ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 233–241

SOME PROPERTIES OF TANGENT DIRAC STRUCTURES OF HIGHER ORDER

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Abstract. Let M be a smooth manifold. The tangent lift of Dirac structure on M was originally studied by T. Courant in [3]. The tangent lift of higher order of Dirac structure L on M has been studied in [10], where tangent Dirac structure of higher order are described locally. In this paper we give an intrinsic construction of tangent Dirac structure of higher order denoted by L^r and we study some properties of this Dirac structure. In particular, we study the Lie algebroid and the presymplectic foliation induced by L^r.

Introduction

Let M be a differential manifold of dimension $m > 0$, in this paper, we denote by $\langle \cdot,\cdot\rangle_M \colon TM \times_M T^*M \to \mathbb{R}$ the usual canonical pairing. In [2], is defined the natural symmetric and skew-symmetric pairings on $TM \oplus T^*M$ by:

$$\langle X\oplus\omega, Y\oplus\mu\rangle_+ = \frac{1}{2}\big(\omega(Y)+\mu(X)\big)$$

$$\langle X\oplus\omega, Y\oplus\mu\rangle_- = \frac{1}{2}\big(\omega(Y)-\mu(X)\big)\,.$$

An almost-Dirac structure, or a Dirac bundle, on a manifold M is a subbundle L of vector bundle $TM\oplus T^*M$ which is maximally isotropic under the symmetric pairing $\langle\cdot\mid\cdot\rangle_+$. We denote by ρ_M and ρ_M^* the natural projection of $TM\oplus T^*M$ onto TM and T^*M respectively. Clearly, $\rho_M(L)$ is a generalized distribution on M. We set

$$\rho_M(L)^* = \bigcup_{x\in M}\big(\rho_M(L_x)\big)^*\,.$$

In [2], is defined a 2-form $\Omega_L\colon \rho_M(L)\to\rho_M(L)^*$ such that:

$$\Omega_L\big(\rho_M(X,\omega)\big)\big(\rho_M(Y,\mu)\big) = \langle X\oplus\omega, Y\oplus\mu\rangle_- = \omega(Y)\,,$$

and the bilinear bracket operation on the sections of $(TM\oplus T^*M\to M)$ by:

$$[X\oplus\omega, Y\oplus\mu] = [X,Y]\oplus\big(\mathcal{L}_X\mu - \mathcal{L}_Y\omega + d(\langle X\oplus\omega, Y\oplus\mu\rangle_-)\big)\,.$$

2010 *Mathematics Subject Classification*: primary 53C15; secondary 53C75, 53D05.
Key words and phrases: Dirac structure, prolongations of vector fields, prolongations of differential forms, Dirac structure of higher order, natural transformations.
Received December 9, 2011, revised June 2012. Editor I. Kolář.
DOI: 10.5817/AM2012-3-233

If $\Gamma(L)$ is closed under this bracket, the author of [2] has said that the almost-Dirac structure L is integrable or L is a Dirac structure on M. This condition is equivalent to $\mathbb{T}_L = 0$, where $\mathbb{T}_L$ is the restriction on L of 3-tensor $\mathbb{T}$ defined on $TM \oplus T^*M$ by:

$$\mathbb{T}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) = \langle [\mathbf{s}_1, \mathbf{s}_2], \mathbf{s}_3 \rangle_+ .$$

Where $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \in \Gamma(TM \oplus T^*M)$.

Theorem 1. *An almost-Dirac structure L is integrable if and only if $(L, [\cdot,\cdot], \rho_M|_L)$ is a Lie algebroid.*

By this theorem, T. Courant in [2] has shown that, if L is an integrable Dirac structure, then the generalized distribution $\rho_M(L)$ generates a generalized foliation on M and by the same way, we have:

Theorem 2. *An integrable Dirac structure has a foliation by presymplectic leaves.*

For the proof of these theorems, see [2].

In [10], we have defined the tangent lift of higher order L^r $(r \geq 1)$ of an almost-Dirac structure L on a manifold M, and we have shown that this lifting is an almost-Dirac structure on a manifold T^rM. We have shown that, L is integrable if and only if L^r is integrable. In this paper we study some properties of L^r namely the structures of Lie algebroid and generalized foliation induced by L^r. The main results of this paper are Theorems 3, 4, 5, 6 and Proposition 3.

All manifolds and maps are assumed to be infinitely differentiable. r will be a natural integer $(r \geq 1)$.

1. Tangent lifts of higher order of some tensor fields revisited

1.1. **Prolongations of sections of vector bundle.** For all $\alpha \in \{0, \ldots, r\}$, we denote by $\chi^{(\alpha)} \colon T^r \to T^r$ the natural transformation defined for all vector bundle (E, M, π) and $\Psi \in C^\infty(\mathbb{R}, E)$ by:

$$\chi_E^{(\alpha)}(j_0^r \Psi) = j_0^r(t^\alpha \Psi) .$$

Where $t^\alpha \Psi$ is the smooth map defined for all $t \in \mathbb{R}$ by: $(t^\alpha \Psi)(t) = t^\alpha \Psi(t)$.

Let $S \colon M \to E$ be a smooth section on E, we define the section $\overline{S}^{(\alpha)}$ of $(T^rE, T^rM, T^r\pi)$ by:

$$\overline{S}^{(\alpha)} = \chi_E^{(\alpha)} \circ T^rS , \qquad 0 \leq \alpha \leq r .$$

For the sake convenience we define $\overline{S}^{(\alpha)} = 0$ for all $\alpha > r$ or $\alpha < 0$.

Definition 1. This section $\overline{S}^{(\alpha)}$ of T^rE is called α-prolongation of order r of S.

Remark 1. Let (E, M, π) be a vector bundle and $\varphi \colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ a local trivialization of E over an open $U \subset M$. For $j = 1, \ldots, n$, we put:

$\varepsilon_j(x) = \varphi^{-1}(x, e_j)$ where $x \in U$ and $(e_j)_{j=1,\ldots,n}$ is the usual basis of $\mathbb{R}^n$.

$(\varepsilon_j)_{j=1,\dots,n}$ is a basis of sections of E over U associated to φ. Using the identification $T^r(U \times \mathbb{R}^n) = T^rU \times \mathbb{R}^{n(r+1)}$, we define a family of sections

$$(\varepsilon_j^\alpha), \qquad 1 \le j \le n, \quad 0 \le \alpha \le r$$

of T^rE over T^rU by:

$$\varepsilon_j^\alpha(\widetilde{x}) = T^r\varphi^{-1}(\widetilde{x}, e_j^\alpha)$$

where $\widetilde{x} \in T^rU$ and (e_j^α) the usual basis of $T^r\mathbb{R}^n = \mathbb{R}^{n(r+1)}$.
We have:

$$\varepsilon_j^\alpha = \overline{\varepsilon_j}^{(\alpha)}, \qquad \text{for all} \quad j = 1, \dots, n \quad \text{and} \quad \alpha = 0, \dots, r. \tag{1}$$

Proposition 1. *Let (E, M, π) be a vector bundle. If Ψ, Ψ' are two tensor fields of type $(0,p)$ on the vector bundle $(T^rE, T^rM, T^r\pi)$ such that for all smooth sections $S_1, \dots, S_p$ on E and $\alpha_1, \dots, \alpha_p \in \{0, 1, \dots, r\}$ the equality*

$$\Psi\big(\overline{S_1}^{(\alpha_1)}, \dots, \overline{S_p}^{(\alpha_p)}\big) = \Psi'\big(\overline{S_1}^{(\alpha_1)}, \dots, \overline{S_p}^{(\alpha_p)}\big)$$

holds, then $\Psi = \Psi'$.

Proof. See [5]. $\square$

For the prolongations of functions, vector fields and differential form of manifold M to manifold T^rM and related properties, see [5] or [11]. From now, we adopt the notations of [11].

1.2. **Prolongations of tensor fields of type (0, p).** Let (E, M, π) be a vector bundle and φ a tensor field of type $(0,p)$ on E. We interpret a tensor φ on E as a p-linear mapping $\varphi\colon E \times_M \cdots \times_M E \to \mathbb{R}$ of the bundle product over M of p-copies of E. For all $\alpha \in \{0, 1, \dots, r\}$, we denote by τ_α the linear form on $J_0^r(\mathbb{R}, \mathbb{R})$ defined by:

$$\tau_\alpha(j_0^r g) = \frac{1}{\alpha!}\frac{d^\alpha}{dt^\alpha}\big(g(t)\big)|_{t=0}\,.$$

We set:

$$\overline{\varphi}^{(\alpha)} = \tau_\alpha \circ T^r\varphi\,; \tag{2}$$

$\overline{\varphi}^{(\alpha)}$ is a tensor field of type $(0,p)$ on $(T^rE, T^rM, T^r\pi)$ called α-prolongation of φ from E to T^rE. When $\alpha = r$, it is denoted by $\overline{\varphi}^{(c)}$ called complete lift of φ from E to T^rE.

Proposition 2. *$\overline{\varphi}^{(\alpha)}$, $0 \le \alpha \le r$, is the only tensor field of type $(0,p)$ on T^rE satisfying:*

$$\overline{\varphi}^{(\alpha)}\big(\overline{S_1}^{(\alpha_1)}, \dots, \overline{S_p}^{(\alpha_p)}\big) = \big(\varphi(S_1, \dots, S_p)\big)^{(\alpha - \sum\limits_{i=1}^{p}\alpha_i)}\,. \tag{3}$$

for all $S_1, \dots, S_p \in \Gamma(E)$ and $\alpha_1, \dots, \alpha_p \in \{0, 1, \dots, r\}$,

Proof. Let $j_0^r\eta \in T^rM$, we have:

$$\begin{aligned}
\overline{\varphi}^{(\alpha)}\big(\overline{S_1}^{(\alpha_1)},\dots,\overline{S_p}^{(\alpha_p)}\big)(j_0^r\eta) &= \overline{\varphi}^{(\alpha)}\big(\chi_E^{(\alpha_1)}\circ T^rS_1(j_0^r\eta),\dots,\chi_E^{(\alpha_p)}\circ T^rS_p(j_0^r\eta)\big)\\
&= \overline{\varphi}^{(\alpha)}\big(j_0^r(t^{\alpha_1}S_1\circ\eta),\dots,j_0^r(t^{\alpha_p}S_p\circ\eta)\big)\\
&= \tau_\alpha\big(j_0^r\varphi(t^{\alpha_1}S_1\circ\eta,\dots,t^{\alpha_p}S_p\circ\eta)\big)\\
&= \tau_\alpha\big(j_0^r t^{\alpha_1+\dots+\alpha_p}\varphi(S_1,\dots,S_p)\circ\eta\big)\\
&= \big(t^{\alpha_1+\dots+\alpha_p}\varphi(S_1,\dots,S_p)\big)^{(\alpha)}(j_0^r\eta)\\
&= \big(\varphi(S_1,\dots,S_p)\big)^{(\alpha-\sum\limits_{i=1}^{p}\alpha_i)}(j_0^r\eta)
\end{aligned}$$

The unicity comes from the equation (1) and Proposition 1. □

2. Tangent Dirac structure of higher order

2.1. **Almost-Dirac structure of higher order.** We denote by $\alpha^r\colon T^*\circ T^r\to T^r\circ T^*$ and $\kappa^r\colon T^r\circ T\to T\circ T^r$ the natural transformations defined in [1] and [5], such that, for all manifold M, we have:

$$\langle\kappa_M^r(u),v^*\rangle_{T^rM}=\langle u,\alpha_M^r(v^*)\rangle'_{T^rM}\,,\qquad (u,v^*)\in T^rTM\oplus T^*T^rM\,,$$

where $\langle\cdot\mid\cdot\rangle'_{T^rM}=\tau_r\circ T^r\langle\cdot\mid\cdot\rangle_M$. Let L be an almost-Dirac structure on m-dimensional manifold defined locally by the bundle morphisms $a\colon U\times\mathbb{R}^m\to TM$ and $b\colon U\times\mathbb{R}^m\to T^*M$. (e_j) denote the canonical basis of $\mathbb{R}^m$. We set:

$$S_i\colon U\to L\,,\quad x\mapsto a(x,e_i)\oplus b(x,e_i)\,,$$

$(S_i)_{1\le i\le n}$ is a basis of sections of L over U. In [10], we have showed that: the almost Dirac structure of order r L^r is determined by the maps a^r and b^r such that:

$$a^r=\kappa_M^r\circ T^ra\quad\text{and}\quad b^r=\varepsilon_M^r\circ T^rb\,;$$

where ε_M^r is the inverse map of α_M^r. The matrix form of a^r and b^r is given by:

$$a^r=\begin{pmatrix} a_j^i & \dots & 0\\ \vdots & \ddots & \vdots\\ (a_j^i)^{(r)} & \dots & a_j^i\end{pmatrix}\qquad\text{and}\qquad b^r=\begin{pmatrix} (b_{ij})^{(r)} & \dots & b_{ij}\\ \vdots & \iddots & \vdots\\ b_{ij} & \dots & 0\end{pmatrix}$$

So that,

$$L^r=(\kappa_M^r\oplus\varepsilon_M^r)(T^rL)\subset TT^rM\oplus T^*T^rM\,.$$

Theorem 3. *Let $X\oplus\omega\in\Gamma(L)$, for all $\alpha\in\{0,...,r\}$, we have $X^{(\alpha)}\oplus\omega^{(r-\alpha)}\in\Gamma(L^r)$.*

Proof. If $(X,\omega)\in\Gamma(L)$ then, they are the maps $\gamma_1,\dots,\gamma_m\in C^\infty(U)$ such that:

$$X\oplus\omega=\sum_{i=1}^{m}\gamma^iS_i.$$

In this case,

$$\begin{cases} X|_U = \gamma^i a_i^j \frac{\partial}{\partial x^j} \\ \omega|_U = \gamma^i b_{ij} dx^j \end{cases}$$

$$X^{(\alpha)} = (\gamma^i)^{(\nu)} (a_i^j)^{(\beta-\alpha-\nu)} \frac{\partial}{\partial x_\beta^j} .$$

We deduce that:

$$X^{(\alpha)} = \begin{pmatrix} a_i^j & 0 & \dots & 0 \\ \dot{a}_i^j & a_i^j & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ (a_i^j)^{(r)} & (a_i^j)^{(r-1)} & \dots & a_i^j \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \gamma^i \\ \vdots \\ (\gamma^i)^{(r-\alpha)} \end{pmatrix}$$

$$\omega^{(r-\alpha)} = (\gamma^i b_{ij})^{(r-\alpha-\beta)} dx_\beta^j = (\gamma^i)^{(r-\nu)} (b_{ij})^{(\nu-\alpha-\beta)} dx_\beta^j .$$

In the same way, we have:

$$\omega^{(r-\alpha)} = \begin{pmatrix} (b_{ij})^{(r)} & (b_{ij})^{(r-1)} & \dots & b_{ij} \\ (b_{ij})^{(r-1)} & (b_{ij})^{(r-2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{ij} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \gamma^i \\ \vdots \\ (\gamma^i)^{(r-\alpha)} \end{pmatrix} .$$

Thus that $(X^{(\alpha)}, \omega^{(r-\alpha)}) \in \Gamma(L^r)$. □

For all $X \oplus \omega \in \Gamma(TM \oplus T^*M) = \mathfrak{X}(M) \oplus \Omega^1(M)$, we set:

$$(X \oplus \omega)^{(\alpha)} = X^{(\alpha)} \oplus \omega^{(r-\alpha)} .$$

Corollary 1. *Let L be an almost-Dirac structure on M.*

(1) *For all $X \oplus \omega, Y \oplus \mu \in \mathfrak{X}(M) \oplus \Omega^1(M)$ and $\alpha, \beta = 0, \dots, r$, we have:*

$$\left[(X \oplus \omega)^{(\alpha)}, (Y \oplus \mu)^{(\beta)}\right] = [X \oplus \omega, Y \oplus \mu]^{(\alpha+\beta)} .$$

(2) *For all $f \in C^\infty(M)$ and $X \oplus \omega \in \mathfrak{X}(M) \oplus \Omega^1(M)$, we have:*

$$\big(f \cdot (X \oplus \omega)\big)^{(\alpha)} = \sum_{\beta=0}^{r-\alpha} f^{(\beta)} \cdot (X \oplus \omega)^{(\alpha+\beta)} .$$

(3) *For all $X \oplus \omega, Y \oplus \mu, Z \oplus \nu \in \Gamma(L)$, we have:*

$$\mathbb{T}_{L^r}\big((X \oplus \omega)^{(\alpha)}, (Y \oplus \mu)^{(\beta)}, (Z \oplus \nu)^{(\gamma)}\big) = \big(\mathbb{T}_L(X \oplus \omega, Y \oplus \mu, Z \oplus \nu)\big)^{(r-\alpha-\beta-\gamma)} ,$$

for all $\alpha, \beta, \gamma \in \{0, 1, \dots, r\}$.

Proof. The proof comes of some properties of tangent lift of higher order of functions, vector fields and differential forms.

□

For all $S \in \Gamma(L)$ and $\alpha \in \{0, 1, \ldots, r\}$, we have:

$$(\kappa_M^r \oplus \varepsilon_M^r)(\overline{S}^{(\alpha)}) = S^{(\alpha)} .$$

Theorem 4. *$\overline{\mathbb{T}_L}^{(c)}$ is a complete lift of $\mathbb{T}_L$ from L to $T^r L$. We denote by η_M^r the inverse map of κ_M^r. We have:*

$$\mathbb{T}_{L^r} = \overline{\mathbb{T}_L}^{(c)} \circ \Big(\bigoplus^3 (\eta_M^r \oplus \alpha_M^r) \Big) . \tag{4}$$

Proof. $\mathbb{T}_{L^r} \circ \big(\bigoplus^3 (\kappa_M^r \oplus \varepsilon_M^r) \big)$ is a tensor field of type $(0,3)$ on $T^r L$. Let $S_1, S_2, S_3 \in \Gamma(L)$ and $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \ldots, r\}$, we have:

$$\begin{aligned}
\mathbb{T}_{L^r} \circ \bigoplus^3 (\kappa_M^r \oplus \varepsilon_M^r) \big(\overline{S_1}^{(\alpha_1)}, \overline{S_2}^{(\alpha_2)}, \overline{S_3}^{(\alpha_3)} \big) &= \mathbb{T}_{L^r} \big(S_1^{(\alpha_1)}, S_2^{(\alpha_2)}, S_3^{(\alpha_3)} \big) \\
&= \big(\mathbb{T}_L (S_1, S_2, S_3) \big)^{(r - \alpha_1 - \alpha_2 - \alpha_3)} \\
&= \overline{\mathbb{T}_L}^{(c)} \big(\overline{S_1}^{(\alpha_1)}, \overline{S_2}^{(\alpha_2)}, \overline{S_3}^{(\alpha_3)} \big) .
\end{aligned}$$

We have the result by the Proposition 2. □

Remark 2. The equation (4) shows that L is integrable if and only if L^r is integrable. Thus, we have given an intrinsic construction of tangent lift of higher order of an almost-Dirac structure, and we have shown independent of any local coordinates system that: this lifting is integrable if and only if the initial almost-Dirac structure is integrable.

Let $X \oplus \omega$, $Y \oplus \mu$ be sections of an almost-Dirac structure L. Define

$$X \bullet (Y \oplus \mu) = [X, Y] \oplus \mathcal{L}_X \mu .$$

Definition 2. L is said invariance under $X \oplus \omega \in \Gamma(L)$ if and only if $X \bullet L \subset L$. When L is integrable this is equivalent to say $d\omega | \rho_M(L) = 0$.

Corollary 2. *If L is an integrable Dirac structure invariant under $X \in \rho_M(\Gamma(L))$, then L^r is invariant under $X^{(\alpha)}$ for all $\alpha = 0, \ldots, r$*

Proof. Let $X \oplus \omega \in \Gamma(L)$, we have $X^{(\alpha)} \oplus \omega^{(r-\alpha)} \in \Gamma(L^r)$ by the equality

$$d\omega^{(r-\alpha)} = (d\omega)^{(r-\alpha)} \quad \text{(see [11])},$$

we deduce that $d\omega^{(r-\alpha)} | \rho(L^r) = 0$. □

2.2. **Admissible functions of L^r.** Let L be an integrable Dirac structure over M. A function f is an admissible relatively to L, if there is vector field X_f such that $(X_f, df) \in \Gamma(L)$. If f and g are two admissible functions, T. Courant defines in [2] their bracket by:

$$\{f, g\} = X_f(g) .$$

Proposition 3. (1) *If f is an admissible function relatively to L, then $f^{(\alpha)}$ is an admissible function relatively to L^r and we have:*

$$X_{f^{(\alpha)}} = (X_f)^{(r-\alpha)} . \tag{5}$$

(2) *For all f, g two admissible functions, $\alpha, \beta = 0, \ldots, r$, we have:*

$$\{f^{(\alpha)}, g^{(\beta)}\} = \{f, g\}^{(\alpha+\beta-r)} . \tag{6}$$

Proof. (1) If f is an admissible function, then $(X_f, df) \in \Gamma(L)$. For all α,

$$\left((X_f)^{(r-\alpha)}, (df)^{(\alpha)}\right) \in \Gamma(L^r) .$$

Since $(df)^{(\alpha)} = df^{(\alpha)}$, it follows that $((X_f)^{(r-\alpha)}, df^{(\alpha)}) \in \Gamma(L^r)$. Thus, $f^{(\alpha)}$ is an admissible function relatively to L^r and $X_{f^{(\alpha)}} = (X_f)^{(r-\alpha)}$.

(2) For $\alpha, \beta = 0, \ldots, r$, we have:

$$\begin{aligned} \{f^{(\alpha)}, g^{(\beta)}\} &= X_{f^{(\alpha)}}(g^{(\beta)}) \\ &= (X_f)^{(r-\alpha)}(g^{(\beta)}) \\ &= \{f, g\}^{(\alpha+\beta-r)} \end{aligned}$$

□

2.3. **The Lie algebroid $(L^r, [\cdot,\cdot], \rho_{T^rM}|_{L^r})$.** For all $\alpha \in \{0, 1, \ldots, r\}$, consider the map

$$\chi^{(\alpha)}_{TM\oplus T^*M} \colon T^r(TM \oplus T^*M) \to T^r(TM \oplus T^*M) ,$$

we have:

$$\chi^{(\alpha)}_{TM\oplus T^*M} = \chi^{(\alpha)}_{TM} \oplus \chi^{(\alpha)}_{T^*M} .$$

In this case, $\chi^{(\alpha)}_L = \chi^{(\alpha)}_{TM} \oplus \chi^{(\alpha)}_{T^*M}|_{T^rL}$.

Proposition 4. *Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid. There is one and only one Lie algebroid structure on T^rE such that: For all $S_1, S_2 \in \Gamma(E)$ and $\alpha, \beta \in \{0, 1, ..., r\}$*

$$\left[\overline{S_1}^{(\alpha)}, \overline{S_2}^{(\beta)}\right] = \overline{[S_1, S_2]}^{(\alpha+\beta)} .$$

The anchor map $\rho^{(r)}$ is given by:

$$\rho^{(r)} = \kappa^r_M \circ T^r\rho .$$

This Lie algebroid structure is called tangent lift of order r of Lie algebroid $(E, [\cdot,\cdot], \rho)$.

Proof. See [9]. □

Theorem 5. *Let L be an integrable Dirac structure on M. The tangent Lie algebroid of order r T^rL, is isomorphic to the Lie algebroid $(L^r, [\cdot], \rho_{T^rM}|_{L^r})$ over T^rM induced by the integrable Dirac structure L^r.*

Proof. Let (S_i) be a basis of sections of L over U.

$$S_i(x) = a(x, e_i) \oplus b(x, e_i) , \quad \forall i = 1, \ldots, m$$

we have $\kappa_M^r \oplus \varepsilon_M^r(\overline{S_i}^{(\alpha)}) = S_i^{(\alpha)}$.
The tangent Lie algebroid of order r T^rL is given by:

$$[\overline{S_i}^{(\alpha)}, \overline{S_j}^{(\beta)}] = \overline{[S_i, S_j]}^{(\alpha+\beta)}$$

$$\begin{aligned}[\kappa_M^r \oplus \varepsilon_M^r(\overline{S_i}^{(\alpha)}), \kappa_M^r \oplus \varepsilon_M^r(\overline{S_j}^{(\beta)})] &= [S_i^{(\alpha)}, S_j^{(\beta)}] = [S_i, S_j]^{(\alpha+\beta)} \\ &= \kappa_M^r \oplus \varepsilon_M^r\left(\overline{[S_i, S_j]}^{(\alpha+\beta)}\right).\end{aligned}$$

It follows that,

$$\kappa_M^r \oplus \varepsilon_M^r|_{T^rL} \colon T^rL \to L^r$$

is a Lie algebroids isomorphism. □

2.4. **Symplectic foliation induced by L^r.** For the tangent lift of higher order of singular foliation of manifold M to T^rM we can see [9]. However, let E be a smooth generalized distribution on M, we denote by $\mathfrak{X}_E$ the set of all local vector fields such that: for all $x \in M$, $X(x) \in E_x$. Let us notice that for a completely integrable distribution E, the family $\mathfrak{X}_E$ is a Lie subalgebra of the Lie algebra of vector fields on M.

Proposition 5. *Let E be a completely integrable generalized distribution on M. Then the distribution E^r generated by the family $\{X^{(\alpha)},\ X \in \mathfrak{X}_E,\ 0 \leq \alpha \leq r\}$ of vector fields on T^rM is completely integrable.*

Proof. See [9]. □

Let $\mathcal{F}$ be a generalized foliation defined by E, the tangent lift of order r of $\mathcal{F}$ denoted by $T^r\mathcal{F}$ is defined by E^r.

Proposition 6. *If a submanifold $F \subset M$ is a leaf of generalized foliation $\mathcal{F}$, then T^rF is a leaf of generalized foliation $T^r\mathcal{F}$.*

Proof. See [9]. □

By the Propositions 5 and 6, we deduce this result.

Theorem 6. *Let L be an integrable Dirac structure, $\mathcal{F}$ the generalized foliation induced by L and F a leaf of $\mathcal{F}$.*

(1) *The generalized foliation induced by L^r is the tangent lift of order r of generalized foliation $\mathcal{F}$.*
(2) *If Ω_F is a presymplectic form on F then $\Omega_F^{(c)}$ is a presymplectic form on the leaf T^rF. Where $\Omega_F^{(c)}$ is a complete lift of differential form Ω_F.*

Proof. Let $X, Y \in \rho_M(\Gamma(L))$ tangent to F, we have:

$$\begin{aligned}\Omega_{T^rF}(X^{(\alpha)}, Y^{(\beta)}) &= \omega^{(r-\alpha)}(Y^{(\beta)}) \\ &= \big(\omega(Y)\big)^{(r-\alpha-\beta)} \\ &= \big(\Omega_F(X, Y)\big)^{(r-\alpha-\beta)} \\ &= \Omega_F^{(c)}(X^{(\alpha)}, Y^{(\beta)})\end{aligned}$$

Thus $\Omega_{T^rF} = \Omega_F^{(c)}$. □

These results generalize the properties of tangent lifting of higher order of Poisson manifold.

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