R)$ is the space of all continuous functions and $w=w(x)$, $x\in[0,1]$, $\lambda=(\alpha,\beta)$. In this case, the bifurcation solutions of equation (2.1) is equivalent to the bifurcation solutions of operator equation

$$F(w,\lambda,\varepsilon_1,\varepsilon_2)=\psi\,,\quad \psi\in M \qquad (2.2)$$

It is clear that when ε_1 and ε_2 are both equal to zero, then the operator F have variational property that is; there exist a functional $V\colon\Omega\to R$ such that $F(w,\lambda,0,0)=\operatorname{grad}_H V(w,\lambda,0)$ or equivalently,

$$\frac{\partial V}{\partial w}(w,\lambda,0)h=\langle f(w,\lambda,0,0),h\rangle_H\,,\quad \forall\ w\in\Omega\,,\quad h\in E$$

where ($\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space H)and

$$V(w,\lambda,\psi)=\int_0^1\Big(\frac{(w'')^2}{2}-\alpha\frac{(w')^2}{2}+\beta\frac{w^2}{2}+\frac{w^3}{3}-w\ \psi\Big)\,dx\,.$$

In this case the solutions of the equation $F(w,\lambda,0,0)=\psi$ are the critical points of the functional $V(w,\lambda,\psi)$, where the critical points of the functional $V(w,\lambda,\psi)$ are the solutions of Euler-Lagrange equation

$$\frac{\partial V}{\partial w}(w,\lambda,0)h=\int_0^1(w''''+\alpha w''+\beta w+w^2)h\ dx=0\,.$$

The bifurcation solutions of problem (1.4) when $\varepsilon_1=\varepsilon_2=0$ have been studied by Abdul Hussain [1], in his work he showed that the discriminate set of problem (1.4) is given by the parameter equation $\tilde{\beta}(\tilde{\beta}^2-q)=0$, where the parameters $\tilde{\beta}$ and q depend on α and β. Also, he showed the existence and stability solutions of a specify problem. If ε_1 and ε_2 are not both equal to zero, then the operator F should be lose the variational property, in this case we go to seek the existence of regular solutions of problem (1.4) in the plane of parameters by using local method of Lyapunov-Schmidt. It is well known that by finite dimensional reduction theorem the solutions of problem (1.4) is equivalent to the solutions of finite dimensional system with $2 = \dim(\ker F_w(0,\lambda))$ variables and $2 = \dim(\mathrm{Coker}\, F_w(0,\lambda))$ equations, so first step in this work we shall find this system and then we analyze the results to find the bifurcation solutions of problem (1.4). Our purpose is to study the bifurcation solutions of problem (1.4) near the bifurcation solutions of the problem

$$\begin{aligned}&\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+w^2=\psi\\&w(0)=w(1)=w''(0)=w''(1)=0\,.\end{aligned}$$

The first step in this reduction is determines the linearized equation corresponding to the equation (2.2), which is given by the following equation:

$$\begin{aligned}&Ah=0\,,\quad h\in E\,,\\&A=\frac{\partial f}{\partial w}(0,\lambda,0,0)=\frac{d^4}{dx^4}+\alpha\frac{d^2}{dx^2}+\beta\,,\quad x\in[0,1]\,,\\&h(0)=h(1)=h''(0)=h''(1)=0\,.\end{aligned}$$

The solution of linearized equation which is satisfied the boundary conditions is given by

$$e_p=c_p\sin(p\pi x)\,,\quad p=1,2,3,\ldots$$

and the characteristic equation corresponding to this solution is

$$p^4\pi^4-\alpha p^2\pi^2+\beta=0\,.$$

This equation gives in the $\alpha\beta$-plane characteristic lines ℓ_p. The characteristic lines ℓ_p consist the points (α,β) in which the linearized equation has non-zero solutions. The point of intersection of characteristic lines in the $\alpha\beta$-plane is a bifurcation point [8]. So for equation (2.2) the point $(\alpha,\beta)=(5\pi^2,4\pi^4)$ is a bifurcation point. Localized parameters α,β as follows,

$$\alpha=5\pi^2+\delta_1\,,\quad \beta=4\pi^4+\delta_2\,,\ \delta_1\,,\ \delta_2\text{are small parameters}$$

lead to bifurcation along the modes

$$e_1(x)=c_1\sin(\pi x)\,,\quad e_2(x)=c_2\sin(2\pi x)$$

where $\| e_1 \|_H=\| e_2 \|_H=1$ and $c_1=c_2=\sqrt{2}$. Let $N=\ker(A)=\mathrm{span}\,\{e_1,e_2\ \}$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E=N\oplus N^\perp\,,\quad N^\perp=\{v\in E: v\perp N\}\,.$$

Similarly, the space M can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$M=N\oplus\tilde{N}^\perp\,,\quad \tilde{N}^\perp=\{v\in M: v\perp N\}\,.$$

There exist two projections $P\colon E\to N$ and $I-P\colon E\to N^\perp$ such that $Pw=u$, $(I-P)w=v$ and hence every vector $w\in E$ can be written in the form

$$w=u+v\,,\quad u=\sum_{i=1}^{2}\xi_ie_i\in N\,,\quad v\in N^\perp\,,\quad \xi_i=\langle w,e_i\rangle\,.$$

Similarly, there exist projections $Q\colon M \to N$ and $I - Q\colon M \to \tilde{N}^{\perp}$ such that

$$QF(w,\lambda,\varepsilon_1,\varepsilon_2) = F_1(w,\lambda,\varepsilon_1,\varepsilon_2)\,,$$
$$(I-Q)F(w,\lambda,\varepsilon_1,\varepsilon_2) = F_2(w,\lambda,\varepsilon_1,\varepsilon_2)\,,$$

and hence,

$$F(w,\lambda,\varepsilon_1,\varepsilon_2) = F_1(w,\lambda,\varepsilon_1,\varepsilon_2) + F_2(w,\lambda,\varepsilon_1,\varepsilon_2)\,,$$
$$F_1(w,\lambda,\varepsilon_1,\varepsilon_2) = \sum_{i=1}^{2} v_i(w,\lambda,\varepsilon_1,\varepsilon_2)e_i \in N\,, \quad F_2(w,\lambda,\varepsilon_1,\varepsilon_2) \in \tilde{N}^{\perp}\,,$$
$$v_i(w,\lambda,\varepsilon_1,\varepsilon_2) = \langle F(w,\lambda,\varepsilon_1,\varepsilon_2), e_i\rangle\,.$$

Since $\psi \in M$ implies that $\psi = \psi_1+\psi_2$, $\psi_1 = t_1e_1+t_2e_2 \in N$, $\psi_2 \in \tilde{N}^{\perp}$. Accordingly, equation (2.2) can be written in the form

$$QF(w,\lambda,\varepsilon_1,\varepsilon_2) = \psi_1\,,$$
$$(I-Q)F(w,\lambda,\varepsilon_1,\varepsilon_2) = \psi_2$$

or

$$QF(u+v,\lambda,\varepsilon_1,\varepsilon_2) = \psi_1\,,$$
$$(I-Q)F(u+v,\lambda,\varepsilon_1,\varepsilon_2) = \psi_2\,.$$

By implicit function theorem, there exists a smooth map $\Phi\colon N \to N^{\perp}$ (depending on λ), such that, $\Phi(w,\lambda,\varepsilon_1,\varepsilon_2) = v$ and

$$(I-Q)F(u+\Phi(w,\lambda,\varepsilon_1,\varepsilon_2),\lambda,\varepsilon_1,\varepsilon_2) = \psi_2\,.$$

To find the solutions of the equation $F(w,\lambda,\varepsilon_1,\varepsilon_2) = \psi$ in the neighbourhood of the point $w = 0$ it is sufficient to find the solutions of the equation

$$QF(u+\Phi(w,\lambda,\varepsilon_1,\varepsilon_2),\lambda,\varepsilon_1,\varepsilon_2) = \psi_1\,. \tag{2.3}$$

Equation (2.3) is called bifurcation equation of the equation (2.1) and then we have the bifurcation equation in the form

$$\Theta(\xi,\lambda,\varepsilon_1,\varepsilon_2) = \psi_1\,, \quad \xi = (\xi_1,\xi_2)\,, \quad \lambda = (\alpha,\beta)$$

where

$$\Theta(\xi,\lambda,\varepsilon_1,\varepsilon_2) = F_1(u+\Phi(w,\lambda,\varepsilon_1,\varepsilon_2),\lambda,\varepsilon_1,\varepsilon_2)\,.$$

Equation (2.1) can be written in the form,

$$\begin{aligned} F(u+v,\lambda,\varepsilon_1,\varepsilon_2) &= A(u+v) + B(u+v) + T(u+v) \\ &= Au + \varepsilon_1 xu + \varepsilon_2 u^{'} + u^2 + \dots \end{aligned}$$

where $B(u+v) = \varepsilon_1 x(u+v) + \varepsilon_2(u+v)'$, $T(u+v) = (u+v)^2$ and the dots denote the terms consists the element v. Hence

$$\begin{aligned} \Theta(\xi,\lambda,\varepsilon_1,\varepsilon_2) &= F_1(u+v,\lambda,\varepsilon_1,\varepsilon_2) \\ &= \sum_{i=1}^{2}\langle Au + \varepsilon_1 xu + \varepsilon_2 u' + u^2, e_i\rangle e_i + \dots = \psi_1 \end{aligned} \tag{2.4}$$

where ($\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space $L^2([0,1],R)$). Equation (2.4) implies that

$$\sum_{i=1}^{2}\langle Au+\varepsilon_1 xu+\varepsilon_2 u'+u^2, e_i\rangle e_i+\cdots=t_1\, e_1+t_2\, e_2\,. \tag{2.5}$$

After some calculations of equation (2.5) we have the following result

$$(A_1\xi_1^2+A_2\xi_2^2+A_3\xi_1-A_4\xi_2)e_1+(B_1\xi_1\xi_2+B_2\xi_1+B_3\xi_2)e_2=t_1\ e_1+t_2\ e_2$$

where $Ae_1=\tilde{\alpha}_1(\lambda)e_1,\quad Ae_2=\tilde{\alpha}_2(\lambda)e_2$

$$A_1=\frac{5}{8}B_1=\frac{8\sqrt{2}}{3\pi}\,,\qquad A_2=\frac{4}{5}A_1=\frac{32\sqrt{2}}{15\pi}\,,\qquad A_3=\frac{\varepsilon_1}{2}+\tilde{\alpha}_1(\lambda)$$
$$A_4=\frac{8\varepsilon_2}{3}+\frac{16\varepsilon_1}{9\pi^2}\,,\qquad B_2=\frac{8\varepsilon_2}{3}-\frac{16\varepsilon_1}{9\pi^2}\,,\qquad B_3=\frac{\varepsilon_1}{2}+\tilde{\alpha}_2(\lambda)$$

and $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ are smooth functions. The symmetry of the function $\psi(x)$ with respect to the involution $I\colon \psi(x)\mapsto\psi(1-x)$ implies that t_2=0, then we have stated the following theorem

Theorem 2.1. *The bifurcation equation*

$$\Theta(\xi,\lambda,\varepsilon_1,\varepsilon_2)=F_1(u+\Phi(w,\lambda,\varepsilon_1,\varepsilon_2),\lambda,\varepsilon_1,\varepsilon_2)=\psi_1$$

corresponding to the equation (2.2) *have the following form*

$$\Theta(\xi,\tilde{\lambda})=\begin{pmatrix}A_1\xi_1^2+A_2\xi_2^2+A_3\xi_1-A_4\xi_2-t_1\\ B_1\xi_1\xi_2+B_2\xi_1+B_3\xi_2\end{pmatrix}+o(|\xi|^2)+O(|\xi|^2)O(\delta)=0$$

where $\xi=(\xi_1,\xi_2)$, $\tilde{\lambda}=(A_3,A_4,B_2,B_4,t_1)\in R^5$, $\delta=(\delta_1,\delta_2)$. *The equation* $\Theta(\xi,\tilde{\lambda})=0$ *is symmetric contact equivalent to the equation*

$$\Theta_0(\tilde{\xi},\gamma)=\begin{pmatrix}\tilde{\xi}_1^2+\tilde{\xi}_2^2+\lambda_1\tilde{\xi}_1+\lambda_2\tilde{\xi}_2+q_1\\ 2\ \tilde{\xi}_1\tilde{\xi}_2+\lambda_3\tilde{\xi}_1+\lambda_4\tilde{\xi}_2\end{pmatrix}+o(|\tilde{\xi}|^2)+O(|\tilde{\xi}|^2)O(\delta)=0$$

where $\gamma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,q_1)\in R^5$.

Contact equivalence implies that the study of the Discriminant set of the equation $\Theta(\xi,\tilde{\lambda})=0$ in the space of parameters A_3, A_4, B_2, B_3, t_1 is similar to the study of the Discriminant set of the equation $\Theta_0(\tilde{\xi},\gamma)=0$ in the space of parameters $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and q_1. The discriminate Σ set of the equation $\Theta_0(\tilde{\xi},\gamma)=0$ is locally equivalent in the neighborhood of the point zero to the discriminate set of the following equation [3],

$$\Theta_1(\tilde{\xi},\gamma)=\begin{pmatrix}\tilde{\xi}_1^2+\tilde{\xi}_2^2+\lambda_1\tilde{\xi}_1+\lambda_2\tilde{\xi}_2+q_1\\ 2\tilde{\xi}_1\tilde{\xi}_2+\lambda_3\tilde{\xi}_1+\lambda_4\tilde{\xi}_2\end{pmatrix}=0 \tag{2.6}$$

this means that, to study the discriminate set of the equation $\Theta_0(\tilde{\xi},\gamma)=0$ it is sufficient to study the discriminate set of the equation $\Theta_1(\tilde{\xi},\gamma)=0$. By changing variables,

$$\eta_1=\tilde{\xi}_1+\frac{\lambda_1}{2}\,,\qquad \eta_2=\tilde{\xi}_2+\frac{\lambda_2}{2}\,,$$

we have equation (2.6) is equivalent to the following equation

$$\Theta_1(\eta,\hat{\lambda}) = \begin{pmatrix} \eta_1^2+\eta_2^2-\beta_1 \\ 2\ \eta_1\eta_2+\tilde{\lambda}_1\eta_1+\tilde{\lambda}_2\eta_2-\beta_2 \end{pmatrix} = 0 \tag{2.7}$$

where $\eta = (\eta_1, \eta_2)$, $\hat{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2) \in R^4$.

3. Analysis of bifurcation

From Section 2 we note that the point $a \in E$ is a solution of equation (2.1) if and only if

$$a = \sum_{i=1}^{2} \bar{\eta}_i e_i + \Phi(\bar{\eta}, \bar{\lambda}),$$

where $\bar{\eta}$ is a solution of the equation

$$\Theta_1(\eta,\hat{\lambda}) = 0, \tag{3.1}$$

also, the Discriminant set of equation (2.1) is equivalent to the Discriminant set of equation (3.1). This section concerning the determination of Discriminant set of equation (3.1) and then the determination of solutions of equation (3.1) as $\hat{\lambda}$ varied. There are two ways to determine the Discriminant set,
1. By finding a relationship between the parameters and variables given in the equation (3.1).
2. By finding the parameters equation, that is; equation of the form,

$$h(\hat{\lambda}) = 0, \quad \hat{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2) \in R^4$$

such that the set of all $\hat{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2)$ in which equation (3.1) has degenerate solutions that satisfy the equation $h(\hat{\lambda}) = 0$, where $h\colon R^4 \to R$ is a map. In this section we used the two ways, the first for the geometric description of the Discriminant set and the second for the theoretical description of the Discriminant set. Let

$$p_1 = \tilde{\lambda}_1, \qquad p_2 = \frac{\tilde{\lambda}_1^2+\tilde{\lambda}_2^2-4\beta_1}{4},$$

$$p_3 = \frac{2\tilde{\lambda}_2\beta_2-4\tilde{\lambda}_1\beta_1}{4}, \qquad p_4 = \frac{\beta_2^2-\tilde{\lambda}_1^2\beta_1}{4},$$

$$\tilde{p}_1 = \frac{\tilde{\lambda}_1}{2}, \qquad \tilde{p}_2 = \frac{\tilde{\lambda}_1^2+\tilde{\lambda}_2^2-16\beta_1}{16},$$

$$\tilde{p}_3 = \frac{-\tilde{\lambda}_1\beta_1}{4}, \qquad \tilde{p}_4 = \frac{4\beta_1^2-\tilde{\lambda}_2\beta_1}{16},$$

$$a_1 = \frac{3\tilde{\lambda}_1}{2}\,, \qquad a_2 = \frac{3\tilde{\lambda}_1^2}{4}\,,$$

$$a_3 = \frac{\frac{\tilde{\lambda}_1\tilde{\lambda}_2^2}{2} - \tilde{\lambda}_2\beta_2 + \frac{\tilde{\lambda}_1^3}{2}}{4}\,, \qquad a_4 = \frac{\frac{\tilde{\lambda}_1\tilde{\lambda}_2\beta_2}{2} - \beta_2^2}{4}\,,$$

$$b_1 = -\frac{\tilde{\lambda}_1}{2}\,, \qquad b_2 = -\frac{3(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2)}{16} - 2\beta_1\,,$$

$$b_3 = \frac{3\tilde{\lambda}_1\beta_1}{4} - \frac{\tilde{\lambda}_2\beta_2}{2}\,, \qquad b_4 = \frac{\beta_1^2 - \beta_2^2 + \tilde{\lambda}_1^2\beta_1 - \frac{\tilde{\lambda}_2\beta_1}{4}}{4}\,,$$

$$c_1 = \frac{\tilde{\lambda}_1}{2}\,, \qquad c_2 = \frac{1 - \tilde{\lambda}_1^2 - \tilde{\lambda}_2^2}{4} - \beta_1\,,$$

$$c_3 = \frac{\frac{\tilde{\lambda}_1\tilde{\lambda}_2^2}{2} - 3\tilde{\lambda}_2\beta_2 + \frac{\tilde{\lambda}_1^3}{2}}{4} + \tilde{\lambda}_1\beta_1\,, \qquad c_4 = \frac{\frac{\tilde{\lambda}_1\tilde{\lambda}_2\beta_2}{4} - \beta_2^2 + \frac{\tilde{\lambda}_1^2\beta_1}{2}}{2}\,,$$

and

$$d_1 = \frac{7(\tilde{\lambda}_1^3 - \tilde{\lambda}_1\tilde{\lambda}_2^2)}{32} - \frac{3\tilde{\lambda}_1\beta_1 - \frac{1}{4}\tilde{\lambda}_1}{2}\,,$$

$$d_2 = \frac{7\tilde{\lambda}_1^2\beta_1 - 5\tilde{\lambda}_1\tilde{\lambda}_2\beta_2 + \frac{\tilde{\lambda}_1^2\tilde{\lambda}_2^2}{2} + \frac{\tilde{\lambda}_1^4}{2}}{8}\,,$$

$$d_3 = \frac{\frac{\tilde{\lambda}_1\beta_1^2}{2} - \frac{\tilde{\lambda}_1\tilde{\lambda}_2\beta_1}{8} - \frac{3\tilde{\lambda}_1\beta_2^2}{2} + \tilde{\lambda}_1^3\beta_1 + \frac{\tilde{\lambda}_1^2\tilde{\lambda}_2\beta_2}{4}}{4}\,,$$

$$k = \sqrt{d_2^2 - 4d_1d_3}\,.$$

Then the following result has been stated

Theorem 3.1. *The Discriminant set of equation* (3.1) *in the space of parameters* $(\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2)$ *is given by the following surface*

$$\begin{aligned}
&[(4(d_1^2\beta_1 - d_2^2) + d_1d_2\tilde{\lambda}_1 + 8d_1d_3)^2 + (4d_2 - d_1\tilde{\lambda}_1)^2k^2 \\
&\qquad - (2d_1^2\beta_1 + d_2^2 - 2d_1d_3)2d_1^2\tilde{\lambda}_2^2]^2 \\
&\qquad - [(2d_1\tilde{\lambda}_1 - 8d_2)(4(d_1^2\beta_1 - d_2^2) + d_1d_2\tilde{\lambda}_1 + 8d_1d_3) + 2d_1^2d_2\tilde{\lambda}_2^2]^2k^2 = 0\,.
\end{aligned}$$

Proof. The set of singular points of the map (2.7) is given by the equation

$$2\eta_1^2 - 2\eta_2^2 + \tilde{\lambda}_2\eta_1 - \tilde{\lambda}_1\eta_2 = 0\,.$$

Hence, the surface can be found by solving the following system in terms of $(\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2)$. The system is

$$\begin{cases} \eta_1^2 + \eta_2^2 + \beta_1 = 0\,, \\ 2\eta_1\eta_2 + \tilde{\lambda}_1\eta_1 + \tilde{\lambda}_2\eta_2 + \beta_2 = 0\,, \\ 2\eta_1^2 - 2\eta_2^2 + \tilde{\lambda}_2\eta_1 - \tilde{\lambda}_1\eta_2 = 0\,. \end{cases} \tag{3.2}$$

By using the substitution and subtracting between the equations of system (3.2) we have the following three quartic equations

$$(3.3)\qquad \begin{cases} \eta_2^4 + p_1\eta_2^3 + p_2\eta_2^2 + p_3\eta_2 + p_4 = 0\,, \\ \eta_2^4 + \tilde{p}_1\eta_2^3 + \tilde{p}_2\eta_2^2 + \tilde{p}_3\eta_2 + \tilde{p}_4 = 0\,, \\ \eta_2^4 + a_1\eta_2^3 + a_2\eta_2^2 + a_3\eta_2 + a_4 = 0\,. \end{cases}$$

Similarly, by using the substitution and subtracting between the equations of system (3.3) we have the following two cubic equations

$$(3.4)\qquad \begin{cases} b_1\eta_2^3 + b_2\eta_2^2 + b_3\eta_2 + b_4 = 0\,, \\ c_1\eta_2^3 + c_2\eta_2^2 + c_3\eta_2 + c_4 = 0\,. \end{cases}$$

System (3.4) gives rise to the quadratic equation of the form $d_1\eta_2^2 + d_2\eta_2 + d_3 = 0$. Solve this equation in terms of η_2, $(d_1 \neq 0)$ and then substitute the result in the third equation of system (3.2) we have the required surface. □

To study the Discriminant set of the equation (3.1) it is convenient to fixed the values of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and then find all sections of the Discriminant set in the $\beta_1\beta_2$-plane. To do this, we used the following parameterization

$$\begin{aligned} \beta_1 &= \eta_1^2 + \eta_2^2\,, \\ \beta_2 &= 2\eta_1\eta_2 + \tilde{\lambda}_1\eta_1 + \tilde{\lambda}_2\eta_2 \end{aligned}$$

and then we describe the Discriminant set of equation (3.1) in the $\beta_1\beta_2$-plane for some values of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ with the number of regular solutions in every region in the following figures, (all figures were drawn by Maple 11).

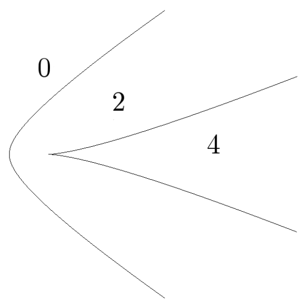

Figure 1: Describe the Discriminant set of equation (3.1) when $\tilde{\lambda}_1 = 0,\ \tilde{\lambda}_2 = 5$.

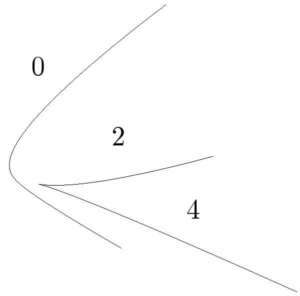

Figure 2: Describe the Discriminant set of equation (3.1) when $\tilde{\lambda}_1 = 3,\ \tilde{\lambda}_2 = 5$.

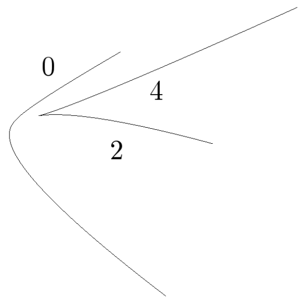

Figure 3: Describe the Discriminant set of equation (3.1) when $\tilde{\lambda}_1 = -3,\ \tilde{\lambda}_2 = 5$.

In figures (1), (2) and (3) the complement of the Discriminant set $\Gamma = R^4 \backslash \Sigma$ is the union of three open subsets $\Gamma = \Gamma_0 \cup \Gamma_2 \cup \Gamma_4$ such that if $\hat{\lambda} \in \Gamma_0$ then equation (3.1) has no regular solutions, if $\hat{\lambda} \in \Gamma_2$ then equation (3.1) has two regular solutions with topological indices 1, -1 and if $\hat{\lambda} \in \Gamma_4$ then equation (3.1) has four regular solutions with topological indices 1,-1,1,-1. If $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 0$ then the complement of the Discriminant set is a union of two open subsets $\Gamma = \Gamma_0 \cup \Gamma_4$ in which we have zero or four regular solutions.

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