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SOME NEW MODIFIED COSINE SUMS AND L^1-CONVERGENCE OF COSINE TRIGONOMETRIC SERIES

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Abstract. In this paper we introduce some new modified cosine sums and then using these sums we study L^1-convergence of trigonometric cosine series.

1. Introduction and preliminaries

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

be cosine trigonometric series and satisfy condition $a_k \to 0$, $k \to \infty$. The partial sum of series (1) we denote by $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$ and let be $f(x) = \lim_{n\to\infty} S_n(x)$.

A sequence (a_k) is said to belong to the class S, or briefly $a_k \in S$, if $a_k \to 0$ as $k \to \infty$, and there exists a sequence of numbers (A_k) such that

$$A_k \downarrow 0\,,$$

$$\sum_{k=1}^{\infty} A_k < \infty\,,$$

and

$$|\Delta a_k| \le A_k\,,$$

for all k, where $\Delta a_k = a_k - a_{k+1}$.

This class of sequences was defined by Sidon in [18] and by Telyakovskiĭ in [21], therefore the class S is sometimes called the Sidon-Telyakovskiĭ class. The class S is generalized later by Tomovski in [22] and by Leindler in [16].

Tomovski defined the class $S_r, r = 1, 2, \dots$ as follows: $\{a_k\}_{k=1}^{\infty} \in S_r$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}_{k=1}^{\infty}$ such that

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Dedicated to my Professor Halil Turku on the occasion of his 79th birthday.

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$\sum_{k=1}^{\infty} k^r A_k < \infty$ and $|\Delta a_k| \leq A_k$ for all k. There was noticed that from $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$ it follows $k^{r+1}A_k = o(1), k \to \infty$. It is clear that $S_{r+1} \subset S_r$ for all $r = 1, 2, \dots$ and for $r = 0$ we get the class $S_0 \equiv S$.

Garret and Stanojević [3] have introduced modified cosine sums

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx\,.$$

Garret and Stanojević [4], Ram [17], Singh and Sharma [20], and Kaur and Bhatia [11], [6], [10] studied the L^1-convergence of this cosine sum under different sets of conditions on the coefficients a_n.

Kumari and Ram [15] introduced new modified cosine and sine sums as

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\Big(\frac{a_j}{j}\Big) k \cos kx\,,$$

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\Big(\frac{a_j}{j}\Big) k \sin kx$$

and have studied their L^1-convergence under the condition that the coefficients a_n belong to different classes of sequences. They deduced some results about L^1-convergence of cosine and sine series as corollaries, as well.

N. Hooda, B. Ram and S. S. Bhatia [5] introduced new modified cosine sums as

$$R_n(x) = \frac{1}{2}\Big(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\Big) + \sum_{k=1}^{n}\Big(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\Big) \cos kx$$

and studied the L^1-convergence of these cosine sums.

K. Kaur [9] introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} \big(\Delta a_{j-1} - \Delta a_{j+1}\big) \sin kx\,,$$

and studied the L^1-convergence of this modified sine sum with semi-convex coefficients. Also, Kaur at al. [12] introduced a new class of numerical sequences as follows:

Definition 1. If $a_k = o(1)$ as $k \to \infty$, and

$$\sum_{k=1}^{\infty} k|\Delta^2 a_{k-1} - \Delta^2 a_{k+1}| < +\infty \qquad (a_0 = 0)$$

then we say that $\{a_k\}$ belongs to the class $\mathbf{K}$.

In their paper they proved the following result regarding to L^1-convergence of the modified sums $K_n(x)$.

Theorem 1. *Let the sequence $\{a_k\}$ belong to the class $\mathbf{K}$, then $K_n(x)$ converges to $f(x)$ in the L^1-norm.*

Later on, Singh and Kaur [19] defined new modified generalized sine sums

$$K_{nr}(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\left(\Delta^r a_{k-1} - \Delta^r a_{k+1}\right)\widetilde{S}_k^{r-1}(x),$$

and a new class of sequences:

Definition 2. Let α be a positive real number. If $a_k = o(1)$ as $k \to \infty$, and

$$\sum_{k=1}^{\infty} k^{\alpha}|\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_{k+1}| < +\infty \qquad (a_0 = 0)$$

then we say that $\{a_k\}$ belongs to the class $\mathbf{K}^{\alpha}$.

They proved the following generalization of Theorem 1.

Theorem 2. *Let the sequence $\{a_k\}$ belong to the class $\mathbf{K}^{\alpha}$, then $K_{nr}(x)$ converges to $f(x)$ in the L^1-norm.*

Some new modified sums are presented in [13] by present author (see also [14]) as follows

$$H_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1} - a_{j+1})\sin jx\right],$$

and also we have proved a new result as below.

Theorem 3. *Let (a_n) be a semi-convex null sequence, then $H_n(x)$ converges to $f(x)$ in L^1-norm.*

The interested reader can find some new results in very recently published papers, [7] where the complex form of the sums $K_n(x)$ is introduced, and paper [8] in which it is studied the L^1-convergence of sine trigonometric series by using a newly introduced modified cosine trigonometric sums under a new class of coefficient sequences (see [8] for details therein).

We recall that with regard to the L^1-convergence of Ress-Stanojević cosine sums $f_n(x)$ to a cosine trigonometric series, belonging to the class S, Ram [17] proved the following theorem:

Theorem 4. *If* (1.1) *belongs to the class S, then $\|f - f_n\|_{L^1} = o(1)$, $n \to \infty$.*

In order to make an advanced study, on this treating topic, now we shall introduce new modified cosine sums as

$$G_n(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n}\sum_{k_2=k_1}^{n}\sum_{k_3=k_2}^{n}\Delta^2\left(a_{k_3}\cos k_3 x\right),$$

where $\Delta^2 a_k = \Delta\left(\Delta a_k\right) = a_k - 2a_{k+1} + a_{k+2}$.

Remark 1. The advantage of introducing of the above modified cosine sums is the following: We have verified that the sums $G_n(x)$ converge in L^1-norm to $f(x)$, without a new class of null-sequences being defined, in contrary what the other authors previously did in their papers (as examples serve classes $\mathbf{K}$, $\mathbf{K}^{\alpha}$, etc.).

The purpose of this paper is to prove analogous statement with Theorem 4 using new modified cosine sums $G_n(x)$ instead of $g_n(x)$ and the L^1-convergence of the series (1.1) will be derived as a corollary.

As usual $D_n(x)$ will denote the real Dirichlet kernel, i.e.

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx \,.$$

For the proof of main result we need the following lemma.

Lemma 1 ([2]). *If $|c_k| \leq 1$, then*

$$\int_0^\pi \Big| \sum_{k=0}^{n} c_k \frac{\sin(k+1/2)\,x}{2\sin\frac{x}{2}} \Big| dx \leq C(n+1),$$

where C is a positive absolute constant.

2. Main results

We establish the following result.

Theorem 5. *Let (1.1) belong to the class S_2, then $\|f - G_n\|_{L^1} = o(1)$, as $n \to \infty$.*

Proof. We have

$$\begin{aligned}
G_n(x) &= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \Delta^2 \left(a_{k_3} \cos k_3 x\right) \\
&= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \big[\Delta\left(a_{k_2} \cos k_2 x\right) - \Delta\left(a_{k_2+1} \cos (k_2+1) x\right) \\
&\quad + \cdots + \Delta\left(a_n \cos nx\right) - \Delta\left(a_{n+1} \cos (n+1)\,x\right)\big] \\
&= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \big[\Delta\left(a_{k_2} \cos k_2 x\right) - \Delta\left(a_{n+1} \cos (n+1)\,x\right)\big] \\
&= \frac{a_0}{2} + \sum_{k_1=1}^{n} \big[a_{k_1} \cos k_1 x - a_{k_1+1} \cos (k_1+1) x + \cdots + a_n \cos nx \\
&\quad - a_{n+1} \cos (n+1)\,x\big] - \Delta\big(a_{n+1} \cos (n+1)\,x\big) \sum_{k_1=1}^{n} (n - k_1 + 1) \\
&= S_n(x) - n a_{n+1} \cos (n+1)\,x - \frac{1}{2} n(n+1) \Delta\big(a_{n+1} \cos (n+1)\,x\big) \\
&= S_n(x) - \frac{1}{2} n(n+3) a_{n+1} \cos (n+1)\,x \\
&\quad + \frac{1}{2} n(n+1) a_{n+2} \cos (n+2)\,x \,. \qquad (2.1)
\end{aligned}$$

From $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^2 A_k < \infty$ follows $k^3 A_k = o(1)$, $k \to \infty$, which gives $k^2 A_k = o(1)$, $k \to \infty$. Therefore from

$$0 \le n^2 |a_n| = n^2 \Big| \sum_{k=n}^{\infty} \Delta a_k \Big| \le \Big| \sum_{k=n}^{\infty} k^2 \Delta a_k \Big| \le \sum_{k=n}^{\infty} k^2 A_k = o(1)\,, \quad n \to \infty$$

follow

$$n^2 a_n = o(1), \quad n a_n = o(1), \qquad n \to \infty\,. \tag{2.2}$$

Also, $\cos(n+1)x$ and $\cos(n+2)x$ are finite in $[0,\pi]$ therefore from (2.1) and (2.2) we get

$$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} S_n(x) = f(x).$$

On the other side, using Abel's transformation we have

$$\begin{aligned} f(x) - G_n(x) &= \lim_{m\to\infty} \Big(\sum_{k=n+1}^{m-1} \Delta a_k D_k(x) + a_m D_m(x) - a_{n+1} D_n(x) \Big) \\ &\quad + \frac{1}{2} n(n+3) a_{n+1} \cos(n+1)\,x - \frac{1}{2} n(n+1) a_{n+2} \cos(n+2)\,x \\ &= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1} D_n(x) + \frac{1}{2} n(n+3) a_{n+1} \cos(n+1)\,x \\ &\quad - \frac{1}{2} n(n+1) a_{n+2} \cos(n+2)\,x\,. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\pi |f(x) - G_n(x)| dx &\le \int_0^\pi \Big| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \Big| dx + |a_{n+1}| \int_0^\pi \big| D_n(x) \big| dx \\ &\quad + \frac{1}{2} n(n+3) |a_{n+1}| \int_0^\pi \big| \cos(n+1)\,x \big|\, dx \\ &\quad + \frac{1}{2} n(n+1) |a_{n+2}| \int_0^\pi \big| \cos(n+2)\,x \big|\, dx \\ &:= \sum_{\nu=1}^{4} B_\nu(n)\,. \end{aligned} \tag{2.3}$$

Since $a_k \in S_2 \subset S_0 \equiv S$ then $\sum_{k=n+1}^{\infty} (k+1) \Delta A_k = o(1)$ as $n \to \infty$, therefore from this fact, Lemma 1, and using Abel's transformation we have

$$\begin{aligned} B_1(n) &= \int_0^\pi \Big| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \Big| dx \le \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \Big| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \Big| dx \\ &= O\Big(\sum_{k=n+1}^{\infty} (k+1) \Delta A_k \Big) = o(1)\,, \quad n \to \infty\,. \end{aligned} \tag{2.4}$$

By well-known Zygmund's theorem (see [20, p. 458]), for n sufficiently large, the following relation holds

$$\int_0^\pi \big|D_n(x)\big|\,dx \sim \log n\,,$$

therefore from the last relation and (2.2) we have

$$B_2(n) = |a_{n+1}|\log n \le n|a_{n+1}| = o(1)\,,\quad n\to\infty\,. \tag{2.5}$$

Moreover, from fact that integrals $\int_0^\pi \big|\cos(n+1)x\big|dx$, $\int_0^\pi \big|\cos(n+2)x\big|dx$ are bounded, and from relation (2.2) we conclude that

$$B_3(n) = O\big(n(n+3)|a_{n+1}|\big) = o(1),\quad n\to\infty \tag{2.6}$$

and similarly

$$B_4(n) = O\big(n(n+1)|a_{n+2}|\big) = o(1),\quad n\to\infty\,. \tag{2.7}$$

Finally, from (2.3)–(2.7) it follows that

$$\|f-G_n\|_{L^1} = o(1)\,,\quad n\to\infty\,.$$

The proof of the Theorem 5 is completed. □

Corollary 1. *Let* (1.1) *belong to the class* S_2*, then* $\|f-S_n\|_{L^1} = o(1)$ *as* $n\to\infty$.

Proof. From Theorem 5, and relations (2.6), (2.7), we have

$$\begin{aligned}\|f-S_n\|_{L^1} &= \|f-G_n+G_n-S_n\|_{L^1}\\ &\le \|f-G_n\|_{L^1} + \|G_n-S_n\|_{L^1}\\ &\le \|f-G_n\|_{L^1} + \frac{1}{2}n(n+3)|a_{n+1}|\int_0^\pi \big|\cos{(n+1)\,x}\big|\,dx\\ &\quad + \frac{1}{2}n(n+1)|a_{n+2}|\int_0^\pi \big|\cos{(n+2)\,x}\big|\,dx = o(1)\end{aligned}$$

as $n\to\infty$, which completely proves the corollary. □

Remark 2. A closer examination of the proofs of Theorem 5 and Corollary 1 reveals that condition $a_k\in S_2$ can be replaced by conditions $a_k\in S$ and $n^2|a_n| = o(1)$. This enables us to formulate Theorem 5 and Corollary 1 in the following form:

Theorem 6. *Let* (a_k) *belong to the class* S *and* $n^2|a_n| = o(1)$*, then* $\|f-G_n\|_{L^1} = o(1)$ *as* $n\to\infty$.

Corollary 2. *Let* (a_k) *belong to the class* S *and* $n^2|a_n| = o(1)$*, then* $\|f-S_n\|_{L^1} = o(1)$ *as* $n\to\infty$.

We would like to finalize this paper with a comment. We have noticed during this study that, if someone tries to introduce some modified sums of the form

$$T_{n,m}(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n}\sum_{k_2=k_1}^{n}\sum_{k_3=k_2}^{n}\cdots\sum_{k_m=k_{m-1}}^{n}\Delta^{m-1}\left(a_{k_m}/k_m\right)k_1\cos k_1x\,,$$

where $m \in N$, $m > 3$, $\Delta a_k = a_k - a_{k+1}$, $\Delta^{m-1} a_k = \Delta \left(\Delta^{m-2} a_k\right)$, which is a natural extension of our results, then several difficulties in the proof of the counterpart of Theorem 5 will be appeared. This is why we are focused only on the case $m = 3$.

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