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ON PROPERTY (B) OF HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we offer criteria for property (B) and additional asymptotic behavior of solutions of the n-th order delay differential equations

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' = q(t)f\left(x(\tau(t))\right).$$

Obtained results essentially use new comparison theorems, that permit to reduce the problem of the oscillation of the n-th order equation to the the oscillation of a set of certain the first order equations. So that established comparison principles essentially simplify the examination of studied equations. Both cases $\int^{\infty} r^{-1/\gamma}(t)\,\mathrm{d}t = \infty$ and $\int^{\infty} r^{-1/\gamma}(t)\,\mathrm{d}t < \infty$ are discussed.

1. Introduction

In this paper, we consider the n-th order ($n \geq 3$) delay differential equations of the form

$$(E) \qquad \left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' = q(t)f\left(x(\tau(t))\right),$$

where we assume that $a, q, \tau, p, \sigma \in C([t_0, \infty))$, $f \in C((-\infty, \infty))$, and

(H_1) γ is the ratio of two positive odd integers,

(H_2) $r(t) > 0$, $q(t) > 0$,

(H_3) $\tau(t) \leq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$, $\tau(t)$ nondecreasing,

(H_4) $xf(x) > 0$, $f'(x) \geq 0$ for $x \neq 0$, $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$.

By a solution of Eq. (E) we mean a function $x(t) \in C^{n-1}((T_x, \infty))$, $T_x \geq t_0$, which has the property $r(t)(x^{(n-1)}(t))^{\gamma} \in C^1((T_x, \infty))$ and satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise, it is called to be nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

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Recently, great attention has been devoted to the oscillation of (E) and the corresponding equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t)f\left(x(\tau(t))\right) = 0\,.$$

The effort of the authors is aimed to obtain such criteria for studied properties that involve the only one main condition. But this effort leads to the strong monotonicity condition, namely,

$$r'(t) \geq 0 \tag{1.1}$$

imposed on the coefficient $r(t)$, see e.g. [5], [15], [21] and even $r(t) \equiv 1$ in [14]. The main novelty of this paper consists in relaxing this nonotonicity condition. What is more, in this paper equation (E) has been studied in both, canonical case, i.e., when the condition

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\,\mathrm{d}s = \infty \tag{1.2}$$

holds as well as the noncanonical case, namely, when

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\,\mathrm{d}s < \infty. \tag{1.3}$$

is satisfied.

Various techniques appeared for the investigation of the higher order differential equations. Our method is based on the establishing new comparison theorems for comparing the $n-th$ order equation (E) with a just one or a couple of the first order delay differential equations in the sense, that the oscillations of these equations yield the oscillation of (E).

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is they are satisfied for all t large enough.

Remark 2. The results of this paper hold true also for the differential equation

$$\left(r(t)\Phi_\gamma\left(u^{(n-1)}(t)\right)\right)' = q(t)f\left(x(\tau(t))\right),$$

where $\Phi_\gamma(u) = |u|^\gamma \operatorname{sgn} u$ and in this case we can replace hypothesis (H_1) by $\gamma \in R_+$.

2. Main results

It is convenient to prove our main results by means of a series of lemmas, as follows. At first, we present basic properties of possible nonoscillatory solutions.

Lemma 1. *If $x(t)$ is a positive solution of (E), then $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is increasing, all derivatives $x^{(i)}(t)$, $1 \leq i \leq n-1$ are of constant signs, and $x(t)$ satisfies either*

$$x^{(n-2)}(t) > 0\,, \quad x^{(n-1)}(t) < 0 \tag{C_1}$$

or

$$x'(t) > 0, \quad \ldots, \quad x^{(n-2)}(t) > 0\,, \quad x^{(n-1)}(t) > 0 \tag{C_2}$$

or, if (1.3) *holds,*

$$(C_3) \qquad x^{(n-2)}(t) < 0\,, \quad x^{(n-1)}(t) > 0\,.$$

Proof. Since $x(t)$ is a positive solution of (E), then it follows from (E) that

$$\big(r(t)\big[x^{(n-1)}(t)\big]^{\gamma}\big)' = q(t)f\big(x(\tau(t))\big) > 0.$$

Thus, $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is increasing, which implies that either $x^{(n-1)}(t) > 0$ or $x^{(n-1)}(t) < 0$ and moreover all lower derivatives are of fixed signs.

At first, we assume that $x^{(n-1)}(t) < 0$, then we are led to $x^{(n-2)}(t) > 0$, because the opposite condition $x^{(n-2)}(t) < 0$ yields that all lower derivatives are negative, which contradicts the positivity of $x(t)$. Therefore, we conclude that $x(t)$ satisfies the case (C_1).

Now, we suppose that $x^{(n-1)}(t) > 0$. Then either $x^{(n-2)}(t) < 0$ or $x^{(n-2)}(t) > 0$. Since $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is increasing and positive, then there exists a constant $c > 0$ such that

$$r(t)\big[x^{(n-1)}(t)\big]^{\gamma} \geq c\,.$$

An integration from t_1 to t, yields

$$x^{(n-2)}(t) \geq x^{(n-2)}(t_1) + c^{1/\gamma}\int_{t_1}^{t} r^{-1/\gamma}(s)\,\mathrm{d}s\,.$$

If (1.2) holds, then the last inequality implies $x^{(n-2)}(t) > 0$, which guarantees that all lower derivatives are positive, i.e., $x(t)$ satisfies the case (C_2). On the other hand, if (1.3), then the event $x^{(n-2)}(t) < 0$ is not eliminated, so that $x(t)$ may satisfy the case (C_3). The proof is complete. □

Remark 3. The cases (C_1) and (C_3) involve various partial cases for lower derivatives, but these details are not important since our comparison method allow to eliminate these subcases en masse.

Definition 1. A nonoscillatory solution $x(t)$ is said to be strongly increasing if $x(t)$ is positive and it satisfies (C_2) or $x(t)$ is negative and $-x(t)$ satisfies (C_2).

Following Kusano and Naito [12], Dzurina [7], we recall the following definition.

Definition 2. Assume that (1.2) holds. We say that (E) enjoys property (B) if every its nonoscillatory solution $x(t)$

(i) for n odd, is strongly increasing;

(ii) for n even, is strongly increasing or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

It is easy to verify that if $x(t)$ is strongly increasing, then the following rate of divergence

$$|x(t)| \geq c\int_{t_0}^{t} r^{-1/\gamma}(s)(t-s)^{n-2}\,\mathrm{d}s\,, \quad c > 0 \tag{2.1}$$

holds.

Our results essential use the following estimate which is due to Philos and Staikos see [18] and [19].

Lemma 2. *Let $z \in C^k([t_0,\infty))$. Assume that $z^{(k)}$ is of fixed sign and not identically zero on a subray of $[t_0,\infty)$. If, moreover, $z(t)>0$, $z^{(k-1)}(t)z^{(k)}(t)\le 0$, and $\lim_{t\to\infty} z(t)\ne 0$, then for every $\delta\in(0,1)$ there exists $t_\delta \ge t_0$ such that*

$$z(t) \ge \frac{\delta}{(k-1)!} t^{k-1} |z^{(k-1)}(t)| \tag{2.2}$$

holds on $[t_\delta,\infty)$.

Our task is to provide criteria for elimination the cases (C_1) and (C_3) to get desired properties of (E).

Theorem 1. *Assume that $\xi(t)\in C([t_0,\infty))$ is such that*

$$\xi(t) \ \textit{ nondecreasing}, \quad \xi(t)>t\,, \quad \textit{and} \quad \tau(\xi(t))<t\,. \tag{2.3}$$

Further assume that $x(t)$ is a positive solution of (E), such that $\lim_{t\to\infty} x(t)\ne 0$. If for some $\delta\in(0,1)$, the first order delay equation

$$y'(t) + r^{-1/\gamma}(t)\Big[\int_t^{\xi(t)} q(s) f\Big(\frac{\delta(\tau(s))^{n-2}}{(n-2)!}\Big)\,\mathrm{d}s\Big]^{1/\gamma} f^{1/\gamma}\big(y\big(\tau(\xi(t))\big)\big) = 0 \tag{E_1}$$

is oscillatory, then $x(t)$ does not satisfy (C_1).

Proof. Assume the contrary, that is, we admit that $x(t)$ satisfies (C_1). Thus, it follows from Lemma 2 that for every $\delta\in(0,1)$

$$x(t) \ge \frac{\delta}{(n-2)!} t^{n-2} x^{(n-2)}(t)\,. \tag{2.4}$$

Setting (2.4) into (E), we get

$$\big(r(t)\big[x^{(n-1)}(t)\big]^\gamma\big)' \ge q(t) f\Big(\frac{\delta(\tau(t))^{n-2}}{(n-2)!}\Big) f\big(x^{(n-2)}(\tau(t))\big)\,. \tag{2.5}$$

An integration from t to $\xi(t)$, yields

$$\begin{aligned} -r(t)\big[x^{(n-1)}(t)\big]^\gamma &\ge \int_t^{\xi(t)} q(s) f\Big(\frac{\delta(\tau(s))^{n-2}}{(n-2)!}\Big) f\big(x^{(n-2)}(\tau(s))\big)\,\mathrm{d}s \\ &\ge f\big(x^{(n-2)}(\tau(\xi(t)))\big) \int_t^{\xi(t)} q(s) f\Big(\frac{\delta(\tau(s))^{n-2}}{(n-2)!}\Big)\,\mathrm{d}s\,, \end{aligned} \tag{2.6}$$

where we have used the monotonicity of $f(x^{(n-2)}(\tau(t)))$. Consequently, $y(t) = x^{(n-2)}(t)$ is a positive solution of the delay differential inequality

$$y'(t) + r^{-1/\gamma}(t)\Big[\int_t^{\xi(t)} q(s) f\Big(\frac{\delta(\tau(s))^{n-2}}{(n-2)!}\Big)\,\mathrm{d}s\Big]^{1/\gamma} f^{1/\gamma}\big(y\big(\tau(\xi(t))\big)\big) \le 0\,.$$

It follows from Theorem 1 in [17], that the corresponding equation (E_1) has also a positive solution. A contradiction and we conclude that $x(t)$ cannot satisfy (C_1). $\square$

Theorem 2. *Let (1.2) hold. If for some constant $\delta\in(0,1)$, the first order differential equation (E_1) is oscillatory then (E) has property (B).*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (E). We may assume that $x(t) > 0$. It follows from Lemma 1 that $x(t)$ satisfies either (C_1) or (C_2).

First assume that n is odd. We shall show that $x(t)$ is strongly increasing, i.e. it satisfies (C_2). Assume the contrary, let $x(t)$ satisfies (C_1). Then it follows from Lemma 1 and (C_1) that $x'(t) > 0$. Therefore, evidently $\lim_{t\to\infty} x(t) \neq 0$. Thus, By Theorem 1, oscillation of (E_1) eliminate the case (C_1) and we conclude that (C_2) holds.

Now, we assume that n is even. We claim that $x(t)$ is strongly increasing or $\lim_{t\to\infty} x(t) = 0$. If we admit $\lim_{t\to\infty} x(t) \neq 0$, then, by Theorem 1 oscillation of (E_1) yields that $x(t)$ is strongly increasing. □

Applying criteria for oscillation of (E_1), we immediately obtain sufficient conditions for property (B) of (E).

Corollary 1. *Let* (1.2) *hold and* $\xi(t)$ *satisfy* (2.3)*. If*

$$f(u^{1/\gamma})/u \geq 1\,, \quad 0 < |u| \leq 1 \tag{2.7}$$

and for some $\delta \in (0,1)$

$$\liminf_{t\to\infty} \int_{\tau(\xi(t))}^{t} r^{-1/\gamma}(u)\Big[\int_u^{\xi(u)} q(s) f\Big(\frac{\delta\,\tau^{n-2}(s)}{(n-2)!}\Big)\,\mathrm{d}s\Big]^{1/\gamma}\mathrm{d}u > \frac{1}{\mathrm{e}}\,, \tag{2.8}$$

then (E) *has property (B).*

Proof. Since condition (2.8) guarantees oscillation of (E_1) the assertion follows from Theorem 2. □

For partial case of (E), we have another result.

Corollary 2. *Let* (1.2) *hold,* $\xi(t)$ *satisfy* (2.3)*,* β *be the ratio of two positive odd integers, and* $\beta < \gamma$*. If*

$$\limsup_{t\to\infty} \int_{\tau(\xi(t))}^{t} r^{-1/\gamma}(u)\Big[\int_u^{\xi(u)} q(s)\big(\tau^{n-2}(s)\big)^{\beta}\,\mathrm{d}s\Big]^{1/\gamma}\mathrm{d}u > 0\,, \tag{2.9}$$

then the differential equation

$$\big(r(t)\big[x^{(n-1)}(t)\big]^{\gamma}\big)' + q(t)x^{\beta}\big(\tau(t)\big) = 0 \tag{E^β}$$

has property (B).

We support our results with couple of illustrative examples. In the first example the condition (1.1) is relaxed, i.e. the opposite condition $r'(t) \leq 0$ holds.

Example 1. Consider the n-th order nonlinear differential equation

$$\Big(\frac{1}{t}\big(x^{(n-1)}(t)\big)^3\Big)' = \frac{b}{t^{3n-1}} x^3(\lambda t) \tag{2.10}$$

where $b > 0$, $0 < \lambda < 1$. We set $\xi(t) = \omega t$ with $\omega = \frac{1+\lambda}{2\lambda}$. Condition (2.8) now reduces to

$$b^{1/3}\lambda^{n-2}\Big(1 - \Big(\frac{2\lambda}{1+\lambda}\Big)^4\Big)^{1/3} \ln\frac{2}{1+\lambda} > \frac{4^{1/3}(n-2)!}{\mathrm{e}}\,, \tag{2.11}$$

which, by Corollary 1, guarantees that (2.10) enjoys property (B). What is more, by (2.1), every strongly increasing solution satisfies $|x(t)| \geq ct^{n+1/3}$, $c > 0$.

It is not difficult to verify that no matter, whether or not, n is even or odd, one strongly increasing solution of (2.10) is $x(t) = t^{\alpha}$, $\alpha > n - 2/3$ such that $[\alpha(\alpha-1)\dots(\alpha+2-n)]^3\,(3\alpha - 3n + 2) = b\lambda^{3\alpha}$.

Moreover, if n is even, one solution satisfying $\lim\limits_{t\to\infty} x(t) = 0$ is $x(t) = t^{-\beta}$, such that $[\beta(\beta+1)\dots(\beta+n-2)]^3\,(3\beta + 3n - 2) = b\lambda^{-3\beta}$.

Theorem 3. *Let $x(t)$ be a positive solution of (E), such that $\lim\limits_{t\to\infty} x(t) \neq 0$. If for some $\delta \in (0,1)$, the first order delay equation*

$$(E_2) \qquad y'(t) + r^{-1/\gamma}(t)\Big[\int_{t_1}^{t} q(s) f\Big(\frac{\delta\,\tau^{n-2}(s)}{(n-2)!}\Big)\,\mathrm{d}s\Big]^{1/\gamma} f^{1/\gamma}\big(y(\tau(t))\big) = 0$$

is oscillatory, then $x(t)$ does not satisfy (C_3).

Proof. Assume the contrary and suppose that $x(t)$ satisfies (C_3). By Lemma 2, we see that for every $\delta \in (0,1)$

$$x(t) \geq -\frac{\delta}{(n-2)!} t^{n-2} x^{(n-2)}(t)\,. \tag{2.12}$$

Combining (2.12) together with(E), we obtain

$$\big(r(t)\big[x^{(n-1)}(t)\big]^{\gamma}\big)' \geq q(t) f\Big(\frac{\delta\,\tau^{n-2}(t)}{(n-2)!}\Big) f\big(-x^{(n-2)}(\tau(t))\big)\,. \tag{2.13}$$

Integrating from t_1 to t, we have

$$\begin{aligned} r(t)\big[x^{(n-1)}(t)\big]^{\gamma} &\geq \int_{t_1}^{t} q(s) f\Big(\frac{\delta\,\tau^{n-2}(s)}{(n-2)!}\Big) f\big(-x^{(n-2)}(\tau(s))\big)\,\mathrm{d}s \\ &\geq f\big(-x^{(n-2)}(\tau(t))\big)\int_{t_1}^{t} q(s) f\Big(\frac{\delta\,\tau^{n-2}(s)}{(n-2)!}\Big)\,\mathrm{d}s\,. \end{aligned} \tag{2.14}$$

Then $y(t) = -x^{(n-2)}(t)$ is a positive solution of the delay differential inequality

$$y'(t) + r^{-1/\gamma}(t)\Big[\int_{t_1}^{t} q(s) f\Big(\frac{\delta\,\tau^{n-2}(s)}{(n-2)!}\Big)\,\mathrm{d}s\Big]^{1/\gamma} f^{1/\gamma}\big(y(\tau(t)\big) \leq 0\,.$$

By Theorem 1 in [17], the corresponding equation (E_2) has also a positive solution. A contradiction and we conclude that $x(t)$ cannot satisfy (C_3). $\square$

Now, we turn our attention to the case when (1.3) holds.

Theorem 4. *Let (1.3) hold. If for some constant $\delta \in (0,1)$, both the first order delay differential equations (E_1) and (E_2) are oscillatory, then every nonoscillatory solution $x(t)$ of (E) is strongly increasing or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (E). We may assume that $x(t) > 0$. It follows from Lemma 1 that $x(t)$ satisfies either (C_1) or (C_2) or (C_3).

Assume that $\lim_{t\to\infty} x(t) \neq 0$. Then by Theorems 1 and 3, oscillations of (E_1) and (E_2) exclude the cases (C_1) and (C_3), respectively. Therefore, we conclude that $x(t)$ has to satisfy (C_1), i.e. $x(t)$ is strongly increasing. □

Employing oscillation criteria for (C_1) and (C_3), we immediately obtain.

Corollary 3. *Let* (1.3) *and* (2.7) *hold and* $\xi(t)$ *satisfy* (2.3). *If for some* $\delta \in (0,1)$ (2.8) *is satisfied and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} r^{-1/\gamma}(u)\Big[\int_{t_1}^{u} q(s) f\Big(\frac{\delta\, \tau^{n-2}(s)}{(n-2)!}\Big)\,\mathrm{d}s\Big]^{1/\gamma} \mathrm{d}u > \frac{1}{\mathrm{e}}\,, \tag{2.15}$$

then every nonoscillatory solution $x(t)$ *of* (E) *is strongly increasing or satisfies* $\lim_{t\to\infty} x(t) = 0$.

We employ another illustrative example. While Example 1 has been intended to show that our result holds for the case when (1.1) is violated, the next example show that our technique is applicable also when (1.1) holds.

Example 2. Consider the n-th order nonlinear differential equation

$$\big(t^6\big(x^{(n-1)}(t)\big)^3\big)' = \frac{b}{t^{3n-8}} x^3(\lambda t) \tag{2.16}$$

with $b > 0$, $0 < \lambda < 1$. We set $\xi(t) = \omega t$ with $\omega = \frac{1+\lambda}{2\lambda}$. Conditions (2.8) and (2.15) reduce to

$$b^{1/3}\lambda^{n-2}\Big(\Big(\frac{1+\lambda}{2\lambda}\Big)^3 - 1\Big)^{1/3} \ln\frac{2}{1+\lambda} > \frac{3^{1/3}(n-2)!}{\mathrm{e}}\,, \tag{2.17}$$

$$b^{1/3}\lambda^{n-2} \ln\frac{1}{\lambda} > \frac{3^{1/3}(n-2)!}{\mathrm{e}}\,, \tag{2.18}$$

respectively. Corollary 3 guarantees that every nonoscillatory solution of (2.16) tis strongly increasing or satisfies $\lim_{t\to\infty} x(t) = 0$, provided that both conditions (2.17) and (2.18) hold.

3. Summary

In this paper, we have presented new comparison theorems for deducing the asymptotic behavior and oscillation of third order delay equation from the oscillation of a set of the suitable first order delay differential equation. Consequently, our method essentially simplifies the examination of the higher order equations and what is more, it supports backward the research on the first order delay differential equations. Our results here extend and complement latest ones mentioned below. The suitable illustrative examples are also provided.

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