ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 207–231

A CHARACTERIZATION OF ISOMETRIES BETWEEN RIEMANNIAN MANIFOLDS BY USING DEVELOPMENT ALONG GEODESIC TRIANGLES

Petri Kokkonen

Abstract. In this paper we characterize the existence of Riemannian covering maps from a complete simply connected Riemannian manifold (M,g) onto a complete Riemannian manifold $(\hat{M},\hat{g})$ in terms of developing geodesic triangles of M onto $\hat{M}$. More precisely, we show that if $A_0\colon T|_{x_0}M \to T|_{\hat{x}_0}\hat{M}$ is some isometric map between the tangent spaces and if for any two geodesic triangles γ, ω of M based at x_0 the development through A_0 of the composite path $\gamma\cdot\omega$ onto $\hat{M}$ results in a closed path based at $\hat{x}_0$, then there exists a Riemannian covering map $f\colon M\to\hat{M}$ whose differential at x_0 is precisely A_0. The converse of this result is also true.

1. Introduction

Consider two Riemannian manifolds (M,g) and $(\hat{M},\hat{g})$ of the same dimension an suppose that one is given an isometry A_0 between given tangent spaces $T|_{x_0}M$ and $T|_{\hat{x}_0}\hat{M}$ of M and $\hat{M}$, respectively. Given a piecewise smooth path $\gamma\colon[0,1]\to M$ starting from x_0, one *develops* this curve onto the tangent space $T|_{x_0}M$ to obtain a curve $\Gamma\colon[0,1]\to T|_{x_0}M$ such that $\Gamma(t)=\int_0^t P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s$ where $P_s^0(\gamma)$ is the parallel transport on M along γ from $\gamma(s)$ to $\gamma(0)=x_0$. Consider the curve $\hat{\Gamma}:=A_0\circ\Gamma$ on $T|_{\hat{x}_0}\hat{M}$ and let $\hat{\gamma}\colon[0,1]\to\hat{M}$ be the unique curve (if it exists) on $\hat{M}$, called the *anti-development of* $\hat{\Gamma}$, starting at $\hat{x}_0$ such that $\hat{\Gamma}(t)=\int_0^t P_s^0(\hat{\gamma})\dot{\hat{\gamma}}(s)\mathrm{d}s$ where $P_s^0(\hat{\gamma})$ is a parallel transport on $\hat{M}$ along $\hat{\gamma}$ from $\hat{\gamma}(s)$ to $\hat{\gamma}(0)=\hat{x}_0$. We say that $\hat{\gamma}$ is the development of γ onto $\hat{M}$ through A_0.

It happens, as it is easy to verify, that if (M,g) and $(\hat{M},\hat{g})$ are isometric through an isomorphism $f\colon M\to\hat{M}$ whose differential at x_0 is A_0, that $\hat{\gamma}=f\circ\gamma$. Thus, in particular, if γ is a loop based at x_0, then $\hat{\gamma}$ will be a loop based at $\hat{x}_0$.

2010 *Mathematics Subject Classification*: primary 53B05; secondary 53C05, 53B21.

Key words and phrases: Cartan-Ambrose-Hicks theorem, development, linear and affine connections, rolling of manifolds.

Received July 18, 2012. Editor J. Slovák.

DOI: 10.5817/AM2012-3-207

This paper addresses the converse of this result: For a given A_0 as above, suppose that for every loop γ based at x_0 its development $\hat{\gamma}$ onto $\hat{M}$ through A_0 is a loop (necessarily based at $\hat{x}_0$), then does there exist an isomorphism $f\colon M \to \hat{M}$ whose differential at x_0 is A_0 ? Under the technical assumptions that (M, g) and $(\hat{M}, \hat{g})$ are complete and simply connected, we are able to answer affirmatively to this question. Indeed, instead of an arbitrary piecewise smooth loop γ based at x_0, it is enough to consider loops γ that are composites of two geodesic triangles based at x_0. Also, the assumptions of simply connectedness can be relaxed; see the main Theorem 3.1 and its Corollary 3.3.

This result is related to, and was originally inspired by, the so-called rolling model of Riemannian manifolds (cf. [1, 4, 5, 7, 6, 11, 13]). Consider two complete, oriented and simply-connected Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of the same dimension and suppose A_0 is an oriented isometry from $T|_{x_0}M$ onto $T|_{\hat{x}_0}\hat{M}$, called an initial *relative orientation* of M and $\hat{M}$ at the initial *contact points* x_0 and $\hat{x}_0$. Let $\gamma\colon [0, 1] \to M$ be a piecewise smooth path on M such that $\gamma(0) = x_0$. Put M and $\hat{M}$ in contact at the points x_0 and $\hat{x}_0$, respectively, (here it might be useful to think of M and $\hat{M}$ as submanifolds of some $\mathbb{R}^N$ and $g, \hat{g}$ being the metrics induced by the Euclidean metric of $\mathbb{R}^N$) and identify the tangent spaces at these points by using A_0. Then let M *roll* against $\hat{M}$ along γ so that the motion contains no instantaneous *spinning* nor *slipping*. The set of contact points on $\hat{M}$ that are produced by this rolling motion form a piecewise smooth curve $\hat{\gamma}(t)$ i.e. at instant $t \in [0, 1]$ the contact point $\hat{\gamma}(t)$ of $\hat{M}$ corresponds to that of $\gamma(t)$ of M. In fact, the model explicitly tells that $P_t^0(\hat{\gamma})\dot{\hat{\gamma}}(t) = A_0 P_t^0(\gamma)\dot{\gamma}(t)$ for all $t \in [0, 1]$ i.e. $\hat{\gamma}$ is nothing more than the development of γ on $\hat{M}$ through A_0 as defined just above. Therefore, to detect if M and $\hat{M}$ are isomorphic, through some isomorphism $f\colon M \to \hat{M}$ with $f_*|_{x_0} = A_0$, it is enough to make M roll against $\hat{M}$ along loops of M based at x_0, identifying initially $T|_{x_0}M$ to $T|_{\hat{x}_0}\hat{M}$ through A_0, and to observe whether or not the paths so traced on $\hat{M}$ by the rolling motion are loops based at $\hat{x}_0$. Indeed, as mentioned above, it is even enough to consider the rolling along loops γ that are composites of two geodesic triangles based at x_0. This is a way of interpreting the main result, Theorem 3.1, of this paper in terms of a mechanical model and "physical experiments".

The outline of the paper is as follows. Section 2 introduces basic concepts, notations and results. The next Section 3 contains the statement of the main Theorem 3.1 of the paper along with its immediate corollaries. The proof of the main theorem is found in Section 4. Actually, there we first prove a technical result (Proposition 4.1) in a more general context of affine manifolds and use it to prove the main theorem. Section 5 relates Theorem 3.1 to the well known Cartan-Ambrose-Hicks theorem ([2, 3, 10, 12]). We give in Theorem 5.2 a total of 8 different characterizations for the existence of a Riemannian covering map between two Riemannian manifolds, one of which is the Cartan-Ambrose-Hicks theorem and one is the main theorem of the paper. Finally, Section 6 contains an application to the main theorem related to the affine holonomy group of a Riemannian manifold ([8]).

2. Notations and basic results

All the manifolds that appear are assumed to be smooth, second countable and Hausdorff (cf. [9, 12]). If $M, \hat{M}$ are manifolds and $x \in M$, $\hat{x} \in \hat{M}$, we write $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ for the set of all $\mathbb{R}$-linear maps $T|_x M \to T|_{\hat{x}}\hat{M}$. We define $T^*M \otimes T\hat{M} := \bigcup_{(x,\hat{x})\in M\times\hat{M}} T|_x M \otimes T|_{\hat{x}}\hat{M}$ for the set of all linear maps between different tangent spaces.

If M is a manifold and $x \in M$, write $\Omega_x(M)$ for the set of all piecewise smooth loops $\gamma\colon [0,1] \to M$ based at x i.e. $\gamma(0) = \gamma(1) = x$. If $\gamma\colon [a,b] \to M$ and $\omega\colon [c,d] \to M$ are paths such that $\gamma(b) = \omega(c)$ we define the *composite path* as

$$\omega \sqcup \gamma\colon [a, b+d-c] \to M\,; \quad \omega \sqcup \gamma(t) = \begin{cases} \gamma(t)\,, & \text{if } \ t \in [a,b] \\ \omega(t-b+c)\,, & \text{if } \ t \in [b, b+d-c]\,. \end{cases}$$

If $a = c = 0$ and $b = d = 1$, i.e. $\gamma, \omega : [0,1] \to M$, then one defines the composite path $\omega \cdot \gamma$ as

$$\omega \cdot \gamma\colon [0,1] \to M\,; \quad \omega \cdot \gamma := \begin{cases} \gamma(2t)\,, & t \in [0, \frac{1}{2}] \\ \omega(2t-1)\,, & t \in [\frac{1}{2}, 1]\,, \end{cases}$$

i.e. $\omega\cdot\gamma = (t \mapsto \omega(2t))|_{[0,1/2]} \sqcup (t \mapsto \gamma(2t))|_{[0,1/2]}$. The inverse path $\gamma^{-1}\colon [a,b] \to M$ of $\gamma\colon [a,b] \to M$ is defined as $\gamma^{-1}(t) = \gamma(b+a-t)$. Observe that if $\gamma\colon [a,b] \to M$, $\omega\colon [c,d] \to M$ and $\Gamma\colon [A,B] \to M$ are three path such that $\gamma(b) = \omega(c)$ and $\omega(d) = \Gamma(A)$, then $(\Gamma \sqcup \omega) \sqcup \gamma = \Gamma \sqcup (\omega \sqcup \gamma)$. However, if γ, ω, $\Gamma\colon [0,1] \to M$ and $\gamma(1) = \omega(0)$, $\omega(1) = \Gamma(0)$, then $\Gamma \cdot (\omega \cdot \gamma) \neq (\Gamma \cdot \omega) \cdot \gamma$. This lack of associativity for '$\cdot$'-operation will not be a handicap for us, as will be explained in Remark 2.9 below, and usually we prefer working with "normalized" paths whose domain of definition is $[0,1]$.

A manifold M equipped with a linear connection ∇ is called an *affine manifold* (M, ∇). If $\gamma\colon [a,b] \to M$ is a piecewise smooth path and (M,∇) is an affine manifold, we write $(P^\nabla)^t_s(\gamma)$, where $t, s \in [a,b]$, for the ∇-parallel transport from $\gamma(s)$ to $\gamma(t)$. Since the connection to be used is usually clear from the context, we write simply $P^t_s(\gamma)$ for $(P^\nabla)^t_s(\gamma)$. The exponential map of (M,∇) at x is written as $\exp^\nabla_x$ and (M,∇) is said to be *geodesically accesible* from $x \in M$ if $\exp^\nabla_x$ is surjective onto M. If $\exp^\nabla_x$ is defined on the whole tangent space $T|_x M$ for all $x \in M$, then (M,∇) is said to be *geodesically complete*. The curvature (resp. torsion) tensor on (M,∇) is denoted by R^∇ (resp. T^∇). If $(\hat{M}, \hat{\nabla})$ is another affine manifold, then a smooth map $f\colon M \to \hat{M}$ is called *affine*, if for any piecewise smooth path $\gamma\colon [a,b] \to M$, one has

$$f_*|_{\gamma(b)} \circ (P^\nabla)^b_a(\gamma) = (P^{\hat{\nabla}})^b_a(f \circ \gamma) \circ f_*|_{\gamma(a)}\,.$$

A manifold M equipped with a positive definite (i.e. Riemannian) metric g is called a Riemannian manifold (M, g). If (M, g) and $(\hat{M}, \hat{g})$ are Riemannian manifolds and if $A \in T^*|_x M \otimes T|_{\hat{x}} \hat{M}$ is such that $\hat{g}(AX, AY) = g(X, Y)$ for all $X, Y \in T|_x M$, we say that A is an *infinitesimal isometry*.

Definition 2.1. Let (M, ∇) be an affine manifold and $k \geq 1$.

(i) A path $\gamma\colon [a, b] \to M$ is called a *k-times broken geodesic*, if there are geodesics $\gamma_0, \dots, \gamma_k$ such that γ_i ends where γ_{i+1} starts from and $\gamma = \gamma_k \sqcup \gamma_{k-1} \sqcup \cdots \sqcup \gamma_1 \sqcup \gamma_0$.

We use $\angle_x(M, \nabla)$ to denote the set of 1-times broken geodesics defined on $[0, 1]$ and starting from $x \in M$.

(ii) A loop $\gamma \in \Omega_x(M)$ based at x is said to be a *geodesic k-polygon* based at x if it is a $(k-1)$-times broken geodesic.

Geodesic 3-polygons (resp. 4-polygons) based at x are called geodesic triangles (resp. quadrilaterals) based x and they constitute a set denoted by $\triangle_x(M, \nabla)$ (resp. $\square_x(M, \nabla)$). We also define

$$\triangle^2_{x_0}(M, \nabla) := \{\gamma \cdot \omega \mid \gamma, \omega \in \triangle_{x_0}(M, \nabla)\}.$$

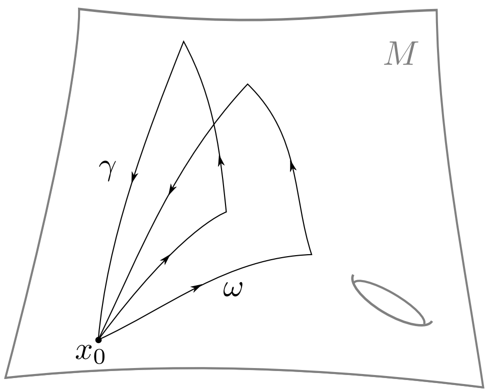

FIG. 1: A typical element $\gamma \cdot \omega$ of $\triangle^2_{x_0}(M, \nabla)$.

Remark 2.2. Notice that a path $\gamma\colon [a, b] \to M$ is a k-times broken geodesic if and only if there is a partition $\{t_0, \dots, t_{k+1}\}$ of $[a, b]$ such that $\gamma|_{[t_i, t_{i+1}]}$ is a ∇-geodesic for $i = 0, \dots, k$.

Definition 2.3. Let (M, ∇) and $(\hat{M}, \hat{\nabla})$ be affine manifolds and let $A \in T^*|_x M \otimes T|_{\hat{x}} \hat{M}$. We define $\mathcal{T}^{(\nabla, \hat{\nabla})}_A\colon \bigwedge^2 T|_x M \to T|_{\hat{x}} \hat{M}$ and $\mathcal{R}^{(\nabla, \hat{\nabla})}_A\colon \bigwedge^2 T|_x M \to T^*|_x M \otimes T|_{\hat{x}} \hat{M}$ called the *relative torsion* and *relative curvature* of (M, ∇), $(\hat{M}, \hat{\nabla})$ at A by

$$\begin{aligned}
\mathcal{T}^{(\nabla, \hat{\nabla})}_A(X, Y) :=& AT^{\nabla}(X, Y) - T^{\hat{\nabla}}(AX, AY) \\
\mathcal{R}^{(\nabla, \hat{\nabla})}_A(X, Y)Z :=& A(R^{\nabla}(X, Y)Z) - R^{\hat{\nabla}}(AX, AY)AZ\,,
\end{aligned}$$

where $X, Y, Z \in T|_x M$. We will often write simply $\mathcal{T}_A$ and $\mathcal{R}_A$ for $\mathcal{T}_A^{(\nabla,\hat{\nabla})}$ and $\mathcal{R}_A^{(\nabla,\hat{\nabla})}$, respectively, when $\nabla, \hat{\nabla}$ are clear from the context.

Definition 2.4. Let (M, ∇) be an affine manifold. For a piecewise smooth $\gamma\colon [a,b] \to M$ we define a piecewise smooth $\Lambda^{\nabla}_{\gamma(a)}(\gamma)\colon [a,b] \to T|_{\gamma(a)} M$ by

$$\Lambda^{\nabla}_{\gamma(a)}(\gamma)(t) = \int_a^t (P^{\nabla})^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s\,, \quad t \in [a,b]\,.$$

We call $\Lambda^{\nabla}_{\gamma(a)}(\gamma)$ the *development of* γ on $T|_{\gamma(a)} M$ with respect to the connection ∇.

In the Riemannian setting, one can characterize the completeness in terms of the development map.

Theorem 2.5 ([8, Theorem IV.4.1])**.** *A Riemannian manifold (M, g), with Levi-Civita connection ∇, is complete if and only if for every $x \in M$ and every piecewise smooth curve $\Gamma\colon [a,b] \to T|_x M$, $\Gamma(a) = 0$, there exists a (necessarily unique) piecewise smooth curve $\gamma\colon [a,b] \to M$ such that $\gamma(a) = x$ and $\Lambda^{\nabla}_x(\gamma) = \Gamma$.*

Definition 2.6. Given (M, ∇), $(\hat{M}, \hat{\nabla})$, $A_0 \in T^*|_{x_0} M \otimes T|_{\hat{x}_0} \hat{M}$ and a piecewise smooth $\gamma\colon [a,b] \to M$ such that $\gamma(a) = x_0$.

(i) We define

$$\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t) := (\Lambda^{\hat{\nabla}}_{\hat{x}_0})^{-1}(A_0 \circ \Lambda^{\nabla}_{x_0}(\gamma))(t)\,,$$

for all $t \in [a,b]$ where defined. If $c \in [a,b]$ is such that $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(c)$ exists, we call $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)|_{[a,c]}$ the *development* of γ onto $\hat{M}$ through A_0 with respect to $(\nabla, \hat{\nabla})$. We will usually write simply $\Lambda_{A_0}(\gamma)$ for $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)$ when there is no risk of confusion.

(ii) If $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t)$ is defined, we define the *relative parallel transport of* A_0 along γ to be the linear map

$$(\mathcal{P}^{(\nabla,\hat{\nabla})})^t_a(\gamma)A_0 : T|_{\gamma(t)} M \to T|_{\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t)} \hat{M}\,;$$

$$(\mathcal{P}^{(\nabla,\hat{\nabla})})^t_a(\gamma)A_0 := (P^{\hat{\nabla}})^t_a\big(\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)\big) \circ A_0 \circ (P^{\nabla})^a_t(\gamma)\,.$$

As before, one writes briefly $\mathcal{P}^t_a(\gamma)A_0$ for $(\mathcal{P}^{(\nabla,\hat{\nabla})})^t_a(\gamma)A_0$ when the connections in question are evident.

Remark 2.7. It is evident that $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t)$ exists for all $t > a$ near a and if $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t)$ exists for some $t > a$ then $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t')$ exists for all $t' \in [a,t]$ as well. By Theorem 2.5, if $(\hat{M}, \hat{g})$ is a complete Riemannian manifold with Levi-Civita connection $\hat{\nabla}$, the development $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t)$ is defined for every $t \in [a,b]$.

We record a lemma whose easy proof we omit.

Lemma 2.8. *Let* (M, ∇) *and* $(\hat{M}, \hat{\nabla})$ *be affine manifold,* $A_0 \in T^*|_{x_0} M \otimes T|_{\hat{x}_0} \hat{M}$ *and* $\gamma\colon [a,b] \to M$ *a piecewise smooth path with* $\gamma(a) = x_0$*.*

(i) *If* $\omega\colon [c,d] \to M$ *is a piecewise smooth path such that* $\gamma(b) = \omega(c)$*, then*

$$\Lambda_{A_0}(\omega \sqcup \gamma) = \Lambda_{\mathcal{P}_a^b(\gamma)A_0}(\omega) \sqcup \Lambda_{A_0}(\gamma)$$
$$\mathcal{P}_a^t(\omega \sqcup \gamma)A_0 = \begin{cases} \mathcal{P}_a^t(\gamma)A_0\,, & \text{if } \ t \in [a,b] \\ \mathcal{P}_c^{t-b+c}(\omega)\mathcal{P}_a^b(\gamma)A_0\,, & \text{if } \ t \in [b, b+d-c]\,. \end{cases}$$

Moreover,

$$\Lambda_{A_0}(\gamma^{-1} \sqcup \gamma)(2b-a) = x_0\,, \quad \mathcal{P}_a^{2b-a}(\gamma^{-1} \sqcup \gamma)A_0 = A_0\,.$$

(ii) *If* $\gamma\colon [a,b] \to M$ *is a k-times broken geodesic on* (M,∇) *and if* $\Lambda_{A_0}(\gamma)(t)$ *exists for all* $t \in [a,b]$ *then* $\Lambda_{A_0}(\gamma)$ *is a k-times broken geodesic on* $(\hat{M}, \hat{\nabla})$*. In particular, if* $\gamma_u(t) := \exp^{\nabla}_{x_0}(tu)$*,* $\hat{\gamma}_{A_0 u}(t) := \exp^{\hat{\nabla}}_{\hat{x}_0}(tA_0 u)$*, then* $\Lambda_{A_0}(\gamma_u) = \hat{\gamma}_{A_0 u}$*.*

(iii) *Let* $\hat{\gamma}\colon [a,b] \to \hat{M}$ *be a piecewise smooth curve such that* $\hat{\gamma}(a) = \hat{x}_0$*. Then* $\hat{\gamma} = \Lambda_{A_0}(\gamma)$ *if and only if*

$$P_t^a(\hat{\gamma})\dot{\hat{\gamma}}(t) = A_0 P_t^a(\gamma)\dot{\gamma}(t)\,, \qquad t \in [a,b]\,.$$

(iv) *If* $X(\cdot)$ *is a vector field along* $\gamma\colon [a,b] \to M$*, then for all* $t \in [a,b]$ *such that* $\hat{\gamma}(t) := \Lambda_{A_0}(\gamma)(t)$ *is defined, one has*

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\big((\mathcal{P}_a^t(\gamma)A_0)X(t)\big) = \big(\mathcal{P}_a^t(\gamma)A_0\big)\nabla_{\dot{\gamma}(t)}X(t)\,.$$

(v) *Suppose* $\phi\colon [\alpha,\beta] \to [a,b]$ *is smooth and* $\dot{\phi}(t) \neq 0$ $\forall t \in [\alpha,\beta]$*. Then the following hold for all* $t \in [\alpha,\beta]$ *such that left or right hand side is defined:*

$$\Lambda_{A_0}(\gamma)\big(\phi(t)\big) = \Lambda_{A_0}(\gamma \circ \phi)(t)$$
$$\mathcal{P}_a^{\phi(t)}(\gamma)A_0 = \mathcal{P}_\alpha^t(\gamma \circ \phi)A_0\,.$$

Remark 2.9. Suppose γ, ω, $\Gamma\colon [0,1] \to M$ are such that $\gamma(1) = \omega(0)$, $\omega(1) = \Gamma(0)$. As remarked earlier, $\Gamma \cdot (\omega \cdot \gamma) \neq (\Gamma \cdot \omega) \cdot \gamma$. Lemma 2.8, however, implies that

$$\Lambda_{A_0}\big((\Gamma \cdot \omega) \cdot \gamma\big)(1) = \Lambda_{A_0}\big(\Gamma \cdot (\omega \cdot \gamma)\big)(1)$$
$$\mathcal{P}_0^1\big((\Gamma \cdot \omega) \cdot \gamma\big)A_0 = \mathcal{P}_0^1\big(\Gamma \cdot (\omega \cdot \gamma)\big)A_0\,.$$

Indeed, the latter (which implies the former) follows by computing

$$\begin{aligned}\mathcal{P}_0^1\big((\Gamma \cdot \omega) \cdot \gamma\big)A_0 &= \mathcal{P}_0^1(\Gamma \cdot \omega)\mathcal{P}_0^1(\gamma)A_0 = \mathcal{P}_0^1(\Gamma)\mathcal{P}_0^1(\omega)\mathcal{P}_0^1(\gamma)A_0 \\ &= \mathcal{P}_0^1(\Gamma)\mathcal{P}_0^1(\omega \cdot \gamma)A_0 = \mathcal{P}_0^1(\Gamma \cdot (\omega \cdot \gamma))A_0\,.\end{aligned}$$

We recall next the Cartan-Ambrose-Hicks theorem (C-A-H Theorem for short) in the context of Riemannian manifolds of equal dimension.

Theorem 2.10 ((C-A-H)[2, 3, 10, 12])**.** *Let* (M,g) *and* $(\hat{M},\hat{g})$ *be complete Riemannian manifolds of the same dimension,* $\dim M = \dim \hat{M}$*, and let* $A_0 \in T^*|_{x_0} M \otimes T|_{\hat{x}_0} \hat{M}$ *be an infinitesimal isometry. Then there exists a complete Riemannian manifold* (N,h)*,* $z_0 \in N$ *and Riemannian covering maps*

$F\colon (N,h)\to(M,g)$, $G\colon (N,h)\to(\hat{M},\hat{g})$ such that $G_*|_{z_0}=A_0\circ F_*|_{z_0}$ if and only if

$$\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma)A_0}=0\,,\quad \forall\,\gamma\in\angle_{x_0}(M,\nabla)\,, \tag{1}$$

where ∇, $\hat{\nabla}$ are the Levi-Civita connections of (M,g) and $(\hat{M},\hat{g})$.

3. Main result

We begin this section with the statement of the main theorem of the paper. The result will then be followed by two corollaries and some remarks. The proof of the theorem is postponed to Section 4.

Theorem 3.1. *Suppose (M,g), $(\hat{M},\hat{g})$ are complete Riemannian manifolds of the same dimension, $\dim M=\dim\hat{M}$, M simply connected and let $A_0\in T^*|_{x_0}M\otimes T|_{\hat{x}_0}\hat{M}$ be an infinitesimal isometry. Then there exists a Riemannian covering map $f\colon M\to\hat{M}$ with $f_*|_{x_0}=A_0$ if and only if*

$$\Lambda^{(\nabla,\hat{\nabla})}_{A_0}\big(\triangle^2_{x_0}(M,\nabla)\big)\subset\Omega_{\hat{x}_0}(\hat{M})\,, \tag{2}$$

where $\nabla,\hat{\nabla}$ are the Levi-Civita connections of (M,g), $(\hat{M},\hat{g})$, respectively.

Remark 3.2. Notice that by Lemma 2.8(ii) the condition (2) is equivalent to

$$\Lambda^{(\nabla,\hat{\nabla})}_{A_0}\big(\triangle^2_{x_0}(M,\nabla)\big)\subset\triangle^2_{\hat{x}_0}(\hat{M},\hat{\nabla})$$

and that it is implied by the condition

$$\Lambda^{(\nabla,\hat{\nabla})}_{A_0}\big(\Omega_{x_0}(M)\big)\subset\Omega_{\hat{x}_0}(\hat{M})\,. \tag{3}$$

If one wishes not to assume M to be simply connected in Theorem 3.1, then the result can be modified as follows:

Corollary 3.3. Suppose (M,g), $(\hat{M},\hat{g})$ are complete Riemannian manifolds of the same dimension, $\dim M=\dim\hat{M}$ and let $A_0\in T^*|_{x_0}M\otimes T|_{\hat{x}_0}\hat{M}$ be and infinitesimal isometry. The condition (2) holds if and only if there exists a complete simply connected Riemannian manifold (N,h), Riemannian covering maps $F\colon N\to M$, $G\colon N\to\hat{M}$ and a $z_0\in N$ such that $G_*|_{z_0}=A_0\circ F_*|_{z_0}$ and

$$\{\Gamma(1)\mid\Gamma\colon[0,1]\to N\text{ continuous}\,,\ \Gamma(0)=z_0,\ F\circ\Gamma\in\triangle^2_{x_0}(M,\nabla)\}\subset G^{-1}(\hat{x}_0)\,, \tag{4}$$

Proof. *Sufficiency.* Let (N,h) and the maps F,G be given as stated and suppose (4) is true. For a $\gamma\in\triangle^2_{x_0}(M,\nabla)$, let Γ be the unique path in N such that $\gamma=F\circ\Gamma$ and $\Gamma(0)=z_0$. It follows that $G\circ\Gamma=\Lambda_{A_0}(\gamma)$ and since $\Gamma(1)\in G^{-1}(\hat{x}_0)$, we have $\Lambda_{A_0}(\gamma)(1)=G(\Gamma(1))=\hat{x}_0$ i.e. (2) holds.

Necessity. Let $F\colon N\to M$ be the universal covering of M and lift the metric g onto N, which we denote by h. As is well known, (N,h) is complete. Fix a point $z_0\in F^{-1}(x_0)$ and write D for the Levi-Civita connection of (N,h). Let $B_0:=A_0\circ F_*|_{z_0}\in T^*|_{z_0}N\otimes T|_{\hat{x}_0}\hat{M}$ which is an infinitesimal isometry and notice that if $\Gamma\colon[0,1]\to N$ is a piecewise smooth path starting from z_0, then

$\Lambda_{B_0}(\Gamma) = \Lambda_{A_0}(F \circ \Gamma)$. In particular, if $\Gamma \in \triangle^2_{z_0}(N, D)$, then $F \circ \Gamma \in \triangle^2_{x_0}(M, \nabla)$ and hence $\Lambda_{B_0}(\Gamma) \in \Omega_{\hat{x}_0}(\hat{M})$ by assumption (2).

Thus Theorem 3.1 implies the existence of a Riemannian covering map $G\colon N \to \hat{M}$ such that $G_*|_{z_0} = B_0 = A_0 \circ F_*|_{z_0}$. To prove (4), let $\Gamma\colon [0,1] \to N$ be such that $\Gamma(0) = z_0$ and $F \circ \Gamma \in \triangle^2_{x_0}(M, \nabla)$. Then $G \circ \Gamma = \Lambda_{B_0}(\Gamma) = \Lambda_{A_0}(F \circ \Gamma) \in \Omega_{\hat{x}_0}(\hat{M})$ so in particular, $G(\Gamma(1)) = \Lambda_{A_0}(F \circ \Gamma)(1) = \hat{x}_0$ i.e. $\Gamma(1) \in G^{-1}(\hat{x}_0)$. The proof is complete. $\square$

Remark 3.4. If the the previous corollary one replaces the condition (2) by (3), then (4) can be replaced by the condition $F^{-1}(x_0) \subset G^{-1}(\hat{x}_0)$. This is clear from the proof of the corollary.

The above theorem has an easy corollary.

Corollary 3.5. Let (M, g), $(\hat{M}, \hat{g})$ be complete Riemannian manifolds of the same dimension, $\dim M = \dim \hat{M}$. Given an infinitesimal isometry $A_0 \in T^*|_{x_0} M \otimes T|_{\hat{x}_0} \hat{M}$ and $x_1 \in M$, $\hat{x}_1 \in \hat{M}$, then there exists a Riemannian covering map $f\colon M \to \hat{M}$ with $f_*|_{x_0} = A_0$ and $f(x_1) = \hat{x}_1$ if and only if

$$\forall \text{ 6-broken geodesic } \gamma\colon [0,1] \to M \text{ s.t. } \gamma(0) = x_0\,,\ \gamma(1) = x_1 \\ \Longrightarrow \Lambda^{(\nabla, \hat{\nabla})}_{A_0}(\gamma)(1) = \hat{x}_1\,.$$

Proof. *Necessity.* Suppose we are given a Riemannian covering map $f\colon M \to \hat{M}$ with $f_*|_{x_0} = A_0$ and $f(x_1) = \hat{x}_1$. Then if γ is a 6-broken geodesic with $\gamma(0) = x_0$, $\gamma(1) = x_1$, it follows that $\Lambda_{A_0}(\gamma) = f \circ \gamma$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(x_1) = \hat{x}_1$.

Sufficiency. Let $\Gamma\colon [0,1] \to M$ be any geodesic from x_0 to x_1 (such a geodesic exists since (M, g) is complete) and define $A_1 := \mathcal{P}^1_0(\Gamma) A_0$. Taking any γ, $\omega \in \triangle_{x_1}(M, \nabla)$, we see that (see Lemma 2.8)

$$\Lambda_{A_1}(\gamma \cdot \omega)(1) = \Lambda_{\mathcal{P}^1_0(\Gamma) A_0}(\gamma \cdot \omega)(1) = \Lambda_{A_0}((\gamma \cdot \omega) \cdot \Gamma)(1) = \hat{x}_1\,,$$

where the last equality follows from the fact that $(\gamma \cdot \omega) \cdot \Gamma$ is a 6-broken geodesic that starts from x_0 and ends to x_1. Thus $\Lambda_{A_1}(\triangle^2_{x_1}(M, \nabla)) \subset \Omega_{\hat{x}_1}(\hat{M})$ and Theorem 3.1 implies that there is a covering map $f\colon M \to \hat{M}$ such that $f_*|_{x_1} = A_1$. In particular, $f(x_1) = \hat{x}_1$. Moreover,

$$f_*|_{\Gamma^{-1}(t)} = \mathcal{P}^t_0(\Gamma^{-1}) A_1 = \mathcal{P}^{1-t}_0 A_0\,,$$

which implies that $f_*|_{x_0} = f_*|_{\Gamma^{-1}(1)} = \mathcal{P}^0_0 A_0 = A_0$. The proof is finished. $\square$

Remark 3.6. The condition of the corollary means that 6-times broken geodesics of M with end points x_0 and x_1 map by the development $\Lambda^{(\nabla, \hat{\nabla})}_{A_0}$ to 6-times broken geodesics of $\hat{M}$ with end points $\hat{x}_0$ and $\hat{x}_1$.

Also observe that this condition is implied by the following stronger one:

$$\forall\ \gamma\colon [0,1] \to M\,,\ \text{piecewise smooth s.t. } \gamma(0) = x_0\,,\ \gamma(1) = x_1 \\ \Longrightarrow \Lambda^{(\nabla, \hat{\nabla})}_{A_0}(\gamma)(1) = \hat{x}_1\,.$$

4. Proof of the main result

The proof of Theorem 3.1 (see p.223) makes use of the following key proposition which we state and prove in a more general setting of affine manifolds.

Proposition 4.1. *Suppose that (M,∇), $(\hat{M},\hat{\nabla})$ are affine manifolds (possibly of different dimensions) and let $A_0\in T^*|_{x_0}M\otimes T|_{\hat{x}_0}\hat{M}$ be given. Let $U\subset T|_{x_0}M$, $\hat{U}\subset T|_{\hat{x}_0}\hat{M}$ be the domains of definitions of $\exp^{\nabla}_{x_0}$, $\exp^{\hat{\nabla}}_{\hat{x}_0}$, respectively, and write $\gamma_u(t)=\exp^{\nabla}_{x_0}(tu)$, $\hat{\gamma}_{\hat{u}}(t)=\exp^{\hat{\nabla}}_{\hat{x}_0}(t\hat{u})$ for $u\in U$, $\hat{u}\in\hat{U}$. Then if*

$$\Lambda^{(\nabla,\hat{\nabla})}_{A_0}\big(\triangle_{x_0}(M,\nabla)\big)\subset\triangle_{\hat{x}_0}(\hat{M},\hat{\nabla})\tag{5}$$

$$\mathcal{T}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma_u)A_0}(\dot{\gamma}_u(1),\cdot)=0\,,\quad\forall u\in U\cap A_0^{-1}(\hat{U})\,,\tag{6}$$

hold, one has for all $u\in U\cap A_0^{-1}(\hat{U})$ that

$$(P^{\hat{\nabla}})^0_1(\hat{\gamma}_{A_0u})\circ(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}\circ A_0=A_0\circ(P^{\nabla})^0_1(\gamma_u)\circ(\exp^{\nabla}_{x_0})_*|_u\tag{7}$$

$$\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma_u)A_0}(\dot{\gamma}_u(1),\cdot)\dot{\gamma}_u(1)=0\,.\tag{8}$$

Remark 4.2. Since $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)$ might not be defined on whole interval $[0,1]$ for every $\gamma\in\triangle_{x_0}(M,\nabla)$, except if e.g. $(\hat{M},\hat{\nabla})$ is geodesically complete, we understand the assumption (5) to mean that if $\gamma\in\triangle_{x_0}(M,\nabla)$ and if $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)$ is defined on $[0,1]$, then $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)\in\triangle_{\hat{x}_0}(\hat{M},\hat{\nabla})$.

Proof. We will not make the assumption Eq. (6) until later on. Notice also that U and $\hat{U}$ are star-shaped around the origin of $T|_{x_0}M$ and hence so is $U\cap A_0^{-1}(\hat{U})$. In the proof we write $\gamma_X(t)=\exp^{\nabla}_x(tX)$ and $\hat{\gamma}_{\hat{X}}(t)=\exp^{\hat{\nabla}}_{\hat{x}}(t\hat{X})$ whenever $x\in M$, $\hat{x}\in\hat{M}$ and $X\in T|_xM$, $\hat{X}\in T|_{\hat{x}}\hat{M}$ and $t\in\mathbb{R}$ are such that these are defined. If they are defined for all $t\in[0,1]$, we assume, by default, that the domains of definitions of γ_X and $\hat{\gamma}_{\hat{X}}$ are the interval $[0,1]$.

Given $u\in U\cap A_0^{-1}(\hat{U})$ and $v\in T|_{x_0}M$ we define a vector field along γ_u by

$$Y_{u,v}(t):=\frac{\partial}{\partial s}\big|_0\exp^{\nabla}_{x_0}\big(t(u+sv)\big)=t(\exp^{\nabla}_{x_0})_*|_{tu}(v)$$

i.e. $Y_{u,v}$ is the unique Jacobi field along γ_u such that $Y_{u,v}(0)=0$, $\nabla_{\dot{\gamma}_u(t)}Y_{u,v}|_{t=0}=v$. Moreover, we write $C^T_{x_0}$ for the set of tangent conjugate points in $T|_{x_0}M$ of $\exp^{\nabla}_{x_0}$ i.e.

$$\begin{aligned}C^T_{x_0}&=\{u\in U\mid\exists v\in T_{x_0}M,\ v\neq 0\text{ s.t. }Y_{u,v}(1)=0\}\\&=\{u\in U\mid\text{rank }(\exp^{\nabla}_{x_0})_*|_u<\dim M\}\,.\end{aligned}$$

Fix, for now, $u\in U\cap A_0^{-1}(\hat{U})$, $v\in T|_{x_0}M$, and assume that $u\notin C^T_{x_0}$. Let V_u be an open neighbourhood of u in $T|_{x_0}M$ such that $\exp^{\nabla}_{x_0}|_{V_u}$ is a diffeomorphism and $V_u\subset A_0^{-1}(\hat{U})$. Define $\omega_{u,w}\in\triangle_{x_0}(M,\nabla)$, for all $w\in T|_{x_0}M$ near enough to the origin such that $\exists \gamma_{Y_{u,w}(1)}(t) \in \exp^{\nabla}_{x_0}(V_u)$ for all $t \in [0,1]$, by

$$\omega_{u,w} := \gamma^{-1}_{Z_{u,w}} \cdot (\gamma_{Y_{u,w}(1)} \cdot \gamma_u),$$

where $Z_{u,w} := (\exp^{\nabla}_{x_0}|_{V_u})^{-1}(\gamma_{Y_{u,w}(1)}(1))$. For such a w we also define

$$\hat{\omega}_{u,w}(t) := \Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\omega_{u,w})(t), \quad t \in [0,1],$$

which exists if w is near enough to the origin in $T|_{x_0}M$. Notice that, by assumption, $\hat{\omega}_{u,w} \in \triangle_{\hat{x}_0}(\hat{M},\hat{\nabla})$.

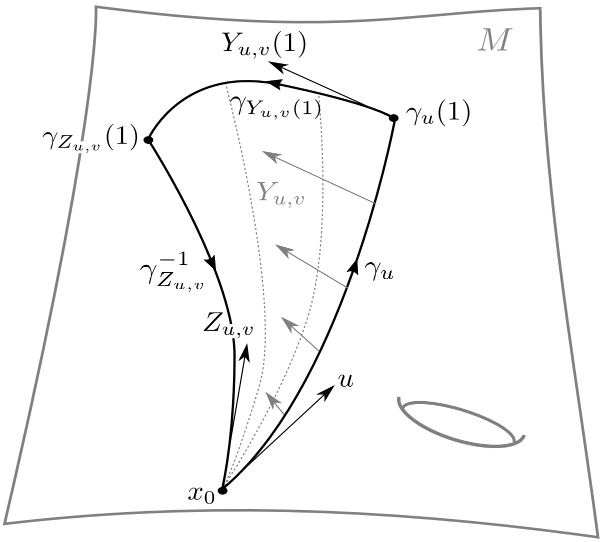

FIG. 2: Construction of the geodesic triangle $\omega_{u,v} = \gamma^{-1}_{Z_{u,v}} \cdot (\gamma_{Y_{u,v}(1)} \cdot \gamma_u)$.

In particular, if $s \in \mathbb{R}$ is near zero, $\omega_{u,sv}$ and $\hat{\omega}_{u,sv}$ are defined and

$$\hat{\omega}_{u,sv}(t) = \Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\omega_{u,sv})(t).$$

It follows that (see Lemma 2.8 case (ii)) for every s near zero, the curve $t \mapsto \hat{\omega}^{-1}_{u,sv}(t/2)$, $t \in [0,1]$, is a geodesic and $\hat{\omega}^{-1}_{u,0}(t/2) = \hat{\gamma}_{A_0 u}(t)$ for $t \in [0,1]$. Therefore a vector field along $\hat{\gamma}_{A_0 u}$ defined by

$$\hat{Y}_{u,v}(t) := \frac{\partial}{\partial s}\Big|_0 \hat{\omega}^{-1}_{u,sv}(t/2), \quad t \in [0,1]$$

is a Jacobi field. Since $\hat{\omega}_{u,sv}^{-1}(0) = \hat{\omega}_{u,sv}(1) = \hat{x}_0$, we have that $\hat{Y}_{u,v}(0) = 0$ which implies that there is a unique $\hat{v}(u,v) \in T|_{\hat{x}_0}\hat{M}$ such that

$$\begin{aligned} \hat{Y}_{u,v}(t) &= \frac{\partial}{\partial s}\Big|_0 \exp^{\hat{\nabla}}_{\hat{x}_0}\big(t(A_0 u + s\hat{v}(u,v))\big) \\ &= t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}\big(\hat{v}(u,v)\big)\,, \quad t \in [0,1]\,, \end{aligned} \tag{9}$$

i.e. $\hat{v}(u,v) = \hat{\nabla}_{\dot{\gamma}_{A_0u}(t)}\hat{Y}_{u,v}(t)|_{t=0}$. Notice that $\hat{Y}_{u,v}(t)$, and hence also $\hat{v}(u,v)$, is well defined for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}) \times T|_{x_0}M$ and $t \in [0,1]$ and it is clear that $\hat{v}$ is a smooth map. $\square$

We will now state and prove three lemmas and come back to the proof of the proposition after them.

Lemma 4.3. *Under the above assumptions, one has*

$$\hat{\nabla}_{\dot{\gamma}_u(t)}\hat{Y}_{u,v}|_{t=1} = \big(\mathcal{P}_0^1(\gamma_u)A_0\big)\nabla_{\dot{\gamma}_u(t)}Y_{u,v}|_{t=1} - \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}\big(\dot{\gamma}_u(1), Y_{u,v}(1)\big)\,. \tag{10}$$

for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}) \times T|_{x_0}M$.

Proof. In the proof we always assume that $s \in \mathbb{R}$ that appears is near zero. Then we may assume that $\omega_{u,sv}$, $\hat{\omega}_{u,sv}$ and $\hat{Y}_{u,sv}$ are defined.

Writing $\partial_t := \frac{\partial}{\partial t}\exp^{\nabla}_{x_0}(t(u+sv))$, $\partial_s := \frac{\partial}{\partial s}\exp^{\nabla}_{x_0}(t(u+sv))$ $\hat{\partial}_t := \frac{\partial}{\partial t}\hat{\omega}^{-1}_{u,sv}(t/2)$, $\hat{\partial}_s := \frac{\partial}{\partial s}\hat{\omega}^{-1}_{u,sv}(t/2)$, we have (here $\frac{\partial}{\partial t}|_{1-}$ means the left hand side derivative at $t=1$)

$$\begin{aligned} &\hat{\nabla}_{\dot{\gamma}_{A_0u}(t)}\hat{Y}_{u,v}|_{t=1} - T^{\hat{\nabla}}(\hat{\partial}_t, \hat{\partial}_s)|_{(t,s)=(1,0)} \\ &\quad = \hat{\nabla}_{\hat{\partial}_t}\frac{\partial}{\partial s}\Big|_0 \hat{\omega}^{-1}_{u,sv}(t/2)\Big|_{t=1-} - T^{\hat{\nabla}}(\hat{\partial}_t, \hat{\partial}_s)|_{(t,s)=(1,0)} \\ &\quad = \hat{\nabla}_{\hat{\partial}_s}\frac{\partial}{\partial t}\Big|_{1-}\hat{\omega}^{-1}_{u,sv}(t/2)\Big|_{s=0} = -\hat{\nabla}_{\hat{\partial}_s}\frac{\partial}{\partial t}\Big|_{1-}\hat{\omega}_{u,sv}(1-t/2)\Big|_{s=0} \\ &\quad = -\hat{\nabla}_{\hat{\partial}_s}\Big(\mathcal{P}_0^{1/2}(\omega_{u,sv})A_0\frac{\partial}{\partial t}\Big|_{1-}\omega_{u,sv}(1-t/2)\Big)\Big|_{s=0} \\ &\quad = \hat{\nabla}_{\hat{\partial}_s}\Big(\mathcal{P}_0^1(\gamma_{Y_{u,sv}(1)}\cdot\gamma_u)A_0\frac{\partial}{\partial t}\Big|_1 \exp^{\nabla}_{x_0}(tZ_{u,sv})\Big)\Big|_{s=0} \end{aligned}$$

where at the second to last equality we used the fact that $\dot{\hat{\omega}}_{u,sv}(t) = (\mathcal{P}_0^t(\omega_{u,sv})A_0)\dot{\omega}_{u,sv}(t)$, $t \in [0,1]$ (using one-sided derivatives at break points); see Lemma 2.8 case (iii). Notice that $Y_{u,sv}(t) = sY_{u,v}(t)$ and so $\gamma_{Y_{u,sv}(1)}(t) = \gamma_{sY_{u,v}(1)}(t) = \gamma_{Y_{u,v}(1)}(ts)$, which leads us to conclude that if $s > 0$,

$$\mathcal{P}_0^1(\gamma_{Y_{u,sv}(1)}\cdot\gamma_u)A_0 = \mathcal{P}_0^1(t \mapsto \gamma_{Y_{u,v}}(ts))\mathcal{P}_0^1(\gamma_u)A_0 = \mathcal{P}_0^s(\gamma_{Y_{u,v}(1)})\mathcal{P}_0^1(\gamma_u)A_0\,.$$

Therefore

$$
\begin{aligned}
&\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0 u}(t)} \hat{Y}_{u,v}|_{t=1} - T^{\hat{\nabla}}(\hat{\partial}_t, \hat{\partial}_s)|_{(t,s)=(1,0)} \\
&\qquad = \hat{\nabla}_{\hat{\partial}_s}\Big((\mathcal{P}_0^s(\gamma_{Y_{u,v}(1)})\mathcal{P}_0^1(\gamma_u)A_0)\frac{\partial}{\partial t}\big|_1 \exp^{\nabla}_{x_0}(tZ_{u,sv})\Big)\Big|_{s=0+} \\
&\qquad = \big(\mathcal{P}_0^1(\gamma_u)A_0\big)\nabla_{\partial_s}\Big(\frac{\partial}{\partial t}\big|_1 \exp^{\nabla}_{x_0}(tZ_{u,sv})\Big)\big|_{s=0} \\
&\qquad = \big(\mathcal{P}_0^1(\gamma_u)A_0\big)\Big(\nabla_{\dot{\gamma}_u(t)}\Big(\frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv})\Big)\big|_{t=1} + T(\partial_s, \partial_t)|_{(t,s)=(1,0)}\Big),
\end{aligned}
$$

where at the second to last equality we used Lemma 2.8 case (iv) (notice that $s \mapsto \frac{\partial}{\partial t}\big|_1 \exp^{\nabla}_{x_0}(tZ_{u,sv})$ is a vector field along $s \mapsto \gamma_{Y_{u,v}(1)}(s) = \gamma_{Y_{u,sv}(1)}(1)$) and at the last equality we noticed that $Z_{u,0} = u$, so $\frac{\partial}{\partial t} \exp^{\nabla}_{x_0}(tZ_{u,sv})\big|_{s=0} = \dot{\gamma}_u(t) = \partial_t$. At this moment we make the following observations:

$$
\begin{aligned}
\hat{\partial}_t|_{(t,s)=(1,0)} &= (\mathcal{P}_0^1(\gamma_u)A_0)\partial_t|_{(t,s)=(1,0)} \\
\partial_s|_{(t,s)=(1,0)} &= Y_{u,v}(1)\,, \\
\hat{\partial}_s|_{(t,s)=(1,0)} &= \hat{Y}_{u,v}(1) = \frac{\partial}{\partial s}\big|_0 \hat{\omega}^{-1}_{u,sv}(1/2) = \frac{\partial}{\partial s}\big|_0 \hat{\omega}_{u,sv}(1/2) \\
&= \frac{\partial}{\partial s}\big|_0 \Lambda_{A_0}(\gamma_{Y_{u,sv}(1)} \cdot \gamma_u)(1) \\
&= \frac{\partial}{\partial s}\big|_0 \Lambda_{\mathcal{P}_0^1(\gamma_u)A_0}(\gamma_{Y_{u,sv}(1)})(1) = \frac{\partial}{\partial s}\big|_0 \hat{\gamma}_{(\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,sv}(1)}(1) \\
&= \frac{\partial}{\partial s}\big|_0 \hat{\gamma}_{(\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,v}(1)}(s) = (\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,v}(1)\,.
\end{aligned}
$$

These allow us to write the above equation into the form

$$
\begin{aligned}
&\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0 u}(t)} \hat{Y}_{u,v}|_{t=1} + \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)) \\
&\qquad\qquad = \big(\mathcal{P}_0^1(\gamma_u)A_0\big)\nabla_{\dot{\gamma}_u(t)}\big(\frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv})\big)\big|_{t=1}\,.
\end{aligned}
$$

Therefore, it remains to show that $\frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv}) = Y_{u,v}(t)$, $\forall t \in [0,1]$. Indeed, $J(t) := \frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv})$ is a Jacobi field along γ_u and it satisfies the boundary conditions $J(0) = 0 = Y_{u,v}(0)$ and

$$
J(1) = \frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(Z_{u,sv}) = \frac{\partial}{\partial s}\big|_0 \gamma_{Y_{u,sv}(1)}(1) = \frac{\partial}{\partial s}\big|_0 \gamma_{Y_{u,v}(1)}(s) = Y_{u,v}(1)\,.
$$

Since $u \notin C^T_{x_0}$, it follows that $J = Y_{u,v}$ and thus the proof is finished. □

From the last proof, we record for later use the following fact:

$$
\hat{Y}_{u,v}(1) = (\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,v}(1)\,, \tag{11}
$$

for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}) \times T|_{x_0}M$.

Lemma 4.4. *Under the above assumptions, for all* $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}) \times T|_{x_0}M$ *the following holds:*

$$
(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0 u}\big(\partial_1 \hat{v}(u,v)(u)\big) = \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1))\,.
$$

Hence in particular,

$$\mathcal{T}_{A_0} = 0\,.$$

Proof. By assumption, $u \in U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}$ and so for all t near 1, one has $tu \in U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}$. In the proof of the first claim, we assume always that t is near enough to 1 so that this is the case.

Since $Y_{tu,v}(1) = \frac{1}{t} Y_{u,v}(t)$, Eq. (11) implies that

$$t\hat{Y}_{tu,v}(1) = t(\mathcal{P}_0^1(\gamma_{tu})A_0)Y_{tu,v}(1) = (\mathcal{P}_0^t(\gamma_u)A_0)Y_{u,v}(t)\,.$$

Writing $\partial_t := \dot{\gamma}_u(t)$, $\hat{\partial}_t := \dot{\hat{\gamma}}_{A_0 u}(t)$ to simplify the notation, we have

$$\begin{aligned}
\hat{Y}_{u,v}(1) + \hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{tu,v}(1)|_{t=1} &= \hat{\nabla}_{\hat{\partial}_t}(t\hat{Y}_{tu,v}(1))|_{t=1} \\
&= \hat{\nabla}_{\hat{\partial}_t}\big((\mathcal{P}_0^t(\gamma_u)A_0)Y_{u,v}(t)\big)|_{t=1} = (\mathcal{P}_0^t(\gamma_u)A_0)\nabla_{\partial_t}Y_{u,v}(t)|_{t=1} \\
&= \hat{\nabla}_{\partial_t}\hat{Y}_{u,v}(t)|_{t=1} + \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1))\,,
\end{aligned}$$

where at the third equality we used Lemma 2.8 case (iv) and at the fourth equality we used (10). But

$$\begin{aligned}
\hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{u,v}(t)|_{t=1} &= \hat{\nabla}_{\hat{\partial}_t}\big(t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1} \\
&= (\exp_{\hat{x}_0})_*|_{A_0u}(\hat{v}(u,v)) + \hat{\nabla}_{\hat{\partial}_t}\big((\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1} \\
&= \hat{Y}_{u,v}(1) + \hat{\nabla}_{\hat{\partial}_t}\big((\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1}\,,
\end{aligned}$$

while

$$\hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{tu,v}(1)|_{t=1} = \hat{\nabla}_{\hat{\partial}_t}\big((\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(tu,v))\big)\big|_{t=1}\,,$$

so combining these three formulas, one gets

$$\begin{aligned}
\mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)) &= \hat{\nabla}_{\hat{\partial}_t}\big((\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(tu,v))\big)\big|_{t=1} \\
&\quad - \hat{\nabla}_{\hat{\partial}_t}\big((\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1} \\
&= \hat{\nabla}_{\hat{\partial}_t}\big((\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}\big(\hat{v}(tu,v) - \hat{v}(u,v)\big)\big)\big|_{t=1}\,.
\end{aligned}$$

Writing $\partial_1\hat{v}(u,v)(X)$ for the differential of $\hat{v}$ at (u,v) with respect to v in the direction X, we have

$$\hat{v}(tu,v) - \hat{v}(u,v) = \int_1^t \frac{\partial}{\partial s}\hat{v}(su,v)\mathrm{d}s = \int_1^t \partial_1\hat{v}(su,v)(u)\mathrm{d}s\,.$$

Notice that $(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u} \in T^*(T|_{\hat{x}_0}\hat{M}) \otimes T\hat{M}$ for $t \in [0,1]$, so if we write $\hat{D}$ for the vector bundle connection on $T^*(T|_{\hat{x}_0}\hat{M}) \otimes T\hat{M} \to T|_{\hat{x}_0}\hat{M} \times \hat{M}$ naturally induced by the canonical connection on vector space $T|_{\hat{x}_0}\hat{M}$ and $\hat{\nabla}$, we get finally

$$\begin{aligned}
\mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot\gamma_u(1),Y_{u,v}(1)) &= \hat\nabla_{\partial_t}\Big((\exp^{\hat\nabla}_{\hat x_0})_*|_{tA_0u}\Big(\int_1^t \partial_1\hat v(su,v)(u)\mathrm{d}s\Big)\Big)\Big|_{t=1} \\
&= \Big(\hat D_{\frac{\mathrm{d}}{\mathrm{d}t}(tA_0u,\gamma_{A_0u}(t))}(\exp^{\hat\nabla}_{\hat x_0})_*|_{tA_0u}\Big)\Big|_{t=1}\int_1^1 \partial_1\hat v(su,v)(u)\mathrm{d}s \\
&\quad + (\exp^{\hat\nabla}_{\hat x_0})_*|_{A_0u}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\int_1^t \partial_1\hat v(su,v)(u)\mathrm{d}s \\
&= (\exp^{\hat\nabla}_{\hat x_0})_*|_{A_0u}(\partial_1\hat v(u,v)(u)),
\end{aligned}$$

which proves the first part of the lemma.

It remains to prove that $\mathcal{T}_{A_0}=0$. Indeed, by what was just proved, we have that for all $u,v\in T|_{x_0}M$ and for all t small,

$$(\exp^{\hat\nabla}_{\hat x_0})_*|_{tu}(\partial_1\hat v(tu,v)(tu)) = \mathcal{T}_{\mathcal{P}(\gamma_{tu},A_0)(1)}\big(\dot\gamma_{tu}(1),Y_{tu,v}(1)\big)$$

holds, i.e., because $Y_{tu,v}(1)=\frac{1}{t}Y_{u,v}(t)$,

$$t(\exp^{\hat\nabla}_{\hat x_0})_*|_{tu}(\partial_1\hat v(tu,v)(u)) = \mathcal{T}_{\mathcal{P}(\gamma_u,A_0)(t)}\big(\dot\gamma_u(t),\frac{1}{t}Y_{u,v}(t)\big)\,.$$

But as $t\to 0$, one has $\frac{1}{t}Y_{u,v}(t)\to\nabla_{\dot\gamma_u(t)}Y_{u,v}|_{t=0}=v$, $\partial_1\hat v(tu,v)(u)\to\partial_1\hat v(0,v)(u)$ and $\mathcal{P}(\gamma_u,A_0)(t)\to A_0$, so in the limit one gets $0=\mathcal{T}_{A_0}(u,v)$. Since $u,v\in T|_{x_0}M$ were arbitrary, the result follows. $\square$

From now on we will make all the assumption in the statement of Proposition 4.1, i.e. we also include the torsion condition Eq. (6).

Lemma 4.5. *Under the above assumptions and for all* $(u,v)\in(U\cap A_0^{-1}(\hat U)\backslash C^T_{x_0})\times T|_{x_0}M$ *and* $t\in[0,1]$, *one has*

$$\hat Y_{u,v}(t) = \frac{\partial}{\partial s}\Big|_0 \exp^{\hat\nabla}_{\hat x_0}\big(tA_0(u+sv)\big) \tag{12}$$

$$\hat Y_{u,v}(t) = \big(\mathcal{P}_0^t(\gamma_u)A_0\big)Y_{u,v}(t)\,. \tag{13}$$

Proof. Write $\hat C^T_{\hat x_0}$ for the tangent conjugate set of $(\hat M,\hat\nabla)$ at $\hat x_0$ defined in the same way as $C^T_{x_0}$. By Lemma 4.4 and condition (6), one has $(\exp^{\hat\nabla}_{\hat x_0})_*|_{A_0u}(\partial_1\hat v(u,v)(u))=0$ for all $(u,v)\in(U\cap A_0^{-1}(\hat U)\backslash C^T_{x_0})\times T|_{x_0}M$. Given such a (u,v), if $A_0u\notin\hat C^T_{\hat x_0}$, then $\partial_1\hat v(u,v)(u)=0$. Otherwise $A_0u\in\hat C^T_{\hat x_0}$, but then $\exists\epsilon>0$ such that $tu\in U\cap A_0^{-1}(\hat U)\backslash C^T_{x_0}$ and $tA_0u\notin\hat C^T_{\hat x_0}$ for all $t\in]1-\epsilon,1+\epsilon[\backslash\{1\}$, hence $\partial_1\hat v(tu,v)(tu)=0$. Letting $t\to 1$ then implies that $\partial_1\hat v(u,v)(u)=0$. Therefore we have shown that

$$\partial_1\hat v(u,v)(u)=0\,,\quad \forall(u,v)\in\big(U\cap A_0^{-1}(\hat U)\backslash C^T_{x_0}\big)\times T|_{x_0}M\,.$$

Now fix $u\in U\cap A_0^{-1}(\hat U)\backslash C^T_{x_0}$ and $v\in T|_{x_0}M$. Notice that the set $S:=\{t\in[0,1]\mid tu\in C^T_{x_0}\}$ is finite or empty. If $S\neq\emptyset$, we write $S=\{t_i\}_{i=1,\dots,N}$ where $0<t_i<t_{i+1}<1$ for all i and we set $t_{N+1}:=1$. In the case where S is empty, we set $t_1:=1$, $N:=0$.

Write $t_0 := 0$ and notice that for $t,\ \tau \in]t_i, t_{i+1}[$ we have

$$\hat{v}(tu, v) = \hat{v}(\tau u, v) + \int_\tau^t \frac{1}{s} \underbrace{\partial_1 \hat{v}(su, v)(su)}_{=0} \mathrm{d}s = \hat{v}(\tau u, v)\,,$$

i.e. the value of $t \mapsto \hat{v}(tu, v)$ is constant on each interval $]t_i, t_{i+1}[$, $i = 0, \dots, N$. Let $\hat{v}_i(u, v)$ be the constant value of $\hat{v}(tu, v)$ for $t \in]t_i, t_{i+1}[$.

Define $\hat{J}_{u,v}(t)$, $t \in [0, 1]\backslash S$, by

$$\hat{J}_{u,v}(t) := t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}\big(\hat{v}(tu, v)\big)\,, \quad \text{if} \quad t \in]t_i, t_{i+1}[\,.$$

Then $\hat{J}_{u,v}$ is a Jacobi field on each interval $]t_i, t_{i+1}[$ since for $t \in]t_i, t_{i+1}[$.

$$\hat{J}_{u,v}(t) = t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}\big(\hat{v}_i(u, v)\big) = \frac{\partial}{\partial s}\Big|_0 \exp^{\hat{\nabla}}_{\hat{x}_0} \big(t(A_0u + s\hat{v}_i(u, v))\big)\,.$$

But we observe that

$$\hat{J}_{u,v}(t) = t\hat{Y}_{tu,v}(1) = \big(\mathcal{P}^t_0(\gamma_u)A_0\big)Y_{u,v}(t)\,, \quad \forall t \in [0, 1]\backslash S\,,$$

hence because $t \mapsto (\mathcal{P}^t_0(\gamma_u)A_0)Y_{u,v}(t)$ is smooth and S is finite, we see that $\hat{J}_{u,v}(t)$ uniquely extends to a Jacobi field along $\hat{\gamma}_{A_0u}$ defined on the whole interval $[0, 1]$. We still denote this Jacobi field by $\hat{J}_{u,v}(t)$ and notice that since

$$\hat{J}_{u,v}(t) = t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}\big(\hat{v}_0(u, v)\big)$$

holds for $t \in]0, t_1[$, it holds for all $t \in [0, 1]$.

To identify $\hat{J}_{u,v}(t)$ once and for all, it remains to compute the value of $\hat{v}_0(u, v)$. We have

$$\begin{aligned}\hat{v}(0, v) &= (\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_0\big(\hat{v}(0, v)\big) = \hat{Y}_{0,v}(1) = \big(\mathcal{P}^1_0(\gamma_0)A_0\big)Y_{0,v}(1)\\ &= A_0Y_{0,v}(1) = A_0(\exp^{\nabla}_{x_0})_*|_0(v) = A_0v\end{aligned}$$

and thus $\hat{v}_0(u, v) = \lim_{t\to 0+} \hat{v}(tu, v) = \hat{v}(0, v) = A_0v$. We have thus shown the following:

$$(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}\big(\hat{v}(u, v)\big) = \hat{Y}_{u,v}(1) = \hat{J}_{u,v}(1) = (\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}(A_0v)\,.$$

We will prove that $\hat{v}(u, v) = A_0v$. Indeed, if $A_0u \notin \hat{C}^T_{\hat{x}_0}$, then the above equation readily implies that $\hat{v}(u, v) = A_0v$. On the other hand, if $A_0u \in \hat{C}^T_{\hat{x}_0}$, then for all $t \neq 1$ near 1, one has $tA_0u \notin \hat{C}^T_{\hat{x}_0}$ and $(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(tu, v) - A_0v) = 0$, which implies that $\hat{v}(tu, v) = A_0v$ and finally $\hat{v}(u, v) = A_0v$ by passing to the limit $t \to 1$.

Since $\hat{v}(u, v) = A_0v$, the claimed Eq. (12) follows from (9). To prove (13), notice that

$$\begin{aligned}\big(\mathcal{P}^t_0(\gamma_u)A_0\big)Y_{u,v}(t) &= t\hat{Y}_{tu,v}(1) = t\frac{\partial}{\partial s}\Big|_0 \exp^{\hat{\nabla}}_{\hat{x}_0} \big(A_0(tu + sv)\big)\\ &= t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(A_0v) = \hat{Y}_{u,v}(t)\,.\end{aligned}$$

This concludes the proof. $\square$

We are now ready to finish the proof of the proposition. Let $u \in U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}$. Because $Y_{u,v}(1) = (\exp^{\nabla}_{x_0})_*|_u(v)$ by definition and since $\hat{Y}_{u,v}(1) = (\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}(A_0v)$, by (12), the formula (7) is an immediate consequence of (13) and Definition 2.6. Since $C^T_{x_0}$ has no interior points in $T|_{x_0}M$, it follows that (7) holds for all $u \in U \cap A_0^{-1}(\hat{U})$.

It remains to prove the formula (8). Let $(u, v) \in U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0} \times T|_{x_0}M$. Taking twice the covariant derivative w.r.t. $\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}$ of both sides of the equation (13), recalling that $Y_{u,v}$, $\hat{Y}_{u,v}$ are Jacobi fields and using Lemma 2.8 case (iv), we get

$$
\begin{aligned}
&R^{\hat{\nabla}}\big(\dot{\hat{\gamma}}_{A_0u}(t), \hat{Y}_{u,v}(t)\big)\dot{\hat{\gamma}}_{A_0u}(t) + \hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\big(T^{\hat{\nabla}}(\dot{\hat{\gamma}}_{A_0u}(t), \hat{Y}_{u,v}(t))\big)\\
&\quad= \hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(\cdot)}\hat{Y}_{u,v}(\cdot) = \big(\mathcal{P}^t_0(\gamma_u)A_0\big)\nabla_{\dot{\gamma}_u(t)}\nabla_{\dot{\gamma}_u(\cdot)}Y_{u,v}(\cdot)\\
&\quad= \big(\mathcal{P}^t_0(\gamma_u)A_0\big)\big(R^{\nabla}(\dot{\gamma}_u(t), Y_{u,v}(t))\dot{\gamma}_u(t) + \nabla_{\dot{\gamma}_u(t)}\big(T^{\nabla}(\dot{\gamma}_u(t), Y_{u,v}(t))\big)\big)\,.
\end{aligned}
$$

Using the last two equations above, the fact that $\dot{\hat{\gamma}}_{A_0u}(t) = \mathcal{P}(\gamma_u, A_0)(t)\dot{\gamma}_u(t)$ and Definition 2.3 we get that, for all $(u, v) \in U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0} \times T|_{x_0}M$,

$$
\mathcal{R}_{\mathcal{P}^t_0(\gamma_u)A_0}\big(\dot{\gamma}_u(t), Y_{u,v}(t)\big)\dot{\gamma}_u(t) = -\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\mathcal{T}_{\mathcal{P}^{\cdot}_0(\gamma_u)A_0}\big(\dot{\gamma}_u(\cdot), Y_{u,v}(\cdot)\big) = 0\,,
$$

since $\mathcal{T}_{\mathcal{P}^t_0(\gamma_u)A_0}(\dot{\gamma}_u(t), Y_{u,v}(t)) = \mathcal{T}_{\mathcal{P}^1_0(\gamma_{tu})A_0}(\dot{\gamma}_{tu}(1), Y_{tu,v}(1)) = 0$, $t \in [0, 1]$, by assumption (6).

Let then $u \in U \cap A_0^{-1}(U)$ and let $0 < t_1 < t_2 < \dots$ be the conjugate times along γ_u (i.e $\{t_1u, t_2u, \dots\} = \{tu \mid t \in [0, 1]\} \cap C^T_{x_0}$). Suppose $X(t)$ is any vector field along γ_u. If $t \neq t_j$ for all j, then there is a $v(t) \in T|_{x_0}M$ such that $Y_{u,v(t)}(t) = X(t)$, and so

$$
\mathcal{R}_{\mathcal{P}^t_0(\gamma_u)A_0}\big(\dot{\gamma}_u(t), X(t)\big)\dot{\gamma}_u(t) = 0\,.
$$

By continuity, this holds for all $t \in [0, 1]$ and hence the result follows once we set $t = 1$.

Remark 4.6. Notice that (5) is equivalent to the condition

$$
\Lambda^{(\nabla,\hat{\nabla})}_{A_0}\big(\triangle_{x_0}(M, \nabla)\big) \subset \Omega_{\hat{x}_0}(\hat{M})\,.
$$

On the other hand, if (5) is replaced by a stronger condition

$$
\Lambda^{(\nabla,\hat{\nabla})}_{A_0}\big(\Omega_{x_0}(M)\big) \subset \Omega_{\hat{x}_0}(\hat{M})\,,
$$

then in the proof one can define $\omega_{u,v}$ to be $\gamma^{-1}_{u+v} \cdot \big((s \mapsto \gamma_{u+sv}(1)) \cdot \gamma_u\big) \in \Omega_{x_0}(M)$. Then $\omega_{u,v}$ and hence $\hat{\omega}_{u,v}$, $\hat{Y}_{u,v}$ and finally $\hat{v}(u, v)$ are defined for all $(u, v) \in (U \cap A_0^{-1}(\hat{U})) \times T|_{x_0}M$. The proof goes through in the same way as above, all the Lemmas 4.3–4.5 being true even with the set $U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}$ replaced with $U \cap A_0^{-1}(\hat{U})$ everywhere. Moreover, the proof becomes slightly easier since one does not need to pay attention to the tangent conjugate set $C^T_{x_0}$.

We will now proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. *Necessity.* If $f\colon M \to \hat{M}$ is a Riemannian covering with $f_*|_{x_0} = A_0$, then for $\gamma \in \triangle^2_{x_0}(M,\nabla)$, one has $\Lambda_{A_0}(\gamma) = f\circ\gamma$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(x_0) = \hat{x}_0$. So $\Lambda_{A_0}(\gamma) \in \Omega_{\hat{x}_0}(\hat{M})$.

Sufficiency. The idea is to prove, using Proposition 4.1, that the condition of C-A-H Theorem 2.10 given by Eq. (1) holds, which then implies the claim.

Define

$$\mathcal{A} := \{\mathcal{P}^1_0(\gamma)A_0 \mid \gamma \in \triangle_{x_0}(M,\nabla)\}$$

and notice that assumption (5) implies that $\mathcal{A} \subset T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ and it is clear that each $A \in \mathcal{A}$ is an infinitesimal isometry.

We claim that

$$\begin{aligned}(14)\qquad & P^0_1(\hat{\gamma}_{Au}) \circ (\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{Au} \circ A = A \circ P^0_1(\gamma_u) \circ (\exp^{\nabla}_{x_0})_*|_u\,, \\ & \forall A \in \mathcal{A},\ u \in T|_{x_0}M\,.\end{aligned}$$

Indeed, fix $A \in \mathcal{A}$ and let $\omega \in \triangle_{x_0}(M,\nabla)$ be arbitrary. Then there is an $\gamma \in \triangle_{x_0}(M,\nabla)$ such that $A = \mathcal{P}^1_0(\gamma)A_0$. But then $\omega\cdot\gamma \in \triangle^2_{x_0}(M,\nabla)$ and hence by Lemma 2.8(i) and the assumptions of the theorem,

$$\Lambda_A(\omega)(1) = \big(\Lambda_{\mathcal{P}^1_0(\gamma)A_0}(\omega)\cdot\Lambda_{A_0}(\gamma)\big)(1) = \Lambda_{A_0}(\omega\cdot\gamma)(1) = \hat{x}_0$$

i.e. $\Lambda_A(\triangle_{x_0}(M,\nabla)) \subset \triangle_{\hat{x}_0}(\hat{M},\hat{\nabla})$. Thus the above claim follows from Proposition 4.1.

For a unit vector $u \in T|_{x_0}M$, let $\tau(u) \in]0,+\infty]$ be cut-time for the geodesic γ_u and set

$$U^T := \{tu \mid u \in T|_{x_0}M,\ \|u\|_g = 1,\ 0 \le t < \tau(u)\}\,,\quad U := \exp_{x_0}(U^T)\,.$$

For every $A \in \mathcal{A}$ one defines a map

$$\phi_A\colon U \to \hat{M}\,;\quad \phi_A = \exp^{\hat{\nabla}}_{\hat{x}_0} \circ A \circ (\exp^{\nabla}_{x_0}|_{U^T})^{-1}\,.$$

We are to show that each ϕ_A is an isometry onto its open image. Indeed, if $x \in U$ and $X \in T|_xM$, let $u = (\exp_{x_0}|_{U^T})^{-1}(x)$ and use (14) to compute

$$\begin{aligned}\|(\phi_A)_*(X)\|_{\hat{g}} &= \big\|(P^1_0(\hat{\gamma}_{Au}) \circ A \circ P^0_1(\gamma_u))X\big\|_{\hat{g}} \\ &= \big\|(A \circ P^0_1(\gamma_u))X\big\|_{\hat{g}} = \big\|P^0_1(\gamma_u)X\big\|_g = \|X\|_g\,.\end{aligned}$$

Since $\dim M = \dim\hat{M}$, it follows that ϕ_A is a diffeomorphism onto its (open) image and is isometric. This settles the claim.

Knowing this, we may now show a property which then allows us (eventually) to call for the C-A-H Theorem 2.10: For all $A \in \mathcal{A}$ and all unit vectors $u \in T|_{x_0}M$,

$$(15)\qquad \mathcal{R}_{\mathcal{P}^t_0(\gamma_u)A} = 0\,,\quad \forall t;\ 0 \le t \le \tau(u)$$

with the understanding that $0 \le t \le \tau(u)$ is replaced by $t \ge 0$ if $\tau(u) = +\infty$. To prove this, notice that since ϕ_A is an isometry onto its open image, one has

$$\begin{aligned}(\phi_A)_*\big(R^{\nabla}((X,Y)Z)\big) &= R^{\hat{\nabla}}(((\phi_A)_*X,(\phi_A)_*Y)((\phi_A)_*Z))\,, \\ &\forall x \in U,\ X,Y,Z \in T|_xM\end{aligned}$$

i.e. $\mathcal{R}_{(\phi_A)_*|_x} = 0$ for all $x \in U$. But we know from (14) that if $0 \le t < \tau(u)$ (whence $tu \in U^T$), one has $(\phi_A)_*|_{\gamma_u(t)} = P_0^t(\gamma_{Au}) \circ A \circ P_t^0(\gamma_u)$, which equals to $\mathcal{P}_0^t(\gamma_u)A$. Hence $\mathcal{R}_{\mathcal{P}_0^t(\gamma_u)A} = 0$ if $0 \le t < \tau(u)$ and by continuity this also holds when $t = \tau(u)$ (if $\tau(u) < +\infty$) which establishes the claim.

We are now ready to finish the proof by appealing to C-A-H Theorem 2.10. Indeed, let $\omega \in \angle_{x_0}(M,\nabla)$. Since (M,g) is complete, there exists a unit vector $u \in T|_{x_0}M$ such that $\gamma_u \colon [0,\tau(u)] \to M$ is a minimal geodesic from x_0 to $\omega(1)$. Because then $\gamma_{\tau(u)u}^{-1} \cdot \omega \in \triangle_{x_0}(M,\nabla)$, one has that $A := \mathcal{P}_0^1(\gamma_{\tau(u)u}^{-1} \cdot \omega)A_0$ is in $\mathcal{A}$ and therefore $\mathcal{R}_{\mathcal{P}_0^{\tau(u)}(\gamma_u)A} = 0$ by (15). But by Lemma 2.8,

$$\begin{aligned}\mathcal{P}_0^{\tau(u)}(\gamma_u)A &= \mathcal{P}_0^1(\gamma_{\tau(u)u})\mathcal{P}_0^1(\gamma_{\tau(u)u}^{-1} \cdot \omega)A_0 \\ &= \mathcal{P}_0^1(\gamma_{\tau(u)u})\mathcal{P}_0^1(\gamma_{\tau(u)u}^{-1})\mathcal{P}_0^1(\omega)A_0 = \mathcal{P}_0^1(\omega)A_0\,,\end{aligned}$$

which proves that $\mathcal{R}_{\mathcal{P}_0^1(\omega)A_0} = 0$ for all $\omega \in \angle_{x_0}(M,\nabla)$.

Therefore the condition (1) of C-A-H Theorem 2.10 is satisfied and hence there exists a complete Riemannian manifold (N,h), $z_0 \in N$ and Riemannian covering maps $F \colon N \to M$, $G \colon N \to \hat{M}$ such that $A_0 = G_*|_{z_0} \circ (F_*|_{z_0})^{-1}$. Since M is simply connected, $F : N \to M$ is a Riemannian isomorphism and setting $f := G \circ F^{-1}$ finishes the proof. $\square$

Remark 4.7. In the case where there are no cut-points on any geodesic of (M,g) emanating from x_0, then one may replace (2) in Theorem 3.1 by the condition

$$\Lambda_{A_0}^{(\nabla,\hat{\nabla})}\big(\triangle_{x_0}(M,\nabla)\big) \subset \Omega_{\hat{x}_0}(\hat{M})\,.$$

Indeed, in this case $\exp_{x_0}^{\nabla} : T|_{x_0}M \to M$ is a diffeomorphism and in the above proof $U^T = T|_{x_0}M$, $U = M$ and so $\phi_{A_0} \colon M \to \hat{M}$ is an isometry onto its open image. It follows from a standard result on Riemannian manifolds that $f := \phi_{A_0}$ is a covering map and obviously $f_*|_{x_0} = A_0$.

5. Different formulations of the Cartan-Ambrose-Hicks Theorem

In this section, we will complement the C-A-H Theorem 2.10 by giving eight equivalent characterizations for the existence of a Riemannian covering map $f \colon (M,g) \to (\hat{M},\hat{g})$ (under specific assumptions).

First we recall a well-known proposition and, for the sake of completeness, give its easy proof.

Proposition 5.1. *Suppose (M,∇), $(\hat{M},\hat{\nabla})$ are affine manifolds such that M is simply connected and geodesically accessible from x_0 and that $(\hat{M},\hat{\nabla})$ is geodesically complete. Given $A_0 \in T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$, there exists an affine map $f : M \to \hat{M}$ such that $f_*|_{x_0} = A_0$ if and only if*

$$(\mathcal{P}^{(\nabla,\hat{\nabla})})_0^1(\gamma)A_0 = A_0\,, \quad \forall \gamma \in \square_{x_0}(M,\nabla)\,. \tag{16}$$

Proof. *Necessity.* If $f\colon M \to \hat{M}$ is an affine map such that $f_*|_{x_0} = A_0$ and if $\gamma \in \square_{x_0}(M,\nabla)$, then $\Lambda_{A_0}(\gamma) = f \circ \gamma$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(x_0) = \hat{x}_0$.

Sufficiency. In the proof we write $\gamma_u(t) = \exp_x^\nabla(tu)$, when $u \in T|_x M$. Let $x \in M$ be given. Let γ, $\omega \in \angle_{x_0}(M,\nabla)$ be such that $\gamma(1) = x$, $\omega(1) = x$, which exist since (M,∇) is geodesically accessible from x_0. Then $\omega^{-1}\cdot\gamma \in \square_{x_0}(M,\nabla)$ and hence

$$\mathcal{P}_0^1(\omega^{-1}\cdot\gamma)A_0 = A_0 .$$

It follows that (see Lemma 2.8 and Remark 2.9)

$$\begin{aligned}\Lambda_{A_0}(\omega)(1) =&\Lambda_{\mathcal{P}_0^1(\omega^{-1}\cdot\gamma)A_0}(\omega)(1) = \Lambda_{A_0}(\omega\cdot(\omega^{-1}\cdot\gamma))(1) = \Lambda_{A_0}((\omega\cdot\omega^{-1})\cdot\gamma)(1)\\ =&\Lambda_{\mathcal{P}_0^1(\gamma)A_0}(\omega\cdot\omega^{-1})(1) = \Lambda_{A_0}(\gamma)(1) .\end{aligned}$$

This shows that if for $x \in M$ one defines

$$f(x) := \{\Lambda_{A_0}(\gamma)(1) \mid \gamma \in \angle_{x_0}(M,\nabla),\ \gamma(1) = x\},$$

then $f(x)$ is a singleton set for all $x \in M$ and hence f can be seen as a map $f\colon M \to \hat{M}$.

We show that f is an affine map. To do that, we first make a construction for its differential that is analogous to that for f above. Let $x \in M$ and let $\gamma,\omega \in \angle_{x_0}(M,\nabla)$ be such that $\gamma(1) = x$, $\omega(1) = x$ as above, then since $\omega^{-1}\cdot\gamma \in \square_{x_0}(M,\nabla)$,

$$\begin{aligned}\mathcal{P}_0^1(\gamma)A_0 &= \mathcal{P}_0^1(\omega\cdot\omega^{-1})\mathcal{P}_0^1(\gamma)A_0 = \mathcal{P}_0^1((\omega\cdot\omega^{-1})\cdot\gamma)A_0\\ &= \mathcal{P}_0^1(\omega)\mathcal{P}_0^1(\omega^{-1}\cdot\gamma)A_0 = \mathcal{P}_0^1(\omega)A_0 ,\end{aligned}$$

and so

$$A(x) := \{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \angle_{x_0}(M,\nabla),\ \gamma(1) = x\}$$

is a singleton set for every $x \in M$ and thus we can view A as a map $M \to T^*M \otimes T\hat{M}$.

We claim that $\exists f_*|_x = A(x)$ for all $x \in M$. Indeed, for any $\gamma \in \angle_{x_0}(M,\nabla)$, one has $f(\gamma(t)) = \Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}f\big(\gamma(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_{A_0}(\gamma)(t) = \big(\mathcal{P}_0^t(\gamma)A_0\big)\dot{\gamma}(t) = A\big(\gamma(t)\big)\dot{\gamma}(t) .$$

Then if $X \in T|_x M$, choose $u \in T|_{x_0}M$ such that $\gamma_u(1) = x$ and notice that $\gamma := \gamma_X.\gamma_u$ is a 1-broken geodesic. Thus the above formula gives by letting $t \to \frac{1}{2}+$,

$$f_*(X) = A(x)X ,$$

showing also that the differential $f_*|_x$ exists.

To show that f is an affine map, it is enough to show that for any geodesic $\Gamma\colon [0,1] \to M$ and any vector field $X(t)$ parallel to it, the vector field $f_*(X(t))$

along $f\circ\Gamma$ is parallel. So choose such $\Gamma=\gamma_v$ and X. Let $u\in T|_{x_0}M$ be such that $\gamma_u(1)=\Gamma(0)$ and notice that $\gamma_{tv}\cdot\gamma_u\in\angle_{x_0}(M,\nabla)$. Then for all $t\in[0,1]$,

$$f(\Gamma(t))=f((\gamma_{tv}\cdot\gamma_u)(1))=\Lambda_{A_0}(\gamma_{tv}\cdot\gamma_u)(1)=\Lambda_{\mathcal{P}_0^1(\gamma_u)A_0}(\Gamma)(t)\,,$$

where the right hand side is a geodesic by Lemma 2.8. Moreover,

$$\begin{aligned}f_*(X(t))&=A\big((\gamma_{tv}\cdot\gamma_u)(1)\big)X(t)\\&=\big(\mathcal{P}_0^1(\gamma_{tv}\cdot\gamma_u)A_0\big)X(t)=\big(\mathcal{P}_0^t(\Gamma)\mathcal{P}_0^1(\gamma_u)A_0\big)X(t)\,,\end{aligned}$$

which, by using $P_t^0(\Gamma)X(t)=X(0)$ and $\Lambda_{\mathcal{P}_0^1(\gamma_u)A_0}(\Gamma)=f\circ\Gamma$, simplifies to

$$f_*\big(X(t)\big)=P_0^t(f\circ\Gamma)\big((\mathcal{P}_0^1(\gamma_u)A_0)X(0)\big)\,.$$

Thus $t\mapsto f_*(X(t))$ is the parallel transport of $(\mathcal{P}_0^1(\gamma_u)A_0)X(0)$ along $f\circ\Gamma$ and the proof is finished. $\square$

We now give the reformulation of the C-A-H Theorem 2.10. The equivalence of (i), (ii), (v), (vi), (vii), (ix) can essentially be found in [11], the Global C-A-H Theorem 4.47.

Theorem 5.2. *Suppose (M,g), $(\hat M,\hat g)$ are complete Riemannian manifolds of the same dimension, $\dim M=\dim\hat M$, M simply connected and let $A_0\in T^*|_{x_0}M\otimes T|_{\hat x_0}\hat M$ be an infinitesimal isometry. Let ∇, $\hat\nabla$ be the Levi-Civita connections of (M,g), $(\hat M,\hat g)$, respectively. Then the following are equivalent (for the sake of clarity we write $\mathcal{P}_a^b(\gamma)$ instead of $(\mathcal{P}^{(\nabla,\hat\nabla)})_a^b(\gamma)$):*

- (i) *There exists a Riemannian covering map $f\colon M\to\hat M$ such that $f_*|_{x_0}=A_0$;*
- (ii) *For all $\gamma\in\angle_{x_0}(M,\nabla)$ one has $\mathcal{R}^{(\nabla,\hat\nabla)}_{\mathcal{P}_0^1(\gamma)A_0}=0$;*
- (iii) $\Lambda^{(\nabla,\hat\nabla)}_{A_0}(\Omega_{x_0}(M))\subset\Omega_{\hat x_0}(\hat M)$;
- (iv) $\Lambda^{(\nabla,\hat\nabla)}_{A_0}(\triangle^2_{x_0}(M,\nabla))\subset\triangle^2_{\hat x_0}(\hat M,\hat\nabla)$;
- (v) $\mathcal{P}_0^1(\gamma)A_0=A_0$ *for all* $\gamma\in\Omega_{x_0}(M)$;
- (vi) $\mathcal{P}_0^1(\gamma)A_0=A_0$ *for all* $\gamma\in\triangle^2_{x_0}(M,\nabla)$;
- (vii) $\mathcal{P}_0^1(\gamma)A_0=A_0$ *for all* $\gamma\in\square_{x_0}(M,\nabla)$;
- (viii) *There exist points $x_1\in M$, $\hat x_1\in\hat M$ such that (pw.='piecewise')*

$$\forall\gamma\colon[0,1]\to M\ \textit{pw. smooth},\ \gamma(0)=x_0,\ \gamma(1)=x_1\implies\Lambda^{(\nabla,\hat\nabla)}_{A_0}(\gamma)(1)=\hat x_1\,.$$

- (ix) *If $\gamma,\omega:[0,1]\to M$ are piecewise smooth, $\gamma(0)=\omega(0)=x_0$ and $\gamma(1)=\omega(1)$, then $\Lambda^{(\nabla,\hat\nabla)}_{A_0}(\gamma)(1)=\Lambda^{(\nabla,\hat\nabla)}_{A_0}(\omega)(1)$.*

Proof. We write $\gamma_u(t)=\exp^{\nabla}_x(tu)$ if $u\in T|_xM$ and $\hat\gamma_{\hat u}(t)=\exp^{\hat\nabla}_{\hat x}(t\hat u)$ if $\hat u\in T|_{\hat x}\hat M$, $t\in[0,1]$.

We will do the following four cycles of deductions: (i) $\iff$ (ii) and (i) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ix) $\Rightarrow$ (viii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Since f_* is a local isometry and $f_*|_{x_0} = A_0$, we have $\Lambda_{A_0}(\gamma) = f\circ\gamma$, $f_*|_{\gamma(1)} = \mathcal{P}_0^1(\gamma)A_0$ and hence if $X, Y, Z \in T|_{\gamma(1)}M$,

$$\mathcal{R}_{\mathcal{P}_0^1(\gamma)A_0}(X, Y)Z = f_*R^{\nabla}(X, Y)Z - R^{\hat{\nabla}}(f_*X, f_*Y)Z = 0\,.$$

(ii) $\Rightarrow$ (i): C-A-H Theorem 2.10

(i) $\Rightarrow$ (v): Again, since f_* is a local isometry and $f_*|_{x_0} = A_0$, then $\mathcal{P}_0^1(\gamma)A_0 = f_*|_{\gamma(1)}$ for any piecewise smooth γ. In particular, $\gamma \in \Omega_{x_0}(M)$ implies $\mathcal{P}_0^1(\gamma)A_0 = f_*|_{\gamma(1)} = f_*|_{x_0} = A_0$.

(v) $\Rightarrow$ (iii): Since $\mathcal{P}_0^1(\gamma)A_0 : T|_{\gamma(1)}M \to T|_{\Lambda_{A_0}(\gamma)(1)}\hat{M}$, and $A_0 : T|_{x_0}M \to T|_{\hat{x}_0}\hat{M}$, it follows that if $\gamma \in \Omega_{x_0}(M)$ and if $\mathcal{P}_0^1(\gamma)A_0 = A_0$, that $\Lambda_{A_0}(\gamma)(1) = \hat{x}_0$ i.e. $\Lambda_{A_0}(\gamma) \in \Omega_{\hat{x}_0}(\hat{M})$.

(iii) $\Rightarrow$ (iv): Obvious (cf. Lemma 2.8).

(iv) $\Rightarrow$ (i): Theorem 3.1 (and the remark that follows it).

(v) $\Rightarrow$ (vi): Obvious.

(vi) $\Rightarrow$ (vii): Let $\Gamma \in \square_{x_0}(M, \nabla)$. After reparameterizing if necessary (see Lemma 2.8 case (v)), we may assume that Γ is $\gamma_4.\gamma_3.\gamma_2.\gamma_1$ with $\gamma_i\colon [0, 1] \to M$, $i = 1, 2, 3, 4$, geodesics. Let $\rho\colon [0, 1] \to M$ be a geodesic from x_0 to $\gamma_2(1) = \gamma_3(0)$. Then $\gamma := \rho^{-1}\cdot(\gamma_2\cdot\gamma_1) \in \triangle_{x_0}(M, \nabla)$ and $\omega := (\gamma_4.\gamma_3)\cdot\rho \in \triangle_{x_0}(M, \nabla)$. Hence by assumption and Lemma 2.8 one has

$$\begin{aligned}\mathcal{P}_0^1(\Gamma)A_0 &= \mathcal{P}_0^1(\gamma_4\cdot\gamma_3)\mathcal{P}_0^1(\gamma_2\cdot\gamma_1)A_0 = \mathcal{P}_0^1(\gamma_4\cdot\gamma_3)\mathcal{P}_0^1(\rho\cdot\rho^{-1})\mathcal{P}_0^1(\gamma_2\cdot\gamma_1)A_0\\ &= \mathcal{P}_0^1(\gamma\cdot\omega)A_0 = A_0\,.\end{aligned}$$

(vii) $\Rightarrow$ (i): By Proposition 5.1 there is an affine map $f\colon M \to \hat{M}$ such that $f_*|_{x_0} = A_0$. Let $x \in M$ and take some geodesic $\gamma\colon [0, 1] \to M$ from x_0 to x. >From the affinity of f and $f_*|_{x_0} = A_0$, it follows that $\mathcal{P}_0^1(\gamma)A_0 = f_*|_{\gamma(1)} = f_*|_x$ and since A_0 is an infinitesimal isometry, then so is $\mathcal{P}_0^1(\gamma)A_0$ and hence f is a local isometry. It follows from a standard result in Riemannian geometry that f is a Riemannian covering map.

(i) $\Rightarrow$ (ix): If $f\colon M \to \hat{M}$ is a Riemannian covering with $f_*|_{x_0} = A_0$ and γ, ω are as stated, then $\Lambda_{A_0}(\gamma) = f\circ\gamma$, $\Lambda_{A_0}(\omega) = f\circ\omega$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(\omega(1)) = \Lambda_{A_0}(\omega)(1)$.

(ix) $\Rightarrow$ (viii): Take any point $x_1 \in M$, fix any piecewise smooth path $\omega : [0, 1] \to M$ from x_0 to x_1 and set $\hat{x}_1 := \Lambda_{A_0}(\omega)(1)$. Then if $\gamma\colon [0, 1] \to M$ is an arbitrary piecewise smooth path from x_0 to x_1, we have $\gamma(1) = x_1 = \omega(1)$ and hence by the assumption, $\Lambda_{A_0}(\gamma)(1) = \Lambda_{A_0}(\omega)(1) = \hat{x}_1$.

(viii) $\Rightarrow$ (i): Corollary 3.5 (and the Remark after it). ☐

Remark 5.3.

(a) Although we don't prove it here, the condition (ii) in the previous theorem (and Eq. (1) in C-A-H Theorem 2.10) can in fact be replaced with

(ii)' For all $\gamma \in \angle_{x_0}(M, \nabla)$ and $X \in T|_{\gamma(1)}M$, one has $\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}_0^1(\gamma)A_0}(\dot{\gamma}(1), X)\dot{\gamma}(1) = 0.$

(b) We point out that the condition (v) (resp. (vi)) is significantly stronger than (iii) (resp. (iv)). To see this, consider the set $Q \subset T^*M \otimes T\hat{M}$ of infinitesimal isometries as a bundle over M (resp. over $\hat{M}$), where the bundle map $\pi_M \colon Q \to M$ (resp. $\pi_{\hat{M}} \colon Q \to \hat{M}$) maps A to x (resp. to $\hat{x}$), if $A \colon T|_x M \to T|_{\hat{x}}\hat{M}$. As a manifold Q has dimension $2n + \frac{n(n-1)}{2}$.

If $\gamma \colon [0,1] \to M$ is a piecewise smooth path that starts from x_0, then $t \mapsto \mathcal{P}_0^t(\gamma)A_0$, $t \in [0,1]$, is a piecewise smooth path in Q that starts from A_0 and $\Lambda_{A_0}(\gamma)(t) = \pi_{\hat{M}}(\mathcal{P}_0^t(\gamma)A_0)$, $\gamma(t) = \pi_M(\mathcal{P}_0^t(\gamma)A_0)$.

The condition (v) says that if γ is a loop of M based at x_0, then $\mathcal{P}_0^t(\gamma)A_0$ is a loop of Q based at A_0 (and therefore automatically $\Lambda_{A_0}(\gamma)$ is a loop of $\hat{M}$ based at $\hat{x}_0$). In other words,

$$\{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \Omega_{x_0}(M)\} = \{A_0\}\,.$$

On the other hand, condition (iii) demands that for any loop γ of M based at x_0, the path $\mathcal{P}_0^t(\gamma)A_0$ comes back to the set fiber $\pi_{\hat{M}}^{-1}(\hat{x}_0)$ (and of course to $\pi_M^{-1}(x_0)$), where it started from i.e.

$$\{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \Omega_{x_0}(M)\} \subset \pi_M^{-1}(x_0) \cap \pi_{\hat{M}}^{-1}(\hat{x}_0)\,.$$

The set $\pi_M^{-1}(x_0) \cap \pi_{\hat{M}}^{-1}(\hat{x}_0)$ is $\frac{n(n-1)}{2}$ dimensional in contrast to $\{A_0\}$ which is 0-dimensional. This can be seen as an illustration of the stringency of condition (v) with respect to (iii).

(c) It is an open problem to determine if actually there is a weaker version of (vii) i.e. if (i)–(ix) are equivalent to the following: $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\square_{x_0}(M,\nabla)) \subset \square_{\hat{x}_0}(\hat{M},\hat{\nabla})$. See also Remark 4.7.

(d) In (viii) the assumption that γ be piecewise smooth can be replaced by the assumption that it be 6-times broken geodesic (see Corollary 3.5).

(d) The condition (ix) can be replaced by the assumption that γ, ω be 1-broken geodesics.

To see this, we use the argument from [11] which is essentially the same as for Proposition 5.1 (but in Riemannian setting). For any $x \in M$, the set

$$\{\Lambda_{A_0}(\gamma)(1) \mid \gamma \in \angle_{x_0}(M,\nabla),\ \gamma(1) = x\}$$

is a singleton set by assumption, so one may define $f(x)$ to be its unique element. This defines $f \colon M \to \hat{M}$. If $X \in T|_x M$, let $\omega \colon [0,1] \to M$ be any geodesic from x_0 to x. Then $\gamma_{tX} \cdot \omega \in \angle_{x_0}(M,\nabla)$, $\gamma_{tX} \cdot \omega(1) = \gamma_X(t)$, and so

$$f\big(\gamma_X(t)\big) = \Lambda_{A_0}(\gamma_{tX} \cdot \omega)(1) = \Lambda_{\mathcal{P}_0^1(\omega)A_0}(\gamma_X)(t) = \hat{\gamma}_{(\mathcal{P}_0^1(\omega)A_0)X}(t)\,.$$

This implies that f is differentiable at x and $f_*(X) = (\mathcal{P}_0^1(\omega)A_0)X$. Since $\mathcal{P}_0^1(\omega)A_0$ is an infinitesimal isometry, this implies that f is a local Riemannian isometry (the smoothness of f is easily established) and therefore a Riemannian covering map, i.e. we arrive at case (i).

(e) We also remark that the condition (ix) is much stronger than (viii). To see this, observe that (viii) can be written in the following way that resembles more condition (ix):

(viii) There exist a point $x_1 \in M$ such that if γ, $\omega\colon [0,1] \to M$ are piecewise smooth, $\gamma(0) = \omega(0) = x_0$ and $\gamma(1) = \omega(1) = x_1$, then $\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(1) = \Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\omega)(1)$.

To put it another way, in (ix) the endpoints $\gamma(1)$ of the curves γ are allowed to move freely on M while in (viii) one only uses curves γ whose endpoints $\gamma(1)$ are fixed to the pre-given point x_1.

Remark 5.4. In [10] the following local version of C-A-H Theorem was proven in the context of affine manifolds: Let (M,∇), $(\hat{M},\hat{\nabla})$ be affine manifolds (possibly of different dimensions), let $A_0 \in T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ and suppose $U \subset T|_{x_0}M$ is an open set containing the origin such that $\exp^{\nabla}_{x_0}|_U$ is a diffeomorphism onto its image and that $\exp^{\hat{\nabla}}_{\hat{x}_0}$ is defined on $A_0(U)$. If

$$\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma_u)A_0}(\dot{\gamma}_u(1),X)Y = 0, \quad \mathcal{T}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma_u)A_0}(\dot{\gamma}_u(1),X) = 0, \tag{17}$$

for all $u \in U$ and X, $Y \in T|_{\gamma_u(1)}M$, then $\exp^{\hat{\nabla}}_{\hat{x}_0} \circ A_0 \circ (\exp^{\nabla}_{x_0}|_U)^{-1}\colon U \to \hat{M}$ is an affine map.

We point out that the conclusion (8) of Proposition 4.1 is not enough to invoke this local of C-A-H Theorem in the general setting of affine manifolds, since (8) gives (17) only in the special case where $Y = \dot{\gamma}_u(1)$. It is an open question whether one is able to reach the former condition in (17) from the assumptions of Proposition 4.1.

6. An application of the main result

Recall that the affine group $\mathrm{Aff}(V)$ of a vector space V is $\mathrm{GL}(V) \times V$ as a set and it is equipped with a group multiplication $\star$ given by

$$(A,v) \star (B,w) := (AB, Aw + v), \quad (A,v),(B,w) \in \mathrm{Aff}(V).$$

Also, there is a natural action $\star$ of $\mathrm{Aff}(V)$ on V given by

$$(A,v) \star w := Aw + v\,, \quad (A,v) \in \mathrm{Aff}(V)\,,\ w \in V\,.$$

Recall also that if (M,∇) is an affine manifold and $x \in M$, then its *affine holonomy group* $\mathcal{A}_x$ at x is a subgroup of the affine group $\mathrm{Aff}(T|_x M)$ given by

$$\mathcal{A}_x = \Big\{ \Big(P^0_1(\gamma), \int_0^1 P^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s \Big) \;\Big|\; \gamma \in \Omega_x(M) \Big\}.$$

As an application of Theorem 3.1 we will give a different proof of Theorem IV.7.2 in [8].

Theorem 6.1. *Suppose (M,g) is a simply connected, complete Riemannian manifold and $x \in M$. If the affine holonomy group $\mathcal{A}_x$ has a fixed point $W \in T|_x M$, then (M,g) is isometric to the Euclidean space.*

Proof. Suppose $W \in T|_x M$ is a fixed point of $\mathcal{A}_x$. Then for all $\gamma \in \Omega_x(M)$ one has $W = \big(P_1^0(\gamma), \int_0^1 P_s^0(\gamma)\dot\gamma(s)\mathrm{d}s\big) \star W$. Write $\gamma_W \colon [0,1] \to M$ for the geodesic with $\dot\gamma_W(0) = W$ and define $x_0 := \gamma_W(1)$.

Then if $\omega \in \Omega_{x_0}(M)$, it follows that $\gamma_W^{-1} \cdot (\omega \cdot \gamma_W) \in \Omega_x(M)$,

$$W = \Big(P_1^0\big(\gamma_W^{-1}\cdot(\omega\cdot\gamma_W)\big), \int_0^1 P_s^0\big(\gamma_W^{-1}\cdot(\omega\cdot\gamma_W)\big)\frac{\mathrm{d}}{\mathrm{d}s}\big(\gamma_W^{-1}\cdot(\omega\cdot\gamma_W)\big)(s)\mathrm{d}s\Big) \star W$$

i.e. if $W' := P_1^0(\gamma_W^{-1})W + \int_0^1 P_s^0(\gamma_W^{-1})\frac{\mathrm{d}}{\mathrm{d}s}\gamma_W^{-1}(s)\mathrm{d}s$,

$$W' = \Big(P_1^0(\omega), \int_0^1 P_s^0(\omega)\dot\omega(s)\mathrm{d}s\Big) \star W'\,.$$

But

$$W' = P_0^1(\gamma_W)W - \int_0^1 P_{1-s}^1(\gamma_W)\dot\gamma_W(1-s)\mathrm{d}s = \dot\gamma_W(1) - \int_0^1 \dot\gamma_W(1)\mathrm{d}s = 0\,,$$

so one has

$$0 = \int_0^1 P_s^0(\omega)\dot\omega(s)\mathrm{d}s, \quad \forall\omega \in \Omega_{x_0}(M)\,. \tag{18}$$

Let A_0 be the identity map $\mathrm{id}_{T|_{x_0}M} : T|_{x_0}M \to T|_{x_0}M$ and define $(\hat M, \hat g) := (T|_{x_0}M, g|_{T|_{x_0}M})$ and $\hat x_0 := 0$, the origin of $T|_{x_0}M$. Then using the natural identification of $T|_{\hat x_0}\hat M = T|_0(T|_{x_0}M)$ with $T|_{x_0}M$, one sees that A_0 is an infinitesimal isometry. For any piecewise smooth $\gamma\colon [0,1] \to M$ such that $\gamma(0) = x_0$ we obviously have

$$\Lambda_{A_0}^{(\nabla,\hat\nabla)}(\gamma) = \int_0^1 P_s^0(\gamma)\dot\gamma(s)\mathrm{d}s\,,$$

with $\nabla, \hat\nabla$ the Levi-Civita connections of (M,g), $(\hat M, \hat g)$, respectively. The above equation (18) shows that $\Lambda_{A_0}^{(\nabla,\hat\nabla)}(\omega)(1) = 0 = \hat x_0$ for all $\omega \in \Omega_{x_0}(M)$ i.e. $\Lambda_{A_0}^{(\nabla,\hat\nabla)}(\Omega_{x_0}(M)) \subset \Omega_{\hat x_0}(\hat M)$ and thus one may invoke Theorem 3.1 (see also the Remark following the Theorem) to obtain a Riemannian covering $f\colon M \to \hat M$. Since $(\hat M, \hat g)$ is an Euclidean space and in particular simply connected, it follows that f is an isometry from (M,g) to the Euclidean space. This completes the proof. □

Remark 6.2. The above result is used e.g. to determine all the possible affine Riemannian holonomy groups from the usual (linear) holonomy groups (see [8]). Moreover, the affine Riemannian holonomy group turn out to determine the orbits of the the control system associated to the rolling (without slipping and spinning) of a Riemannian manifold onto its tangent plane (see [5]).

Acknowledgement. The work of the author is part of his PhD thesis and is supported by Finnish Academy of Sciences and Letters, KAUTE foundation and Institut français de Finlande. The author also wants to thank his advisor Y. Chitour for having suggested the problem.

References

[1] Alouges, F., Y., Chitour, Long, R., *A motion planning algorithm for the rolling–body problem*, IEEE Trans. on Robotics (2010).

[2] Blumenthal, R., Hebda, J., *The generalized Cartan–Ambrose–Hicks Theorem*, Geom. Dedicata **29** (2) (1989), 163–175.

[3] Cheeger, J., Ebin, D. G., *Comparison theorems in Riemannian geometry*, North–Holland Math. Library **9** (1975).

[4] Chelouah, A., Chitour, Y., *On the controllability and trajectories generation of rolling surfaces*, Forum Math. **15** (2003), 727–758.

[5] Chitour, Y., Kokkonen, P., *Rolling Manifolds: Intrinsic Formulation and Controllability*, preprint, arXiv:1011.2925v2 [math.DG], 2011.

[6] Godoy Molina, M., Grong, E., *Geometric conditions for the existence of an intrinsic rolling*, preprint, arXiv:1111.0752v1 [math.DG], 2011.

[7] Godoy Molina, M., Grong, E., Markina, I., Silva Leite, F., *An intrinsic formulation to the rolling manifolds problem*, preprint, arXiv:1008.1856v1 [math.DG], 2010.

[8] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, vol. 1, Wiley–Interscience, 1996.

[9] Lee, J., *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer–Verlag, New York, 2003.

[10] Pawel, K., Reckziegel, H., *Affine Submanifolds and the Theorem of Cartan–Ambrose–Hicks*, Kodai Math. J. **25** (3) (2002), 341–356.

[11] Robbin, J. W., Salamon, D. A., *Introduction to Differential Geometry*, ETH, Lecture Notes, preliminary version, January 2011, `http://www.math.ethz.ch/~salamon/PREPRINTS/diffgeo2011.pdf`.

[12] Sakai, T., *Riemannian Geometry*, Transl. Math. Monogr., **149**, American Mathematical Society, Providence, RI, 1996.

[13] Sharpe, R. W., *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics, vol. 166, Springer–Verlag, New York, 1997.

L2S, Université Paris-Sud XI, CNRS and Supélec,
Gif-sur-Yvette, 91192, France
and
University of Eastern Finland, Department of Applied Physics,
70211, Kuopio, Finland
E-mail: `petri.kokkonen@lss.supelec.fr`