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g-NATURAL METRICS OF CONSTANT CURVATURE ON UNIT TANGENT SPHERE BUNDLES

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Abstract. We completely classify Riemannian g-natural metrics of constant sectional curvature on the unit tangent sphere bundle T_1M of a Riemannian manifold (M,g). Since the base manifold M turns out to be necessarily two-dimensional, weaker curvature conditions are also investigated for a Riemannian g-natural metric on the unit tangent sphere bundle of a Riemannian surface.

1. Introduction and main results

A classical research field in Riemannian geometry is represented by the study of relationships between the geometry of a Riemannian manifold (M,g), and the one of its unit tangent sphere bundle T_1M, equipped with some Riemannian metric. Usually, T_1M has been equipped with one of the following Riemannian metrics:

a) either the *Sasaki metric* g^S, induced by the Sasaki metric of the tangent bundle TM, or

b) the metric $\bar{g} = \frac{1}{4}g^S$ of the *standard contact metric structure* $(\eta, \bar{g})$ of T_1M, or

c) the *Cheeger-Gromoll metric* g_{CG}; (T_1M, g_{CG}), is isometric to the tangent sphere bundle $T_\rho M$, with suitable radius $\rho = \frac{1}{\sqrt{2}}$, equipped with the metric induced by the Sasaki metric of TM, the isometry being explicitly given by $\Phi\colon T_1M \to T_{\frac{1}{\sqrt{2}}}M, (x,u) \mapsto (x, u/\sqrt{2})$.

Geometries determined by the three metrics above are very much similar to one another, and they often showed a quite "rigid" behaviour, in the sense that many curvature properties on T_1M, equipped with one of these metrics, imply strong restrictions on the base manifold itself. Surveys on the geometry of (T_1M, g^S) and $(T_1M, \eta, \bar{g})$ can be found in [6] and [7], respectively.

The first author and M. Sarih [5] investigated geometric properties of "g-natural" metrics on the tangent bundle TM. In [1], the authors introduced a three-parameter

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family of "g-natural" contact metric structures on T_1M, and investigated how their contact metric properties, expressible in terms of the Levi-Civita connection, are reflected by the geometry of the base manifold. The study of curvature properties of "g-natural" contact metric structures on T_1M was realized in [2], where general formulae for the curvature of an arbitrary g-natural Riemannian metric on T_1M were given.

In this paper, we start to attack the problem of understanding the geometry of a general g-natural Riemannian metric on T_1M, from the most natural and restrictive assumption: constant sectional curvature.

For the Sasaki metric g^S, it is well known that (T_1M, g^S) has constant sectional curvature if and only if the base manifold (M, g) is two-dimensional and either flat or of constant Gaussian curvature equal to 1 [6]. When we replace g^S by the most general g-natural Riemannian metric $\tilde{G}$, we again find that (M, g) is necessarily two-dimensional and of constant Gaussian curvature $\bar{c}$, but we have much more freedom concerning the possible values of $\bar{c}$. Indeed, we have

Theorem 1.1. *Let $\tilde{G} = a \cdot \widetilde{g^s} + b \cdot \widetilde{g^h} + c \cdot \widetilde{g^v} + d \cdot \widetilde{k^v}$ be a Riemannian g-natural metric on T_1M. Then, $(T_1M, \tilde{G})$ has constant sectional curvature $\tilde{K}$ if and only if the base manifold is a Riemannian surface (M^2, g) of constant Gaussian curvature $\bar{c}$ and one of the following cases occurs:*

(i) *$d = 0$ and $\bar{c} = 0$. In this case, $\tilde{K} = 0$.*

(ii) *$b = 0$ and $\bar{c} = \dfrac{d}{a}$. In this case, $\tilde{K} = \dfrac{d}{a\varphi}$, where $\varphi = a + c + d$.*

(iii) *$b = 0$, $d = a + c$ and $\bar{c} = \dfrac{a+c}{a} > 0$. In this case, $\tilde{K} = \dfrac{1}{2a} > 0$.*

From Theorem 1.1, we obtain at once the following classification of Riemannian g-natural metrics of constant sectional curvature in the unit tangent sphere bundle of a Riemannian surface (M^2, g).

Corollary 1.1. *Let (M^2, g) be a Riemannian surface of constant sectional curvature $\bar{c}$. The following are all and the ones g-natural Riemannian metrics of constant sectional curvature on T_1M^2:*

- *if $\bar{c} = 0$, then g-natural Riemannian metrics of the form $\tilde{G} = a \cdot \widetilde{g^s} + b \cdot \widetilde{g^h} + c \cdot \widetilde{g^v}$, $a > 0$, $a(a+c) - b^2 > 0$, have constant sectional curvature $\tilde{K} = 0$.*
- *if $\bar{c} > 0$, then g-natural Riemannian metrics of the form either $\tilde{G} = a \cdot \widetilde{g^s} + c \cdot \widetilde{g^v} + (\bar{c}a) \cdot \widetilde{k^v}$, $a > 0$, $a + c > 0$, or $\tilde{G} = a \cdot \widetilde{g^s} + a(\bar{c}-1) \cdot \widetilde{g^v} + (\bar{c}a) \cdot \widetilde{k^v}$, $a > 0$, have constant sectional curvature $\tilde{K} > 0$.*
- *if $\bar{c} < 0$, then g-natural Riemannian metrics of the form $\tilde{G} = a \cdot \widetilde{g^s} + c \cdot \widetilde{g^v} + (\bar{c}a) \cdot \widetilde{k^v}$, $a > 0$, $c > -a(\bar{c}+1)$, have constant sectional curvature $\tilde{K} < 0$.*

Now, by Theorem 1.1, only unit tangent sphere bundles of two-dimensional Riemannian manifolds of constant Gaussian curvature can admit g-natural Riemannian metrics of constant sectional curvature. Moreover, by Corollary 1.1 only some g-natural metrics, over a Riemannian surface (M^2, g) of constant Gaussian curvature $\bar{c}$, have constant sectional curvature. Therefore, it is natural to investigate some milder curvature conditions for a g-natural Riemannian metric $\tilde{G}$ on T_1M^2.

A Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *curvature homogeneous* if, for any points x, $y \in M$, there exists a linear isometry $f\colon T_xM \to T_yM$ such that $f_{*x}(R_x) = R_y$. A locally homogeneous space is curvature homogeneous, but there are many well-known examples of curvature homogeneous Riemannian manifolds which are not locally homogeneous. We may refer to [9] for further results and references concerning curvature homogeneous manifolds, especially in dimension three. If $\dim \bar{M} = 3$, then curvature homogeneity is equivalent to the constancy of the Ricci eigenvalues. In particular, a curvature homogeneous manifold $(\bar{M}, \bar{g})$ has constant scalar curvature $\bar{\tau}$. The constancy of the scalar curvature is itself a well-known curvature condition, which naturally appears in many fields of Riemannian Geometry.

Concerning g-natural Riemannian metrics on T_1M^2, we can prove the following

Theorem 1.2. *Let (M^2, g) be a Riemannian surface. The following properties are equivalent:*

(i) *(M^2, g) has constant Gaussian curvature,*

(ii) *T_1M^2 admits a g-natural Riemannian metric of constant scalar curvature,*

(iii) *T_1M^2 admits a curvature homogeneous g-natural Riemannian metric.*

Moreover, when one of the properties above is satisfied, then all *g-natural Riemannian metrics on T_1M^2 are curvature homogeneous.*

Remark 1.1. We explicitly note that Theorem 1.2 can be used to build many examples of three-dimensional curvature homogeneous Riemannian manifolds, as unit tangent sphere bundles over Riemannian surfaces of constant Gaussian curvature, equipped with a g-natural Riemannian metric.

The paper is organized in the following way. We shall first recall the definition and properties of g-natural metrics on TM and T_1M in Section 2. In Section 3, we shall prove our main results.

2. Riemannian g-natural metrics on TM and T_1M

Let (M, g) be a connected Riemannian manifold and ∇ its Levi-Civita connection. The Riemannian curvature R of g is taken with the sign convention

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,.$$

If we write $p_M : TM \to M$ for the natural projection and F for the natural bundle with $FM = p_M^*(T^* \otimes T^*)M \to M$, then $Ff(X_x, g_x) = (Tf \cdot X_x, (T^* \otimes T^*)f \cdot g_x)$ for all manifolds M, local diffeomorphisms f of M, $X_x \in T_xM$ and $g_x \in (T^* \otimes T^*)_xM$. The sections of the canonical projection $FM \to M$ are called *F-metrics* in literature. So, if we denote by $\oplus$ the fibered product of fibered manifolds, then the F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third argument.

For a given F-metric δ on M, there are three distinguished constructions of metrics on the tangent bundle TM [10]:

(a) If δ is symmetric, then the *Sasaki lift* δ^s of δ is defined by

$$\begin{cases} \delta^s_{(x,u)}(X^h,Y^h) = \delta(u;X,Y)\,, & \delta^s_{(x,u)}(X^h,Y^v) = 0, \\ \delta^s_{(x,u)}(X^v,Y^h) = 0\,, & \delta^s_{(x,u)}(X^v,Y^v) = \delta(u;X,Y)\,, \end{cases}$$

for all X, $Y \in M_x$. When δ is non degenerate and positive definite, so is δ^s.

(b) The *horizontal lift* δ^h of δ is a pseudo-Riemannian metric on TM, given by

$$\begin{cases} \delta^h_{(x,u)}(X^h,Y^h) = 0\,, & \delta^h_{(x,u)}(X^h,Y^v) = \delta(u;X,Y)\,, \\ \delta^h_{(x,u)}(X^v,Y^h) = \delta(u;X,Y)\,, & \delta^h_{(x,u)}(X^v,Y^v) = 0\,, \end{cases}$$

for all X, $Y \in M_x$. If δ is positive definite, then δ^s is of signature (m,m).

(c) The *vertical lift* δ^v of δ is a degenerate metric on TM, given by

$$\begin{cases} \delta^v_{(x,u)}(X^h,Y^h) = \delta(u;X,Y)\,, & \delta^v_{(x,u)}(X^h,Y^v) = 0\,, \\ \delta^v_{(x,u)}(X^v,Y^h) = 0\,, & \delta^v_{(x,u)}(X^v,Y^v) = 0\,, \end{cases}$$

for all X, $Y \in M_x$. The rank of δ^v is exactly that of δ.
If $\delta = g$ is a Riemannian metric on M, then these three lifts of δ coincide with the three well-known classical lifts of the metric g to TM.

The three lifts above of *natural* F-metrics generate the class of g-natural metrics on TM. These metrics were first introduced by Kowalski and Sekizawa in [10] (see also [4] for the definition of g-natural metrics and [8] for the general definition of naturality). On unit tangent sphere bundles, the restrictions of g-natural metrics possess a simpler form. Precisely, we have

Proposition 2.1 ([3])**.** *Let (M,g) be a Riemannian manifold. For every Riemannian metric $\tilde{G}$ on T_1M induced from a Riemannian g-natural metric G on TM, there exist four constants a, b, c and d, with $a>0$, $a(a+c)-b^2>0$ and $a(a+c+d)-b^2>0$, such that $\tilde{G} = a\cdot\widetilde{g^s} + b\cdot\widetilde{g^h} + c\cdot\widetilde{g^v} + d\cdot\widetilde{k^v}$, where*

* *k is the natural F-metric on M defined by*

  $$k(u;X,Y) = g(u,X)g(u,Y)\,, \quad \textit{for all} \quad (u,X,Y)\in TM\oplus TM\oplus TM\,,$$

* *$\widetilde{g^s}$, $\widetilde{g^h}$, $\widetilde{g^v}$ and $\widetilde{k^s}$ are the metrics on T_1M induced by g^s, g^h, g^v and k^v, respectively.*

It is worth mentioning that such a metric $\tilde{G}$ on T_1M is necessarily induced by a metric on TM of the form $G = a\cdot g^s + b\cdot g^h + c\cdot g^v + \beta\cdot k^v$, where a, b, c are constants and $\beta\colon [0,\infty)\to\mathbb{R}$ is a C^∞-function depending on the norm of $u\in TM$, such that

$$a>0\,,\ \alpha := a(a+c)-b^2>0\,, \ \text{ and } \ \phi(t) := a\big(a+c+t\beta(t)\big)-b^2>0\,, \tag{2.1}$$

for all $t\in[0,\infty)$ (see [3] for such a choice). Inequalities (2.1) express the fact that G is Riemannian (cf. [3]). We may refer to [4] for the formulae concerning the Levi-Civita connection and the curvature tensor of a g-natural Riemannian metric on TM of the form $G = a\cdot g^s + b\cdot g^h + c\cdot g^v + \beta\cdot k^v$.

Next, as it is well known, the *tangent sphere bundle of radius* $\rho > 0$ over a Riemannian manifold (M, g), is the hypersurface $T_\rho M = \{(x,u) \in TM | g_x(u,u) = \rho^2\}$. The tangent space of $T_\rho M$, at a point $(x,u) \in T_\rho M$, is given by

$$(T_\rho M)_{(x,u)} = \left\{X^h + Y^v / X \in M_x, Y \in \{u\}^\perp \subset M_x\right\}.$$

When $\rho = 1$, $T_1 M$ is called *the unit tangent (sphere) bundle.*

Let $G = a \cdot g^s + b \cdot g^h + c \cdot g^v + \beta \cdot k^v$ be a Riemannian g-natural metric on TM, that is, a g-natural metric satisfying (2.1), and $\tilde{G}$ the metric on $T_1 M$ induced by G. Note that $\tilde{G}$ only depends on the value $d := \beta(1)$ of β at 1 (see also [3]).

Using the Schmidt's orthonormalization process, a simple calculation shows that the vector field on TM defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}} \left[-b \cdot u^h + (a+c+d) \cdot u^v\right],$$

for all $(x,u) \in TM$, is normal to $T_1 M$ and unitary at any point of $T_1 M$. Here ϕ is, by definition, the quantity $\phi(1) = a(a+c+d) - b^2$.

Now, we define the "tangential lift" X^{t_G} – with respect to G – of a vector $X \in M_x$ to $(x,u) \in T_1 M$ as the tangential projection of the vertical lift of X to (x,u) – with respect to N^G –, that is,

$$\begin{aligned} X^{t_G} &= X^v - G_{(x,u)}\big(X^v, N^G_{(x,u)}\big)\, N^G_{(x,u)} \\ &= X^v - \sqrt{\frac{\phi}{a+c+d}}\, g_x(X,u)\, N^G_{(x,u)}. \end{aligned} \tag{2.2}$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

The tangent space $(T_1 M)_{(x,u)}$ of $T_1 M$ at (x,u) is spanned by vectors of the form X^h and Y^{t_G}, where X, $Y \in M_x$. Hence, the Riemannian metric $\tilde{G}$ on $T_1 M$, induced from G, is completely determined by the identities

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c) g_x(X,Y) + d g_x(X,u) g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= b g_x(X,Y), \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= a g_x(X,Y) - \frac{\phi}{a+c+d} g_x(X,u) g_x(Y,u), \end{cases} \tag{2.3}$$

for all $(x,u) \in T_1 M$ and X, $Y \in M_x$. It should be noted that, by (2.3), horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$ if and only if $b = 0$.

Convention 2.1. By (2.2) it follows that the tangential lift to $(x,u) \in T_1 M$ of the vector u is given by $u^{t_G} = \frac{b}{a+c+d}\, u^h$, that is, it is a horizontal vector. Therefore, the tangent space $(T_1 M)_{(x,u)}$ coincides with the set

$$\{X^h + Y^{t_G} / X \in M_x, Y \in \{u\}^\perp \subset M_x\}.$$

For this reason, the operation of tangential lift from M_x to a point $(x,u) \in T_1 M$ will be always applied only to vectors of M_x which are orthogonal to u.

The Levi-Civita connection $\tilde{\nabla}$ of $(T_1 M, \tilde{G})$ was calculated in [1]. The Riemannian curvature of $(T_1 M, \tilde{G})$ was determined in [2], were the authors proved the following result:

Proposition 2.2 ([2]). *Let (M,g) be a Riemannian manifold and let $G = a\cdot g^s + b\cdot g^h + c\cdot g^v + \beta\cdot k^v$, where a, b and c are constants and $\beta\colon [0,\infty)\to\mathbb{R}$ is a function satisfying* (2.1). *Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. If we denote by $\tilde{R}$ the Riemannian curvature tensor of $(T_1M,\tilde{G})$, then:*

(i) $$\tilde{R}(X^h,Y^h)Z^h = \Big\{R(X,Y)Z + \frac{ab}{2\alpha}\,[2(\nabla_u R)(X,Y)Z - (\nabla_Z R)(X,Y)u]$$
$$+\frac{a^2}{4\alpha}\,[R(R(Y,Z)u,u)X - R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z]$$
$$+\frac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z$$
$$+R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X]$$
$$+\frac{ad(\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z]$$
$$+\frac{ab^2}{2\alpha^2}\Big[-\frac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u) + d\,g(Y,u)g(Z,u)\Big]R_uX$$
$$-\frac{ab^2}{2\alpha^2}\Big[-\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\Big]R_uY$$
$$+\frac{d}{4\alpha}\Big[-\frac{2b^2}{a+c+d}\,g(R(Y,u)Z,u) + d\,g(Y,u)g(Z,u)\Big]X$$
$$-\frac{d}{4\alpha}\Big[-\frac{2b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\Big]Y$$
$$+\frac{d}{4\alpha(a+c+d)}\Big\{-4abg((\nabla_u R)(X,Y)Z,u) + a^2\,[g(R(Y,Z)u,R(X,u)u)$$
$$-g(R(X,Z)u,R(Y,u)u) - 2g(R(X,Y)u,R(Z,u)u)]$$
$$+\frac{a^2b^2}{\alpha}\,[g(R(Y,u)Z+R(Z,u)Y,R(X,u)u) - g(R(X,u)Z+R(Z,u)X,R(Y,u)u)]$$
$$-\Big[\frac{ad(b^2-\alpha)}{\alpha} + \frac{2b^2d(\phi+2b^2)}{\phi(a+c+d)} + \frac{4b^2\alpha}{\phi}\Big]\,[g(X,u)g(R(Y,u)Z,u)$$
$$-g(Y,u)g(R(X,u)Z,u)] - 3a(a+c)g(R(X,Y)Z,u) + (a+c)d[g(X,u)g(Y,Z)$$
$$-g(Y,u)g(X,Z)]\Big\}u\Big\}^h + \Big\{-\frac{b^2}{\alpha}\,(\nabla_u R)(X,Y)Z + \frac{a(a+c)}{2\alpha}(\nabla_Z R)(X,Y)u$$
$$-\frac{ab}{4\alpha}\,[R(R(Y,Z)u,u)X - R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z$$
$$-R(X,R(Y,u)Z)u - R(X,R(Z,u)Y)u + R(Y,R(X,u)Z)u + R(Y,R(Z,u)X)u]$$
$$-\frac{ab^3}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z$$

$$
\begin{aligned}
&- R(Y,u)R(Z,u)X] - \frac{ab^3}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z \\
&+ R(X,u)R(Z,u)Y + R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X] \\
&- \frac{bd(3\alpha - b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z] \\
&+ \frac{b(b^2-\alpha)}{2\alpha^2}\Big[\frac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u) - d\,g(Y,u)g(Z,u)\Big]R_uX \\
&- \frac{b(b^2-\alpha)}{2\alpha^2}\Big[\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) - d\,g(X,u)g(Z,u)\Big]R_uY \\
&+ \frac{(a+c)bd}{2\alpha(a+c+d)}\,[g(R(Y,u)Z,u)X - g(R(X,u)Z,u)Y]\Big\}^{t_G},
\end{aligned}
$$

(ii) $$
\begin{aligned}
\tilde{R}(X^h,Y^{t_G})Z^h = &\Big\{ -\frac{a^2}{2\alpha}\,(\nabla_X R)(Y,u)Z + \frac{ab}{2\alpha}[R(X,Y)Z + R(Z,Y)X] \\
&+ \frac{a^3b}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z - R(Y,u)R(Z,u)X] \\
&+ \frac{a^2bd}{4\alpha^2}\,[g(X,u)R(Y,u)Z - g(Z,u)R(X,Y)u] \\
&- \frac{ab}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\,g(R(Y,u)Z,u) + \alpha d\,g(Y,Z)]R_uX \\
&+ \frac{a^2b}{2\alpha^2}\,\left[\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) - d\,g(X,u)g(Z,u)\right]R_uY \\
&- \frac{bd}{4\alpha(a+c+d)}\,[a\,g(R(Y,u)Z,u) + (2(a+c)+d)\,g(Y,Z)]X \\
&+ \frac{b}{\alpha}\Big[-\frac{ad+b^2}{2(a+c+d)}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\Big]Y \\
&- \frac{bd}{2\alpha}\,g(X,Y)Z + \frac{d}{4\alpha(a+c+d)}\Big\{2a^2\,g((\nabla_X R)(Y,u)Z,u) \\
&+ \frac{a^3b}{\alpha}\,[g(R(Y,u)Z,R(X,u)u) - g(R(X,u)Z + R(Z,u)X,R(Y,u)u)] \\
&+ ab\Big[-\frac{\alpha+\phi}{\alpha} + \frac{d}{a+c+d}\Big]\,g(X,u)g(R(Y,u)Z,u) \\
&- 2ab\,[2g(R(X,Y)Z,u) + g(R(Z,Y)X,u)] \\
&+ bd\Big[\Big(3 - \frac{d}{a+c+d}\Big)\,g(X,u)g(Y,Z) + 2\,g(Z,u)g(X,Y)\Big]\Big\}u\Big\}^h \\
&+ \Big\{\frac{ab}{2\alpha}\,(\nabla_X R)(Y,u)Z + \frac{a^2}{4\alpha}\,R(X,R(Y,u)Z)u
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{a^2b^2}{4\alpha^2}\left[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z - R(Y,u)R(Z,u)X\right] \\
&- \frac{b^2}{\alpha} R(X,Y)Z + \frac{a(a+c)}{2\alpha} R(X,Z)Y \\
&+ \frac{ad(\alpha-b^2)}{4\alpha^2}\left[g(X,u)R(Y,u)Z - g(Z,u)R(X,Y)u\right] \\
&- \frac{\alpha-b^2}{4\alpha^2(a+c+d)}\left[a(ad+b^2)\, g(R(Y,u)Z,u) + \alpha d\, g(Y,Z)\right]R_uX \\
&+ \frac{ab^2}{2\alpha^2}\Big[-\frac{ad+b^2}{a+c+d}\, g(R(X,u)Z,u) + d\, g(X,u)g(Z,u)\Big]R_uY \\
&+ \frac{(a+c)d}{4\alpha(a+c+d)}\left[a\, g(R(Y,u)Z,u) + (2(a+c)+d)\, g(Y,Z)\right]X \\
&+ \frac{1}{4\alpha}\Big[2b^2\Big(2-\frac{d}{a+c+d}\Big)\, g(R(X,u)Z,u) \\
&- d(4(a+c)+d)\, g(X,u)g(Z,u)\Big]Y + \frac{(a+c)d}{2\alpha}\, g(X,Y)Z\Big\}^{t_G},
\end{aligned}
$$

(iii) $$\tilde{R}(X^{t_G},Y^{t_G})Z^{t_G} = \frac{1}{2\alpha(a+c+d)}\Big\{\big\{a^2b\left[g(Y,Z)R_uX - g(X,Z)R_uY\right]$$

$$- b(\alpha+\phi)[g(Y,Z)X - g(X,Z)Y]\big\}^h + \{-ab^2\left[g(Y,Z)R_uX - g(X,Z)R_uY\right]$$

$$+ [(a+c)(\alpha+\phi)+\alpha d]\left[g(Y,Z)X - g(X,Z)Y\right]\big\}^{t_G}\Big\},$$

for all $x \in M$, $(x,u) \in T_1M$ and all arbitrary vectors X, Y and $Z \in M_x$ satisfying Convention 2.1*, where $R_uX = R(X,u)u$ denotes the* Jacobi operator *associated to u.*

3. Proofs of the main results

Proof of Theorem 1.1. We shall first show that the case when dim$M \geq 3$ can not occur, and then we shall treat the case dim$M = 2$.

Step 1: Obstructions when M is not two-dimensional.
Let $(x,u) \in T_1M$. For any pair (W,Z) of linearly independent vectors tangent to T_1M at (x,u), we shall denote by $\tilde{K}_u(W,Z)$ the sectional curvature of the plane spanned by W and Z. Since dim$M \geq 3$, we can consider an orthonormal triplet $\{u,X,Y\}$ of vectors in M_x. Using (2.3) and Proposition 2.2, long but standard calculations yield

$$a\varphi\tilde{K}_u(u^h,X^{t_G}) = -\frac{a^2d}{2\alpha}K(X,u) + \frac{a^3}{4\alpha}\|R_uX\|^2 + d\Big(1+\frac{ad}{4\alpha}\Big), \tag{3.1}$$

$$\alpha\tilde{K}_u(X^h, X^{t_G}) = \Big[\frac{ad}{2\varphi} + \frac{a(a+c)b^2}{\alpha\varphi} - \frac{b^4}{2\alpha\varphi}\Big]K(X,u) - \frac{a^2(ad+b^2)}{4\alpha\varphi}K(X,u)^2 + \frac{a^3}{4\alpha}\|R_uX\|^2 - \frac{d(4\varphi-d)}{4\varphi}\,, \tag{3.2}$$

$$(a+c)^2\tilde{K}_u(X^h, Y^h) = (a+c)K(X,Y) + bg((\nabla_u R)(X,Y)Y, X) - \frac{3a}{4}||R(X,Y)u||^2 + \frac{ab^2}{4\alpha}\|R(X,u)Y + R(Y,u)X\|^2 + \frac{b^2(ad+b^2)}{\alpha\varphi}[K(X,u)K(Y,u) - g(R_uX,Y)^2]\,, \tag{3.3}$$

$$a(a+c)\tilde{K}_u(X^h, Y^{t_G}) = \frac{a^3}{4\alpha}\|R(Y,u)X\|^2 - \frac{a^2(ad+b^2)}{4\alpha\varphi}g(R_uX,Y)^2 + \frac{b^2(2\alpha+b^2)}{2\alpha\varphi}K(X,u)\,, \tag{3.4}$$

$$a^2\tilde{K}_u(X^{t_G}, Y^{t_G}) = \frac{\phi}{\varphi}\,, \tag{3.5}$$

where $R_uX = R(X,u)u$ and $K(X,u)$ is the sectional curvature of the plane of M_x spanned by X and u. Note that (3.1) and (3.2) also hold in the two-dimensional case.

Assume now that $(T_1M, \tilde{G})$ has constant sectional curvature $\tilde{K}$. By (3.5), we get

$$\tilde{K} = \frac{\phi}{a^2\varphi}\,. \tag{3.6}$$

Note that, since $\phi > 0$, (3.6) implies that $\tilde{K} \neq 0$.

We shall show that (M,g) has constant sectional curvature k, and we deduce that this case cannot occur, which will give the required obstruction for the non two-dimensional case of M.

In order to show that (M,g) has constant sectional curvature, we shall prove that on M the sectional curvature of all two-planes (at all points) has the same constant value. Using (3.6) into (3.1) and (3.2), we then have

$$\begin{cases} 0 = \dfrac{a^2}{4\alpha\varphi}\|R_uX\|^2 - \dfrac{ad}{2\alpha\varphi}K(X,u) + \dfrac{d}{a\varphi}\big(1 + \frac{ad}{4\alpha}\big) - \dfrac{\phi}{a^2\varphi}\,, \\ 0 = \dfrac{a^3}{4\alpha}\|R_uX\|^2 - \dfrac{a^2(ad+b^2)}{4\alpha\varphi}K(X,u)^2 \\ \quad + \Big[\dfrac{ad}{2\varphi} + \dfrac{b^2(2\alpha+b^2)}{2\alpha\varphi}\Big]K(X,u) - \dfrac{d(4\varphi-d)}{4\varphi} - \dfrac{\alpha\phi}{a^2\varphi}\,. \end{cases} \tag{3.7}$$

Multiplying the first equation of (3.7) by $a\varphi$, and comparing the two obtained equations, we get

$$0=\frac{a^2(ad+b^2)}{4\alpha\varphi}K(X,u)^2-\frac{(\alpha+\phi)(ad+b^2)+b^4}{2\alpha\varphi}K(X,u)$$
$$+2d+(ad+b^2)\Big[\frac{d^2}{4\alpha\varphi}-\frac{\phi}{a^2\varphi}\Big]. \tag{3.8}$$

We treat separately the cases $ad+b^2\neq 0$ and $ad+b^2=0$.

First case: $ad+b^2\neq 0$.
Sectional curvature K of (M,g) may be regarded as a real-valued C^∞-function, defined on the Grassmann manifold $G_2(M)$ of two-planes over M. M being connected, $G_2(M)$ itself is connected. Since $ad+b^2\neq 0$, (3.8) is a second order equation with constant coefficients and so, K can assume at most two distinct (constant) values, depending on a, b, c and d. Therefore, it is globally constant on $G_2(M)$.

Second case: $ad+b^2=0$.
Then, (3.8) reduces to

$$-\frac{b^4}{2\alpha\varphi}K(X,u)+2d=0\,,$$

or equivalently, since $b^2=-ad$,

$$-\frac{a^2d^2}{2\alpha\varphi}K(X,u)+2d=0\,. \tag{3.9}$$

If $d\neq 0$, then (3.9) implies at once that $K(X,u)$ is constant. In the remaining case $d=0$, from $ad+b^2=0$ it also follows $b=0$. Then, from (3.3) and (3.4), we respectively obtain

$$\tilde{K}=\frac{1}{a+c}K(X,Y)-\frac{3a}{4(a+c)^2}\|R(X,Y)u\|^2\,, \tag{3.10}$$

$$\tilde{K}=\frac{a}{(a+c)^2}\|R(Y,u)X\|^2\,, \tag{3.11}$$

for any orthonormal triplet $\{u,X,Y\}$ of tangent vectors at $x\in M$, and for all x. Because of (3.11), $\|R(Y,u)X\|^2$ takes the same constant value for any orthonormal triplet $\{u,X,Y\}$. Therefore, $||R(X,Y)u||^2$ is constant and so, by (3.10), $K(X,Y)$ is constant, that is, (M,g) has constant sectional curvature.

Finally, since (M,g) has constant sectional curvature,then $\|R_uX\|^2=k^2$ (and obviously, $R(U,V)W=0$ for any mutually orthogonal vectors U,V,W). Replacing into equations (3.1)–(3.4) and taking into account (3.6), we get an overdetermined system of algebraic equations for k, with no solutions, as we also checked by computer work. Hence, this case cannot occur.

Step 2: Two-dimensional case.
We now assume $\dim M=2$, and hence, T_1M^2 is three-dimensional. Let $(x,u)\in T_1M^2$. We first build a basis of vectors tangent to T_1M at (x,u). Let $(x,v)\in T_1M^2$ such that $\{u,v\}$ is an orthonormal basis of M^2_x. It is easy to show that $\{u^h,v^h,v^v\}$ forms a basis of vectors tangent to T_1M^2 at (x,u). We can compute the curvature $\tilde{R}$

both from Proposition 2.2 and using the fact that $(T_1M^2, \tilde{G})$ has constant sectional curvature $\tilde{K}$. For example, using Proposition 2.2, we easily get

$$\begin{aligned}\tilde{R}(u^h, v^h)v^h = &\Big\{ -\frac{ab}{2\alpha}u(\bar{c}) + \frac{3a^2}{4\alpha}\bar{c}^2 - \Big(1+\frac{ad}{2\alpha}\Big)\bar{c} - \frac{d^2}{4\alpha}\Big\}v^h \\ &+ \Big\{\frac{b^2-\alpha}{2\alpha}u(\bar{c}) - \frac{ab}{\alpha}\bar{c}^2 + \frac{bd}{\alpha}\bar{c}\Big\}v^{t_G}\,.\end{aligned} \tag{3.12}$$

On the other hand, since $(T_1M^2, \tilde{G})$ has constant sectional curvature $\tilde{K}$, we also have

$$\tilde{R}(u^h, v^h)v^h = \tilde{K}\{\tilde{G}(u^h, v^h)u^h - \tilde{G}(u^h, u^h)v^h\} = -\varphi\tilde{K}v^h\,. \tag{3.13}$$

Thus, comparing (3.12) and (3.13), we find

$$-\frac{ab}{2\alpha}u(\bar{c}) + \frac{3a^2}{4\alpha}\bar{c}^2 - \Big(1+\frac{ad}{2\alpha}\Big)\bar{c} - \frac{d^2}{4\alpha} = -\varphi\tilde{K} \quad\text{and}\quad \frac{b^2-\alpha}{2\alpha}u(\bar{c}) - \frac{ab}{\alpha}\bar{c}^2 + \frac{bd}{\alpha}\bar{c} = 0\,.$$

We proceed exactly in the same way by comparing other formulae for $\tilde{R}$ coming from Proposition 2.2 with the corresponding formulae expressing the fact that $(T_1M^2, \tilde{G})$ has constant sectional curvature $\tilde{K}$. Taking into account the facts that (x, u) is arbitrary and $\{u^h, v^h, v^{t_G}\}$ is a basis of vectors tangent to T_1M^2 at (x, u), we eventually obtain that $(T_1M^2, \tilde{G})$ has constant sectional curvature $\tilde{K}$ if and only if the following system is satisfied:

$$\begin{cases} -\dfrac{ab}{2\alpha}u(\bar{c}) + \dfrac{3a^2}{4\alpha}\bar{c}^2 - \Big(1+\dfrac{ad}{2\alpha}\Big)\bar{c} - \dfrac{d^2}{4\alpha} = -\varphi\tilde{K}\,, \\ \dfrac{b^2-\alpha}{2\alpha}u(\bar{c}) - \dfrac{ab}{\alpha}\bar{c}^2 + \dfrac{bd}{\alpha}\bar{c} = 0\,, \\ \dfrac{a}{2\varphi}v(\bar{c}) = 0\,, \\ \dfrac{a(b^2-3\alpha)}{4\alpha\varphi}\bar{c}^2 + \Big(1 - \dfrac{a(a+c)d}{2\alpha\varphi}\Big)\bar{c} + \dfrac{(a+c)d^2}{4\alpha\varphi} = (a+c)\tilde{K}\,, \\ -\dfrac{a^2}{2\alpha}u(\bar{c}) - \dfrac{ab}{\alpha}\bar{c} + \dfrac{bd}{\alpha} = 0\,, \\ \dfrac{ab}{2\alpha}u(\bar{c}) - \dfrac{a^2}{4\alpha}\bar{c}^2 + \dfrac{ad+2b^2}{2\alpha}\bar{c} - \dfrac{d[4(a+c)+d]}{4\alpha} = -\varphi\tilde{K}\,, \\ b\Big[\dfrac{a^2}{4\alpha\varphi}\bar{c}^2 + \dfrac{ad+b^2}{2\alpha\varphi}\bar{c} - \dfrac{d}{2\alpha}\Big(1+\dfrac{d}{2\varphi}\Big)\Big] = b\tilde{K}\,, \\ -\dfrac{a^2(a+c)}{4\alpha\varphi}\bar{c}^2 + \Big[\dfrac{d}{2\varphi}\Big(\dfrac{b^2}{\alpha}-1\Big) - \dfrac{b^2}{\alpha}\Big]\bar{c} + \dfrac{(a+c)d}{2\alpha}\Big(1+\dfrac{d}{2\varphi}\Big) = -(a+c)\tilde{K}\,, \end{cases} \tag{3.14}$$

for all $\{u, v\}$ orthonormal basis of M^2_x, $x \in M^2$. Since $a > 0$, the third equation in (3.14) implies at once $v(\bar{c}) = 0$. Therefore, $\bar{c}$ is constant and (3.14) easily reduces to

$$
\begin{cases}
3a^2\bar{c}^2 - 2(2\alpha + ad)\bar{c} - d^2 = -4\alpha\varphi\tilde{K}\,, \\
a(b^2 - 3\alpha)\bar{c}^2 + 2[2\alpha\varphi - a(a+c)d]\bar{c} + (a+c)d^2 = 4(a+c)\alpha\varphi\tilde{K}\,, \\
b(a\bar{c} - d) = 0\,, \\
a^2\bar{c}^2 - 2(ad + 2b^2)\bar{c} + d[4(a+c) + d] = 4\alpha\varphi\tilde{K}\,, \\
b[a^2\bar{c}^2 + 2(ad + b^2)\bar{c} - d(2\varphi + d)] = 4b\alpha\varphi\tilde{K}\,, \\
-a^2(a+c)\bar{c}^2 + 2[d(b^2 - \alpha) - 2b^2\varphi]\bar{c} + (a+c)d(2\varphi + d) \\
\quad = -4(a+c)\alpha\varphi\tilde{K}\,.
\end{cases} \tag{3.15}
$$

The fourth equation in (3.15) implies at once that either $b = 0$ or $\bar{c} = \dfrac{d}{a}$. We shall treat these two cases separately.

a) If $\bar{c} = \dfrac{d}{a}$, then from (3.15) it follows at once

$$
\begin{cases}
d = a\varphi\tilde{K}\,, \\
bd = 0\,.
\end{cases} \tag{3.16}
$$

Therefore, one of the following cases must occur:

- either $d = 0$, $\bar{c} = 0$ and $\tilde{K} = 0$, or
- $b = 0$, $\bar{c} = \dfrac{d}{a}$ and $\tilde{K} = \dfrac{d}{a\varphi}$.

b) If $b = 0$, then $\alpha = a(a+c)$ and (3.15) reduces to

$$
\begin{cases}
3a^2\bar{c}^2 - 2a[2(a+c) + d]\bar{c} - d^2 = -4a(a+c)\varphi\tilde{K}\,, \\
a^2\bar{c}^2 - 2ad\bar{c} + d[4(a+c) + d] = 4a(a+c)\varphi\tilde{K}\,, \\
a^2\bar{c}^2 + 2ad\bar{c} - d[4(a+c) + 3d] = 4a(a+c)\varphi\tilde{K}\,.
\end{cases} \tag{3.17}
$$

Summing the first two equations of (3.17), we find

$$a^2\bar{c}^2 - a(a+c+d)\bar{c} + d(a+c) = 0\,,$$

whose roots are $\bar{c} = \dfrac{d}{a}$ and $\bar{c} = \dfrac{a+c}{a}$. We already treated the case $\bar{c} = \dfrac{d}{a}$ for any value of b. Hence, it is enough to consider the case when $\bar{c} = \dfrac{a+c}{a}$. Replacing $\bar{c}$ by $\dfrac{d}{a}$ in (3.17), we easily obtain either $d = a+c$ or $d = -2(a+c)$. However, the latter can not occur, since it implies $\varphi = a+c+d = -(a+c) < 0$. Hence, $d = a+c$ and, again by (3.17), $\tilde{K} = \dfrac{\varphi}{4a(a+c)} = \dfrac{1}{2a}$. Summarizing, in this case we have

- $b = d - (a+c) = 0$, $\bar{c} = \dfrac{a+c}{a} > 0$ and $\tilde{K} = \dfrac{1}{2a}$,

and this completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Let $\tilde{G} = a \cdot \widetilde{g^s} + b \cdot \widetilde{g^h} + c \cdot \widetilde{g^v} + d \cdot \widetilde{k^v}$ be an arbitrary g-natural Riemannian metric $\tilde{G}$ on T_1M^2. We shall first build a smooth local moving frame $\{e_1, e_2, e_3\}$ on T_1M^2. Consider the vector field e_1 defined on T_1M^2 by $e_1(x,u) = \frac{1}{\sqrt{\varphi}} u^h_{(x,u)}$. Using the Schmidt orthonormalization process, we can choose, in a neighborhood $W := p^{-1}(V) \cap T_1M^2$ of any point of T_1M^2, a horizontal vector field e_2 defined on W, such that $\{e_1, e_2\}$ is $\tilde{G}$-orthonormal on W. Next, we define a vector field e_3 on W by

$$e_3(x,u) = \frac{1}{\sqrt{\alpha}} \big[-b[p_* e_2]^h_{(x,u)} + (a+c)[p_* e_2]^{t_G}_{(x,u)} \big] ,$$

for all $(x,u) \in W$. Hence, $\{e_1, e_2, e_3\}$ is a smooth moving frame on W, and we can now compute the components of the Ricci tensor $\widetilde{\mathrm{Ric}}$ with respect to it. In fact, by the definition of the Ricci tensor, we have

$$\widetilde{\mathrm{Ric}}(Z,W) = -\sum_{i=1}^{3} \tilde{G}(\tilde{R}(Z,e_i)W, e_i) ,$$

for any $(x,u) \in T_1M^2$ and Z, W tangent vectors to T_1M^2 at (x,u). Long but standard calculations lead to the following formulae:

$$\widetilde{\mathrm{Ric}}_{(x,u)}(e_1,e_1) = -\frac{a^2}{2\alpha\varphi}\bar{c}^2 + \frac{b^2-\alpha}{\alpha\varphi}\bar{c} + \frac{d[2(a+c)+d]}{2\alpha\varphi} , \tag{3.18}$$

$$\begin{aligned} \widetilde{\mathrm{Ric}}_{(x,u)}(e_2,e_2) = {} & \frac{b}{(a+c)\varphi}u(\bar{c}) + \frac{a(b^2-\alpha)}{2(a+c)\alpha\varphi}\bar{c}^2 \\ & + \Big[\frac{1}{a+c} + \frac{b^2(2\alpha+b^2)}{2\alpha^2\varphi}\Big]\bar{c} - \frac{d[2(a+c)+d]}{2\alpha\varphi} , \end{aligned} \tag{3.19}$$

$$\begin{aligned} \widetilde{\mathrm{Ric}}_{(x,u)}(e_3,e_3) = {} & -\frac{b}{(a+c)\varphi}u(\bar{c}) + \frac{a(\alpha-b^2)}{2(a+c)\alpha\varphi}\bar{c}^2 \\ & + \frac{b^2}{(a+c)\alpha}\Big[1 + \frac{(a+c)(2b^2-\alpha)(b^2+2\alpha)}{2\alpha^2\varphi}\Big]\bar{c} \\ & + \frac{d^2(b^2-\alpha)}{2\alpha^2\varphi} . \end{aligned} \tag{3.20}$$

$$\widetilde{\mathrm{Ric}}_{(x,u)}(e_1,e_2) = -\frac{ab}{2\alpha\sqrt{\varphi}}[p_* e_2](\bar{c}) , \tag{3.21}$$

$$\widetilde{\mathrm{Ric}}_{(x,u)}(e_1,e_3) = -\frac{a}{2\sqrt{\alpha\varphi}}[p_* e_2](\bar{c}) , \tag{3.22}$$

$$\widetilde{\mathrm{Ric}}_{(x,u)}(e_2,e_3) = \frac{1}{(a+c)\varphi\sqrt{\alpha}}\{\alpha\, u(\bar{c}) + ab\,\bar{c}^2 - bd\,\bar{c}\} . \tag{3.23}$$

From (3.18)–(3.20), we get at once the scalar curvature $\widetilde{\tau}$ of $(T_1M^2, \tilde{G})$:

$$(3.24)\quad \widetilde{\tau} = \sum_{i=1}^{3} \widetilde{\mathrm{Ric}}(e_i, e_i) = \frac{1}{2\alpha\varphi}\Big\{-a^2\bar{c}^2+2\Big[\alpha+\phi+\frac{b^4(2\alpha+b^2)}{\alpha^2}\Big]\bar{c}+\frac{d^2(b^2-\alpha)}{\alpha}\Big\}.$$

We now proceed to prove that (i)–(iii) are equivalent.

(i) ⇒(iii): If (M^2, g) has constant Gaussian curvature $\bar{c}$, then, by (3.18)–(3.23) we get that all components of the Ricci tensor, with respect to $\{e_1, e_2, e_3\}$, are constant. So, $(T_1M^2, \tilde{G})$ is curvature homogeneous.

(iii) ⇒(ii): It holds for any Riemannian manifold.

(ii) ⇒(i): Suppose $\tilde{G}$ is a g-natural Riemannian metric of constant scalar curvature $\widetilde{\tau}$ on T_1M^2. By equation (3.24), the Gaussian curvature $\bar{c}$ of (M^2, g) can only attain two constant real values, since all the coefficients in (3.24) are constant and $a > 0$. Being M^2 connected and $\bar{c}$ a continuous function defined on M^2, we can conclude that $\bar{c}$ is constant.

Finally, when one of conditions (i)–(iii) is satisfied, then the Gaussian curvature $\bar{c}$ is constant. So, by (3.18)–(3.23), we have that, for *any* g-natural Riemannian metric $\tilde{G}$ on T_1M^2, the components of the Ricci tensor, with respect to $\{e_1, e_2, e_3\}$, are constant. Hence, $(T_1M^2, \tilde{G})$ is curvature homogeneous, for all $\tilde{G}$. □

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