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ON UNITARY CONVEX DECOMPOSITIONS OF VECTORS IN A JB^*-ALGEBRA

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Abstract. By exploiting his recent results, the author further investigates the extent to which variation in the coefficients of a unitary convex decomposition of a vector in a unital JB^*-algebra permits the vector decomposable as convex combination of fewer unitaries; certain C^*-algebra results due to M. Rørdam have been extended to the general setting of JB^*-algebras.

Introduction

It is well known that the class of JB^*-algebras includes all C^*-algebras (cf. [15]). The present author developed a theory of unitary isotopes of a unital JB^*-algebra and by applying the same he has obtained various results on the geometry of the unit ball in a JB^*-algebra with his special interest in studying decompositions of vectors as convex combinations of unitaries, called unitary convex decompositions (cf. [6]–[14]); these include generalization of some well known results due to B. Russo, H. A. Dye, R. V. Kadison, C. L. Olsen, G. K. Pedersen and M. Rørdam. In turn, we proved in [7] generalization of some C^*-algebra results on asymmetric decompositions of vector, due to R. V. Kadison and G. K. Pedersen appeared in [3], for general JB^*-algebras. Now, with our recent results appeared in [6, 8, 14], we further investigate the extent to which variation in the coefficients of a unitary convex decomposition of a vector in a unital JB^*-algebra $\mathcal{J}$ permits the vector decomposable as convex combination of fewer unitaries in $\mathcal{J}$. In the course of our analysis we also obtain extension of certain C^*-algebra results due to M. Rørdam to the general setting of JB^*-algebras. As we have seen in [6, 8, 14], the set $\mathcal{V}(x)$ (see below) plays a basic role in the study of convex combinations of unitaries; this set is an interval and also gives major information about those convex combinations of unitaries which can represent an element of the unit ball of a JB^*-algebra. Inspired by the work of M. Rørdam [5], we give some further results about the set $\mathcal{V}(x)$ in relation to the unitary convex decompositions of vectors in the unit ball of a unital JB^*-algebra.

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Isotopes of Jordan algebras

We recall (from [2], for instance) that a commutative (not necessarily associative) algebra $(\mathcal{J},\circ)$ over a field of characteristic $\neq 2$ is called a Jordan algebra if for all x, $y \in \mathcal{J}$, $x^2 \circ (x \circ y) = (x^2 \circ y) \circ x$; a Jordan algebra with unit is called unital. An element x in a Jordan algebra $\mathcal{J}$ with unit e is said to be invertible if there exists $x^{-1} \in \mathcal{J}$, called the inverse of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ is symbolized as $\mathcal{J}_{\mathrm{inv}}$.

For any Jordan algebra $\mathcal{J}$ and $x \in \mathcal{J}$, the x-homotope of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear space structure as $\mathcal{J}$ but a different product, denoted by "$\cdot_x$", defined by $a \cdot_x b = \{axb\}$ for all $a, b \in \mathcal{J}$, where $\{pqr\}$ denotes the Jordan triple product of p, q, r given by $\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p$.

If $\mathcal{J}$ is a unital Jordan algebra and $x \in \mathcal{J}_{\mathrm{inv}}$, then by x-isotope of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, we mean the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$; clearly, x acts as the unit for $\mathcal{J}^{[x]}$. For any invertible element a in $\mathcal{J}$ (unital Jordan algebra), $\mathcal{J}_{\mathrm{inv}} = \mathcal{J}^{[a]}_{\mathrm{inv}}$ (see [11, Lemma 4.2]); thus, the set of invertible elements in a unital Jordan algebra is invariant on passage to any of its isotopes.

Unitary isotopes of JB^*-algebras

A real or complex Jordan algebra $\mathcal{J}$ with product "$\circ$" is called a Banach Jordan algebra if there is a complete norm $\|\cdot\|$ on $\mathcal{J}$ satisfying $\|a \circ b\| \leq \|a\|\|b\|$ for all a, $b \in \mathcal{J}$. If, in addition, $\mathcal{J}$ has a unit e with $\|e\| = 1$ then it is called a unital Banach Jordan algebra; throughout the sequel, we will only be considering complex unital Banach Jordan algebras.

In this article, we are interested in a special class of Banach Jordan algebras, called JB^*-algebras (originally, Jordan C^*-algebras); this includes all C^*-algebras as a proper subclass: a complex Banach Jordan algebra $\mathcal{J}$ with involution "$*$" is called a JB^*-algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$ (cf. [15, 16]). We denote the open and closed unit balls in a JB^*-algebra $\mathcal{J}$ by $(\mathcal{J})_1^\circ$ and $(\mathcal{J})_1$, respectively. A JB^*-algebra $\mathcal{J}$ is said to be of topological stable rank 1 (in short, tsr 1) if $\mathcal{J}_{\mathrm{inv}}$ is norm dense in $\mathcal{J}$; for some interesting properties of such algebras see [11].

An element u in a unital JB^*-algebra $\mathcal{J}$ is called a unitary if $u^* = u^{-1}$ (the inverse of u); in such a case, $\|u\| = 1$. The set of all unitary elements in a JB^*-algebra $\mathcal{J}$ is denoted by $\mathcal{U}(\mathcal{J})$; the algebraic convex hull of $\mathcal{U}(\mathcal{J})$ is symbolized by $\mathrm{co}\,\mathcal{U}(\mathcal{J})$ and its norm closure by $\overline{\mathrm{co}}\,\mathcal{U}(\mathcal{J})$. For any unitary element u of JB^*-algebra $\mathcal{J}$, the isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of $\mathcal{J}$; which indeed is a JB^*-algebra with u as its unit with respect to the original norm and the new involution $*_u$ given as $x^{*_u} = \{ux^*u\}$ (see [1], for instance). For any unitary element u in unital JB^*-algebra $\mathcal{J}$, $\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]})$ (see [11, Theorem 4.6]); thus, the set of unitaries is invariant on passage to unitary isotopes.

Unitary convex decompositions

In papers [6] and [7], the author presented several applications of the theory of unitary isotopes of JB^*-algebras; these include a new proof of the celebrated

Russo-Dye theorem for JB^*-algebras and various other results on means and convex combinations of unitaries.

For any element x in a unital JB^*-algebra $\mathcal{J}$, The numbers $u_c(x)$ and $u_m(x)$ are defined by

$$u_c(x) = \min\Big\{n : x = \sum_{j=1}^{n} \alpha_j u_j \quad \text{with} \quad u_j \in \mathcal{U}(\mathcal{J}),\ \alpha_j \geq 0,\ \sum_{j=1}^{n} \alpha_j = 1\Big\},$$

$$u_m(x) = \min\Big\{n : x = \frac{1}{n}\sum_{j=1}^{n} u_j,\ u_j \in \mathcal{U}(\mathcal{J})\Big\}.$$

If x has no unitary convex decomposition, the $u_c(x)$ is defined to be the ∞. Indeed, each convex combination of unitaries in a unital JB^*-algebra $\mathcal{J}$ is the mean of the same number of unitaries in $\mathcal{J}$. Hence, $u_m(x) = u_c(x)$ (see [13, Corollary 4.5], also see [3, Corollary 15] for C^*-algebras); this number is called unitary rank of x and is symbolized simply by $u(x)$.

Let $\mathcal{J}$ be a unital JB^*-algebra. For every positive integer n, the set $\mathrm{co}_n\,\mathcal{U}(\mathcal{J})$ in $\mathcal{J}$ is defined by

$$\mathrm{co}_n\,\mathcal{U}(\mathcal{J}) = \Big\{\sum_{i=1}^{n} \alpha_i u_i :\ u_i \in \mathcal{U}(\mathcal{J}), \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\Big\}.$$

Hence, $\mathrm{co}_n\,\mathcal{U}(\mathcal{J}) = \{x \in \mathcal{J} : u(x) \leq n\}$. Another related set $\mathrm{co}_{n+}\,\mathcal{U}(\mathcal{J})$ is defined as the set of vectors x in $\mathcal{J}$ such that for each real number $\epsilon > 0$ there is a convex decomposition $\sum_{i=1}^{n+1} \alpha_i u_i$ of x with $u_i \in \mathcal{U}(\mathcal{J})$ and $\alpha_{n+1} < \epsilon$.

Asymmetric decompositions

Inspired by the work of R. V. Kadison and G. K. Pedersen [3] and M. Rørdam [5]. In [7] the present author began a study of asymmetric decompositions of vectors in a unitary JB^*-algebra. From [7], we know the following results on unitary convex decompositions (see [3], for C^*-algebra case):

Lemma 1. *Let $\mathcal{J}$ be a unital JB^*-algebra and $x \in \mathcal{J}$ such that $\|x\| \leq 1-\epsilon$ for some $\epsilon \in (0, (n+1)^{-1})$. If $\mathrm{dist}(x, \mathrm{co}_n\,\mathcal{U}(\mathcal{J})) < \frac{\epsilon^2}{1-\epsilon}$, then there exist unitaries $u_i \in \mathcal{U}(\mathcal{J})$, $i = 1, \ldots, n+1$, such that*

$$x = \sum_{i=1}^{n} \alpha_i u_i + \epsilon u_{n+1}\,,$$

where $\alpha_k \geq 0$ and $\sum_{i=1}^{n} \alpha_i + \epsilon = 1$.

Lemma 2. *For any unital JB^*-algebra $\mathcal{J}$,*

$$(\mathcal{J})_1^\circ \cap \bar{\mathrm{co}}_n\mathcal{U}(\mathcal{J}) = (\mathcal{J})_1^\circ \cap \mathrm{co}_{n+}\,\mathcal{U}(\mathcal{J})\,.$$

By using the above Lemmas 1 and 2, the author obtained the following generalization of [3, Proposition 18]; this gives us two characterizations of JB^*-algebras of tsr 1 (see [7, Theorem 11]):

Theorem 3. *For any unital JB^*-algebra $\mathcal{J}$, the following statements are equivalent:*

(i) $\mathcal{J}$ *is of* tsr 1*;*
(ii) $\frac{1}{2}\mathcal{U}(\mathcal{J}) + \frac{1}{2}\mathcal{U}(\mathcal{J})$ *is norm dense in* $(\mathcal{J})_1$*;*
(iii) $(\mathcal{J})_1^\circ \subseteq \mathrm{co}_{2+}\mathcal{U}(\mathcal{J})$.

Generally, it is not possible to replace $\mathrm{co}_{2+}\mathcal{U}(\mathcal{J})$ by $\mathrm{co}_2\mathcal{U}(\mathcal{J})$ in the statement (iii) of the above theorem: this follows from the fact that any C^*-algebra can be considered as a JB^*-algebra and the illustration given with the C^*-algebra of convergent complex number sequences, by Kadison and Pedersen in [3].

For vectors of unitary rank ≥ 3, we know the following fact from [7, Theorem 13]:

Theorem 4. *Let $\mathcal{J}$ be a unital JB^*-algebra and $x \in \mathcal{J}$ such that $u(x) = n \geq 3$. Suppose $x = \sum_{i=1}^n \alpha_i u_i$, where $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{J})$ and $\alpha_1, \ldots, \alpha_n$ are non-negative real numbers with sum equal to 1. Then*

(i) $\alpha_i \leq \alpha_j + \alpha_k$, *(for $j \neq k$);*
(ii) $\frac{1}{n-1} \leq \alpha_j + \alpha_k$, *(for $j \neq k$);*
(iii) $\alpha_j \leq \frac{2}{n+1}$, $\forall j$.

More about asymmetric decompositions

We continue the study of asymmetric decompositions. By using our recent results appeared in [6, 8, 14], we keep on investigating the extent to which variation in the coefficients of a unitary convex decomposition of a vector in a unital JB^*-algebra $\mathcal{J}$ permits the vector decomposable as convex combination of fewer unitaries in $\mathcal{J}$. In the sequel, we extend certain C^*-algebra results due to M. Rørdam to the general setting of JB^*-algebras.

Let $\mathcal{J}$ be a unital JB^*-algebra. For each real number $\delta \geq 1$, consider the subset $\mathrm{co}_\delta\mathcal{U}(\mathcal{J})$ of $\mathrm{co}\,\mathcal{U}(\mathcal{J})$ given by

$$\mathrm{co}_\delta\mathcal{U}(\mathcal{J}) = \left\{\delta^{-1}\sum_{i=1}^{n-1} u_i + \delta^{-1}(1+\delta-n)u_n : \ u_j \in \mathcal{U}(\mathcal{J}), j = 1, \ldots, n\right\},$$

where n is the integer given by $n - 1 < \delta \leq n$. Clearly, $\mathrm{co}_\delta\mathcal{U}(\mathcal{J}) = \mathrm{co}_n\mathcal{U}(\mathcal{J})$ if $\delta = n$. For any element $x \in (\mathcal{J})_1$, we can define the set $\mathcal{V}(x)$ as follows:

$$\mathcal{V}(x) = \{\beta \geq 1 \ : \ x \in \mathrm{co}_\beta\mathcal{U}(\mathcal{J})\}.$$

As mentioned above, the set $\mathcal{V}(x)$ is an interval in the real line and it plays a vital role in the study of convex combinations of unitaries (cf. [6, 8, 14]). We shall see below that the set $\mathcal{V}(x)$ also gives major information about those convex combinations of unitaries which can represent an element of the unit ball of a JB^*-algebra.

Theorem 5. *Let $(a_1, \ldots, a_m) \in \Re^m$, $a_j \geq 0$ and $\sum_{j=1}^m a_j = 1$. Let $x \in (\mathcal{J})_1$ and $a_o = \max\{a_1, \ldots, a_m\}$.*

(i) *If $\mathcal{V}(x) = [\gamma, \infty)$, then there exist $u_1, \ldots, u_m \in \mathcal{U}(\mathcal{J})$ such that $x = \sum_{j=1}^m a_j u_j$ if and only if $a_o \leq \gamma^{-1}$.*

(ii) *If $\mathcal{V}(x) = (\gamma, \infty)$, then there exist $u_1, \ldots, u_m \in \mathcal{U}(\mathcal{J})$ such that $x = \sum_{j=1}^m a_j u_j$ if $a_o < \gamma^{-1}$ and only if $a_o \leq \gamma^{-1}$.*

Proof. Without any loss of generality, we suppose $a_o = a_1 \geq a_2 \geq \ldots \geq a_m$. Assuming $x = \sum_{j=1}^m a_j u_j$ for some $u_j \in \mathcal{U}(\mathcal{J})$, we have $\|a_1^{-1}x - u_1\| \leq a_1^{-1}(a_2 + \ldots + a_m) = a_1^{-1}(1 - a_1) = a_1^{-1} - 1$. Hence, by [14, Theorem 2.2], $(a_1^{-1}, \infty) \subseteq \mathcal{V}(x)$ so that $a_o \leq \gamma^{-1}$. For the other hand, suppose $a_o \leq \gamma^{-1}$ if $\mathcal{V}(x) = [\gamma, \infty)$ (and $a_o < \gamma^{-1}$ if $\mathcal{V}(x) = (\gamma, \infty)$). Let $n \in \mathcal{N}$ such that $n - 1 < a_o^{-1} \leq n$. Since $ma_o \geq 1$, $n \leq m$ and $x = a_o(v_1 + \ldots + v_{n-1}) + a_o(a_o^{-1} + 1 - n)v_n + 0v_{n+1} + \ldots + 0v_m = b_1 v_1 + \ldots + b_m v_m$ (say) where $b_{n+1} = \ldots = b_m = 0$. By [3, Lemma 13] and [13, Theorem 4.4], there exist $u_1, \ldots, u_m \in \mathcal{U}(\mathcal{J})$ with $x = \sum_{j=1}^m a_j u_j$ if $\sum_{j=1}^k a_j \leq \sum_{j=1}^k b_j$ for all $k = 1, \ldots, m-1$. However, $\sum_{j=1}^k a_j \leq 1 = \sum_{j=1}^k b_j$ for $k \geq n$; otherwise $\sum_{j=1}^k a_j \leq ka_o = \sum_{j=1}^k b_j$. □

From the examples appearing in [5, Remark 3.13]), we know the existence of at least two operators x with $\mathcal{V}(x) = (2, \infty)$: one of the operators has no representation as a convex combination of unitaries having $\frac{1}{2}$ as a coefficient; whereas in case of the second operator $\frac{1}{2}$ appears as a coefficient in such a representation.

The next result gives us more information in determining the convex combinations which represent a given element of the unit ball of a JB^*-algebra.

Theorem 6. *Let $(a_1, \ldots, a_n) \in \Re^n$ and $(b_1, \ldots, b_m) \in \Re^m$ be such that $a_j \geq 0$, $b_j \geq 0$ and $\sum_{j=1}^n a_j = \sum_{k=1}^m b_k = 1$. Let $a_o = \max\{a_1, \ldots, a_n\}$ and $b_o = \max\{b_1, \ldots, b_m\}$. Then for any unital JB^*-algebra $\mathcal{J}$,*

$$a_1\mathcal{U}(\mathcal{J}) + \ldots + a_n\mathcal{U}(\mathcal{J}) \subseteq b_1\mathcal{U}(\mathcal{J}) + \ldots + b_m\mathcal{U}(\mathcal{J})$$

$$\textit{if} \quad a_o > b_o \quad \textit{or if} \quad a_o = b_o \quad \textit{and} \quad l \geq l'$$

where $l = \text{card}\{j : a_j = a_o\}$ and $l' = \text{card}\{k : b_k = b_o\}$.

Proof. Without any loss of generality, assume $a_o = a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_o = b_1 \geq b_2 \geq \ldots \geq b_m$. Let $x \in a_1\mathcal{U}(\mathcal{J}) + \ldots + a_n\mathcal{U}(\mathcal{J})$. Then there exist $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{J})$ such that $x = \sum_{j=1}^n a_j u_j$. So, $\|a_1^{-1}x - u_1\| \leq a_1^{-1} - 1$ since $a_1 \geq a_2 \geq \ldots \geq a_n$. Hence, by [14, Theorem 2.2], $(a_o^{-1}, \infty) \subseteq \mathcal{V}(x)$. If $a_o > b_o$ then, by previous theorem, $x = b_1\mathcal{U}(\mathcal{J}) + \ldots + b_m\mathcal{U}(\mathcal{J})$. Now, assume $a_o = b_o$ and $l \geq l'$. Then $\sum_{i=l'+1}^n a_i = 1 - l'b_o = \sum_{i=l'+1}^m b_i$ and there exist $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{J})$ such that

$$\text{(i)} \qquad x = \sum_{i=1}^n a_i u_i = b_o \sum_{j=1}^{l'} u_j + (b_{l'+1} + \ldots + b_m)y$$

where $y = (1 - l'a_o)^{-1}(a_{l'+1}u_{l'+1} + \ldots + a_n u_n) \in (\mathcal{J})_1$. For any $k > l'$, we have $0 \leq b_o^{-1}b_k < 1$ and so by [13, Theorem 2.6] (repeating $m - l'$ times) we get

$$\text{(ii)} \qquad u_{l'} + (b_o^{-1}b_{l'+1} + \ldots + b_o^{-1}b_m)y = v_{l'} + b_o^{-1}b_{l'+1}v_{l'+1} + \ldots + b_o^{-1}b_m v_m$$

with $v_{l'}, \ldots, v_m \in \mathcal{U}(\mathcal{J})$. It follows from above observations (i) and (ii) that $x \in b_1\mathcal{U}(\mathcal{J}) + \ldots + b_m\mathcal{U}(\mathcal{J})$. □

In [5], Rørdam remarks that for certain C^*-algebras "if" in the above theorem (the C^*-algebra version) can be replaced with "if and only if"; for more details see [4].

Corollary 7. *For any unital JB^*-algebra $\mathcal{J}$, if $a_j \geq 0$ and $b_k \geq 0$ with $\sum_{j=1}^n a_j = 1 = \sum_{k=1}^m b_k$, then one of the sets $a_1\mathcal{U}(\mathcal{J})+\ldots+a_n\mathcal{U}(\mathcal{J})$ and $b_1\mathcal{U}(\mathcal{J})+\ldots+b_m\mathcal{U}(\mathcal{J})$ contains the other.*

Proof. Follows immediately from the previous theorem. □

Next, we consider the extension $\mathrm{co}_{\gamma+}\,\mathcal{U}(\mathcal{J})$ of the construct $\mathrm{co}_{n+}\,\mathcal{U}(\mathcal{J})$ (see above) given for any number $\gamma \geq 1$ by $\mathrm{co}_{\gamma+}\,\mathcal{U}(\mathcal{J}) = \cap_{\delta>\gamma}\,\mathrm{co}_\delta\,\mathcal{U}(\mathcal{J})$.

From [14, Corollary 2.3], it follows that $\mathrm{co}_\gamma\,\mathcal{U}(\mathcal{J}) \subseteq \mathrm{co}_{\gamma+}\,\mathcal{U}(\mathcal{J})$, and $x \in \mathrm{co}_{\gamma+}\,\mathcal{U}(\mathcal{J})$ if and only if $(\gamma, \infty) \subseteq \mathcal{V}(x)$.

Next, we need to recall the following result on invertibles appeared in our paper [14, Corollary 2.8]:

Lemma 8. *If x is an invertible element of the unit ball in a JB^*-algebra, then $\mathcal{V}(x) = [2(1 + \|x^{-1}\|^{-1})^{-1}, \infty)$; in particular, the interval $\mathcal{V}(x)$ is closed.*

Theorem 9. *Let $\mathcal{J}$ be a unital JB^*-algebra. Let $(a_1, \ldots, a_n) \in \Re^n$ be such that $a_j \geq 0$ and $\sum_{j=1}^n a_j = 1$, and put $a = \max\{a_1, \ldots, a_n\}$.*
(i) $\mathrm{co}_{a^{-1}}\,\mathcal{U}(\mathcal{J}) \subseteq a_1\mathcal{U}(\mathcal{J}) + \ldots + a_n\mathcal{U}(\mathcal{J}) \subseteq \mathrm{co}_{a^{-1}+}\,\mathcal{U}(\mathcal{J})$.
(ii) *For $1 \leq \delta < 2$, $\mathrm{co}_\delta\,\mathcal{U}(\mathcal{J}) = \bar{\mathrm{co}}_{\delta+}\mathcal{U}(\mathcal{J})$.*

Proof. (i) Follows immediately from Theorem 6.

(ii) We observe that $x \in \mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J})$ together with $1 \leq \delta < 2$ gives $x = \alpha u_1 + (1-\alpha)u_2$ for some unitaries $u_1, u_2 \in \mathcal{U}(\mathcal{J})$ with $0 \leq \alpha < \frac{1}{2}$, by the construction of $\mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J})$. So that $\|x - u_2\| < 1$ since $\|u\| = 1$ for all $u \in \mathcal{U}(\mathcal{J})$. Hence, $x \in \mathcal{J}_{\mathrm{inv}}^{[u_2]}$ by the construction of the unitary isotope $\mathcal{J}^{[u_2]}$ (see above) and [11, Lemma 2.1]. This together with [11, Lemma 4.2] gives that $x \in \mathcal{J}_{\mathrm{inv}}$. Thus, $\mathcal{V}(x)$ is a closed interval by the previous lemma. However, $x \in co_{\delta+}\mathcal{U}(\mathcal{J})$ clearly gives the set inclusion $(\delta, \infty) \subseteq \mathcal{V}(x)$. Therefore, $[\delta, \infty) \subseteq \mathcal{V}(x)$; this in turn gives $x \in \mathrm{co}_\delta\,\mathcal{U}(\mathcal{J})$. Thus, the required result follows from [14, Corollary 2.9]. □

The next couple of results describe the set $\mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J})$ for the case $\delta \geq 2$. Here, $\alpha(x)$ denotes the distance from vector $x \in \mathcal{J}$ to the set $\mathcal{J}_{\mathrm{inv}}$ of invertibles in the JB^*-algebra $\mathcal{J}$; see [11] for some of its properties.

Theorem 10. *Let $\mathcal{J}$ be a unital JB^*-algebra and $\delta \geq 2$. Then:*
(i) $\mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J}) \subseteq \{x \in (\mathcal{J})_1 : \alpha(x) \leq 1 - 2\delta^{-1}\}$.
(ii) $\mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J}) \subseteq \bar{\mathrm{co}}_\delta\mathcal{U}(\mathcal{J})$.

Proof. (i) Suppose $x \in (\mathcal{J})_1$ and $\alpha(x) > 1 - 2\delta^{-1} \geq 0$. Then x is non-invertible (otherwise, $\alpha(x)$ must be zero). From this and the fact $\alpha \leq \|x\|$ (cf. [11]), we get $\mathcal{V}(x) \subseteq [\beta_x, \infty)$ with $\beta_x = 2(1 - \alpha(x))^{-1}$ by [8, Theorem 30]. However, $\delta < \beta_x$. Therefore, $(\delta, \infty) \not\subseteq \mathcal{V}(x)$. This proves the part (i).
(ii) Let $x \in \mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J})$ and let $m \in \mathcal{N}$ be given by $m \leq \delta < m+1$. Let $\epsilon > 0$ and choose $\gamma > \delta$ such that $|\gamma - \delta| < (2m)^{-1}\delta^2\epsilon$ and $\gamma < m+1$. It follows from the definition of $\mathrm{co}_{\delta+}\,\mathcal{U}(\mathcal{J})$ that $x \in \mathrm{co}_\gamma\,\mathcal{U}(\mathcal{J})$, and hence $x = \gamma^{-1}(\sum_{j=1}^m u_j + (\gamma -$

$m)u_{m+1})$ with $u_1, \dots, u_{m+1} \in \mathcal{U}(\mathcal{J})$. Setting $y = \delta^{-1}(\sum_{j=1}^m u_j + (\delta - m)u_{m+1})$, we see that $y \in \mathrm{co}_\delta\, \mathcal{U}(\mathcal{J})$ satisfying

$$\|x - y\| \le m|\delta^{-1} - \gamma^{-1}| + |\gamma^{-1}(\gamma - m) - \delta^{-1}(\delta - m)|$$

$$= (\delta\gamma)^{-1}(m|\gamma - \delta| + |\delta(\gamma - m) - \gamma(\delta - m)|) \le 2m\delta^{-2}|\gamma - \delta| < \epsilon\,.$$

From this, we immediately conclude $x \in \bar{\mathrm{co}}_\delta \mathcal{U}(\mathcal{J})$. □

From [14], we recall that the λ_u-function is defined on the closed unit ball $(\mathcal{J})_1$ of unital JB^*-algebra $\mathcal{J}$ as follows:

$$\lambda_u(x) := \sup\{0 \le \lambda \le 1 : x = \lambda v + (1 - \lambda)y \quad \text{with} \quad v \in \mathcal{U}(\mathcal{J}), y \in (\mathcal{J})_1\}\,,$$

for all $x \in (\mathcal{J})_1$. In [6], we introduced the following Λ-conditions for elements x in a JB^*-algebra, where β_x denotes the number $2(1 - \alpha(x))^{-1}$:
(Λ_1): $(\beta_x, \infty) \subseteq \mathcal{V}(x)$.
(Λ_2): $(\lambda_u(x))^{-1} = \inf \mathcal{V}(x) = \beta_x$.
(Λ_3): For all $\gamma > \beta_x$, there exists $u \in \mathcal{U}(\mathcal{J})$ such that $\|\gamma x - u\| \le \gamma - 1$.
(Λ_4): $\lambda_u(x) \ge \beta_x^{-1}$.

For any $x \in (\mathcal{J})_1$ with $\alpha(x) < 1$, the above conditions are equivalent to each other (see [6, Theorem 3.6]). The next result improves the previous theorem, provided that any one of the above conditions (Λ_1)–(Λ_4) is satisfied:

Theorem 11. *Let $\mathcal{J}$ be a unital JB^*-algebra such that for any $x \in (\mathcal{J})_1$ with $\alpha(x) < 1$, x satisfies any one of the conditions (Λ_1)–(Λ_4) and let $\delta \ge 2$. Then*
(i) $\mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J}) = \{x \in (\mathcal{J})_1 : \alpha(x) \le 1 - 2\delta^{-1}\}$.
(ii) $\bar{\mathrm{co}}_\delta \mathcal{U}(\mathcal{J}) = \mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J})$.

Proof. (i) Let $x \in (\mathcal{J})_1$ and $\alpha(x) \le 1 - 2\delta^{-1}$ with $\delta \ge 2$. If x is invertible, then $(\delta, \infty) \subseteq \mathcal{V}(x)$ by [14, Corollary 2.8]. On the other hand, if x is non-invertible, then under the hypothesis we have $(\delta, \infty) \subseteq (\beta_x, \infty) \subseteq \mathcal{V}(x)$ because $\delta \ge \beta_x$. Thus, in any case, $x \in \mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J})$ by the construction of $\mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J})$ (see above). Hence, the part (i) follows from the previous theorem.
(ii) If $\delta \ge 2$, then by the part (i) we have $\mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J}) = \{x \in (\mathcal{J})_1 : \alpha(x) \le 1 - 2\delta^{-1}\}$. By the continuity of $\alpha(x)$ (cf. [11, Lemma 6.2]), the set $\{x \in (\mathcal{J})_1 : \alpha(x) \le 1 - 2\delta^{-1}\}$, and hence also the $\mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J})$ are closed. We conclude that $\bar{\mathrm{co}}_\delta \mathcal{U}(\mathcal{J}) = \mathrm{co}_{\delta+}\, \mathcal{U}(\mathcal{J})$. □

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