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ON THE KOLÁŘ CONNECTION

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To the memory of my father Jan Mikulski on his 100th birthday

Abstract. Let $Y \to M$ be a fibred manifold with m-dimensional base and n-dimensional fibres and $E \to M$ be a vector bundle with the same base M and with n-dimensional fibres (the same n). If $m \geq 2$ and $n \geq 3$, we classify all canonical constructions of a classical linear connection $A(\Gamma, \Lambda, \Phi, \Delta)$ on Y from a system $(\Gamma, \Lambda, \Phi, \Delta)$ consisting of a general connection Γ on $Y \to M$, a torsion free classical linear connection Λ on M, a vertical parallelism $\Phi\colon Y \times_M E \to VY$ on Y and a linear connection Δ on $E \to M$. An example of such $A(\Gamma, \Lambda, \Phi, \Delta)$ is the connection $(\Gamma, \Lambda, \Phi, \Delta)$ by I. Kolář.

0. Introduction

A general connection on a fibred manifold $Y \to M$ is a section $\Gamma\colon Y \to J^1Y$ of the first jet prolongation J^1Y of $Y \to M$. Equivalently, $\Gamma\colon Y \times_M TM \to TY$ is a lifting map or a projection tensor field $\Gamma\colon TY \to TY$ or it is a decomposition $TY = VY \oplus H^\Gamma Y$, e.t.c. If Y is a vector bundle and $\Gamma\colon Y \to J^1Y$ is a vector bundle map (over id_M), then Γ is called a linear connection on $Y \to M$. A linear connection on $Y = TM \to M$ (the tangent bundle of M) is called a classical linear connection on M. There are several equivalent definitions of classical linear connection on M (a differentiation $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, a right invariant connection $PM \to J^1PM$ on the linear frame bundle PM of M, a system of Christoffel symbols, e.t.c.). A classical linear connection ∇ is torsion free if its torsion tensor $T(X_1, X_2) = \nabla_{X_1}X_2 - \nabla_{X_2}X_1 - [X_1, X_2]$ is equal to 0.

If N is a manifold and V is a vector bundle, $\dim(N) = \dim(V)$, a parallelism on N, is a fibred diffeomorphism $P\colon N \times V \to TN$ over id_N such that for any $z \in N$ the map $P_z\colon V \to T_zN$, $P_z(v) = P(z, v)$, is linear.

If $Y \to M$ is a fibred manifold and $E \to M$ is a vector bundle such that $\dim Y_x = \dim E_x$, $x \in M$, a vertical parallelism on $Y \to M$ is a vector bundle isomorphism $\Phi\colon Y \times_M E \to VY$, i.e. it is a system of parallelism $\Phi_x\colon Y_x \times E_x \to TY_x$ on Y_x for any $x \in M$.

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In [4], I. Kolář constructed a classical linear connection $\Psi = (\Gamma, \Lambda, \Phi, \Delta)\colon TY \to J^1(TY \to Y)$ from a system consisting of a general connection $\Gamma\colon Y \to J^1Y$ on $Y \to M$, a classical linear connection $\Lambda\colon TM \to J^1(TM \to M)$ on M, a vertical parallelism $\Phi\colon Y \times_M E \to VY$ on Y and a linear connection $\Delta\colon E \to J^1(E \to M)$ on $E \to M$ as follows. "We decompose $Z \in T_yY$ into the horizontal part $h(Z) = \Gamma(y, Z_o)$, $Z_o \in T_xM$, $x = p(y)$ and the vertical part $vZ = \Phi(y, Z_1)$, $Z_1 \in E_x$. We take a vector field X on M such that $j^1_xX = \Lambda(Z_o)$ and construct its Γ-lift $\Gamma X\colon Y \to TY$. Further, we consider a section s of E such that $j^1_xs = \Delta(Z_1)$. For every $Z \in T_yY$ we define

$$\psi(Z) = j^1_y\bigl(\Gamma X + \varphi(s)\bigr) \,."$$

Here $\varphi(s)\colon Y \to VY$ is defined by $\varphi(s)(y) = \Phi(y, s(p(y)))$.

The above construction is a generalization of the construction H of a classical linear connection $H(D, \Lambda)$ on E from a linear connection D in a vector bundle $E \to M$ by means of a classical linear connection Λ on M presented by J. Gancarzewicz [2]. It is also a generalization of a construction N of a classical linear connection $N(\Gamma, \Lambda)$ on P from a principal (right invariant) connection on a principal bundle $P \to M$ by means of a classical linear connection Λ considered in [5, p. 415].

In the present paper we study the problem how to construct a classical linear connection $A(\Gamma, \Lambda, \Delta, \Phi)$ on Y from a system $(\Gamma, \Lambda, \Phi, \Delta)$ consisting of a general connection Γ on $Y \to M$, a torsion free classical linear connection Λ on M, a vertical parallelism $\Phi\colon Y \times_M E \to VY$ and a linear connection Δ on $E \to M$.

In Section 2, modifying the torsion tensor field $\mathcal{T}or(\Gamma, \Lambda, \Phi, \Delta)$ of the Kolář connection $(\Gamma, \Lambda, \Phi, \Delta)$, the torsion field $\tau\Phi\colon Y \to \bigwedge^2 V^*Y \otimes VY$ of Φ and the covariant differential $D_{(\Gamma,\Delta)}\Phi\colon Y \times_M E \to VY \otimes T^*M$, we construct tensor fields $\tau_i(\Gamma, \Lambda, \Phi, \Delta)$ of type $T^* \otimes T^* \otimes T$ on Y canonically depending on $(\Gamma, \Lambda, \Phi, \Delta)$, $i = 1, \dots, 12$.

The main result of the present paper can be written in the form of the following theorem.

Theorem A. *If $m \geq 2$ and $n \geq 3$, any canonical construction A in question is of the form*

$$A(\Gamma, \Lambda, \Phi, \Delta) = (\Gamma, \Lambda, \Phi, \Delta) + \sum_{i=1}^{12} \lambda_i \tau_i(\Gamma, \Lambda, \Phi, \Delta)$$

for some (uniquely determined by A) real numbers $\lambda_1, \dots, \lambda_{12}$.

Classifications of constructions on connections has been studied in many papers, e.g. [3], [1], e.t.c.

All manifolds considered in the paper are assumed to be Hausdorff, second countable, without boundary, finite dimensional and smooth (of class C^∞). Maps between manifolds are assumed to be smooth (infinitely differentiable).

1. Natural operators

Let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred (local) diffeomorphisms. Let $\mathcal{VB}_{m,n}$ be the category of vector bundles with m-dimensional bases and n-dimensional fibres and their (local) vector bundle isomorphisms.

Definition 1. A (gauge) $\mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n}$-natural operator A sending systems $(\Gamma, \Lambda, \Phi, \Delta)$ consisting of general connections on fibred manifolds $Y \to M$, torsion free classical linear connections Λ on M, vertical parallelisms $\Phi\colon Y \times_M E \to VY$ on Y and linear connections Δ on vector bundles $E \to M$ into classical linear connections $A_{Y,E}(\Gamma, \Lambda, \Phi, \Delta)$ on Y is an $\mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n}$-invariant system of regular operators

$$A_{Y,E}\colon\ Con(Y) \times Con^o_{clas}(M) \times Par(Y \times_M E) \times Con_{lin}(E) \to Con_{clas}(Y)$$

for any pair (Y, E) consisting of a $\mathcal{FM}_{m,n}$-object $Y = (p_Y\colon Y \to M)$ and a $\mathcal{VB}_{m,n}$-object $E = (p_E\colon E \to M)$ (the same base M), where $Con(Y)$ is the set of general connections Γ on $p_Y\colon Y \to M$, $Con^o_{clas}(M)$ is the set of torsion free classical linear connections Λ on M, $Par(Y \times_M E)$ is the set of vertical parallelisms $\Phi\colon Y \times_M E \to VY$ on Y, $Con_{lin}(E)$ is the set of linear connections Δ on $p_E\colon E \to M$ and $Con_{clas}(Y)$ is the set of classical linear connections on Y.

Remark 1. The invariance of A means that if $(\Gamma, \Lambda, \Phi, \Delta) \in Con(Y) \times Con^o_{clas}(M) \times Par(Y \times_M E) \times Con_{lin}(E)$ is (f, g)-related to $(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1) \in Con(Y_1) \times Con^o_{clas}(M_1) \times Par(Y_1 \times_{M_1} E_1) \times Con_{lin}(E_1)$, where $f\colon Y \to Y_1$ is a $\mathcal{FM}_{m,n}$-map covering $\underline{f}\colon M \to M_1$ and $g\colon E \to E_1$ is a $\mathcal{VB}_{m,n}$-map covering also $\underline{f}\colon M \to M_1$, then $A_{Y,E}(\Gamma, \Lambda, \Phi, \Delta)$ and $A_{Y_1,E_1}(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)$ are f-related. A tuple $(\Gamma, \Lambda, \Phi, \Delta)$ is (f, g)-related to $(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)$ if Γ is f-related to Γ_1, Λ is $\underline{f}$-related to Λ_1, Φ is (f, g)-related to Φ_1 and Δ is g-related to Δ_1. In particular, Φ is (f, g)-related to Φ_1 if $Vf \circ \Phi = \Phi_1 \circ (f \times_{\underline{f}} g)$.

Remark 2. The regularity of A means that $A_{Y,E}$ transforms smoothly parametrized families into smoothly parametrized families.

For simplicity, we will omit the indexes Y and E on $A_{Y,E}$.

Remark 3. One can show standardly, that if $germ_y(\Gamma_1) = germ_y(\Gamma)$, $germ_x(\Lambda_1) = germ_x(\Lambda)$, $germ_y(\Phi_1) = germ_y(\Phi)$, $germ_x(\Delta_1) = germ_x(\Delta)$, $y \in Y_x$, $x \in M$, then $A(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)(y) = A(\Gamma, \Lambda, \Phi, \Delta)(y)$. That is why, A is in fact defined for locally defined $(\Gamma, \Lambda, \Phi, \Delta)$, too.

One can verify that the Kolář connection $(\Gamma, \Lambda, \Phi, \Delta)$ (mentioned in Introduction) defines a natural operator A in the sense of Definition 1, where $A(\Gamma, \Lambda, \Phi, \Delta) := (\Gamma, \Lambda, \Phi, \Delta)$.

So, to classify all natural operators in the sense of Definition 1 it suffices to classify all natural operators in the sense of the following definition.

Definition 2. A (gauge) $\mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n}$-natural operator A sending systems $(\Gamma, \Lambda, \Phi, \Delta)$ consisting of general connections Γ on fibred manifolds $Y \to M$, torsion free classical linear connections Λ on M, vertical parallelisms $\Phi : Y \times E \to VY$ on Y and linear connections Δ on vector bundles $E \to M$ into tensor fields $A(\Gamma, \Lambda, \Phi, \Delta)$ of type $T^* \otimes T^* \otimes T$ on Y is an $\mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n}$-invariant system of regular operators

$$A\colon \ Con(Y) \times Con^o_{clas}(M) \times Par(Y \times_M E) \times Con_{lin}(E) \to \mathcal{T}en^{(1,2)}(Y$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$ and any $\mathcal{VB}_{m,n}$-object $E \to M$ (the same M), where $\mathcal{T}en^{(1,2)}(Y)$ is the space of tensor fields of type $\otimes^2 T^* \otimes T$ on Y.

A simple example of a natural operator A in the sense of Definition 2 is given by the torsion of the Kolář connection $(\Gamma, \Lambda, \Phi, \Delta)$ (mentioned above).

Any natural operator A in the sense of Definition 1 is of the form

$$A(\Gamma, \Lambda, \Phi, \Delta) = (\Gamma, \Lambda, \Phi, \Delta) + A^1(\Gamma, \Lambda, \Phi, \Delta)\,,$$

where A^1 is a (uniquely determined) natural operator in the sense of Definition 2. That is why, from now on we study natural operators in the sense of Definition 2, only. Several examples of natural operators in the sense of Definition 2 are presented in the next section.

From now on, we can understand any natural operator A in the extended version as in Remark 3.

2. The main examples of natural operators

Let $p_Y : Y \to M$ be a fibred manifold and $p_E \colon E \to M$ be a vector bundle. Let $(\Gamma, \Lambda, \Phi, \Delta)$ be a 4-tuple consisting of a general connection Γ on $p_Y \colon Y \to M$, a classical linear connection Λ on M, a vertical parallelism $\Phi\colon Y \times_M E \to VY$ and of a linear general connection Δ on $p_E \colon E \to M$.

According to the usual Γ-decomposition $TY = VY \oplus_Y H^\Gamma Y$ we have the decomposition

$$\begin{aligned} T^*Y \otimes TY = (V^*Y \otimes VY) \oplus_Y (V^*Y \otimes H^\Gamma Y) \\ \oplus_Y \big((H^\Gamma)^* \otimes VY\big) \oplus_Y \big((H^\Gamma)^* \otimes H^\Gamma\big)\,. \end{aligned}$$

Let id_{HY} be the tensor field of type $T^* \otimes T$ on Y being the $(H^\Gamma Y)^* \otimes H^\Gamma Y$-component of the identity tensor field id_{TY} on Y (the other 3 component of id_{HY} are zero). Let id_{VY} be the tensor field of type $T^* \otimes T$ on Y being the $V^*Y \otimes VY$-component of id_{TY} (the other 3 components of id_{VY} are zero).

Quite similarly, we have the decomposition

$$\begin{aligned} T^*Y \otimes T^*Y \otimes TY &= (V^*Y \otimes V^*Y \otimes VY) \oplus_Y (V^*Y \otimes V^*Y \otimes H^\Gamma Y) \\ &\oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes VY) \oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y) \\ &\oplus_Y \big((H^\Gamma Y)^* \otimes V^*Y \otimes VY\big) \oplus_Y \big((H^\Gamma Y)^* \otimes V^*Y \otimes H^\Gamma Y\big) \\ &\oplus_Y \big((H^\Gamma Y)^* \otimes H^\Gamma Y\big)^* \otimes VY) \oplus_Y \big((H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y\big)\,. \end{aligned}$$

Let $\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$ be the $(H^\Gamma Y)^*\otimes V^*Y\otimes VY$-component of the torsion tensor field $\mathcal{T}or(\Gamma,\Lambda,\Phi,\Delta)$ of the Kolář connection $(\Gamma,\Lambda,\Phi,\Delta)$ (mentioned in Introduction). This components can be treated as the tensor field of type $T^*\otimes T^*\otimes T$ on Y (the other 7 components of it are zero). Taking contraction C^1_2 we produce tensor field $C^1_2\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$ of type T^* on Y. Let $\mathcal{T}or^{H^*\otimes H^*\otimes V)}(\Gamma,\Lambda,\Phi,\Delta)$ be the $(H^\Gamma Y)^*\otimes(H^\Gamma Y)^*\otimes VY$-component of $\mathcal{T}or(\Gamma,\Lambda,\Phi,\Delta)$. Thus we have the following tensor fields of type $T^*\otimes T^*\otimes T$ on Y canonically depending on $(\Gamma,\Lambda,\Phi,\Delta)$ (i.e. we have the corresponding natural operators in the sense of Definition 2).

Example 1. $\tau_1(\Gamma,\Lambda,\Phi,\Delta):=\mathcal{T}or^{H^*\otimes H^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$.

Example 2. $\tau_2(\Gamma,\Lambda,\Phi,\Delta):=\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$.

Example 3. $\tau_3(\Gamma,\Lambda,\Phi,\Delta):=\mathrm{id}_{HY}\otimes C^1_2\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$.

Example 4. $\tau_4(\Gamma,\Lambda,\Phi,\Delta):=C^1_2\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)\otimes\mathrm{id}_{HY}$.

Example 5. $\tau_5(\Gamma,\Lambda,\Phi,\Delta):=C^1_2\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)\otimes\mathrm{id}_{VY}$.

Example 6. $\tau_6(\Gamma,\Lambda,\Phi,\Delta):=\mathrm{id}_{VY}\otimes C^1_2\mathcal{T}or^{H^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$.

In the above examples (and from now on) we identify tensor fields τ of type $T^*\otimes T\otimes T^*$ with tensor fields $\tilde\tau$ of type $T^*\otimes T^*\otimes T$ and with tensor fields $\overline\tau$ of type $T\otimes T^*\otimes T^*$ by $\overline\tau(\omega,X_1,X_2)=\tilde\tau(X_1,X_2,\omega)=\tau(X_1,\omega,X_2)$. Moreover, tensor fields of types $T\otimes T^*\otimes T^*$ or $T^*\otimes T\otimes T^*$, we will always understand as the equivalent ones of type $T^*\otimes T^*\otimes T$. That is why, the contraction C^1_2 is clear.

In general, if $P\colon N\times V\to TN$ is a parallelism on a manifold N and $v\in N$, the vector field $\tilde v\colon N\to TN$, $\tilde v(z)=P(z,v)$ is called the constant vector field corresponding to v. One can show easily that there is a unique classical linear connection $\nabla=\nabla^P$ on N such that $\nabla_{\tilde v}\tilde w=0$ for any constant vector fields on N. The torsion tensor of ∇ will be denoted by $\tau(P)$ and called the torsion tensor field of P (thus $\tau(P)(X_1,X_2)=\nabla_{X_1}X_2-\nabla_{X_2}X_1-[X_1,X_2]$). If $\Phi\colon Y\times_M E\to VY$ is a vertical parallelism, we have the torsion tensor field $\tau\Phi$ of Φ given by

$$\tau\Phi=\bigcup_{x\in M}\tau(\Phi_x)\colon Y\to\bigwedge^2 V^*Y\otimes VY\,.$$

(The concept of a vertical parallelism and its torsion was introduced by I. Kolář in [K].) We can treat $\tau\Phi$ as the tensor field of type $T^*\otimes T^*\otimes T$ on Y (the other components of it in the decomposition we define to be 0). Thus we have the following tensor fields of type $T^*\otimes T^*\otimes T$ on Y canonically depending on $(\Gamma,\Lambda,\Phi,\Delta)$.

Example 7. $\tau_7(\Gamma,\Lambda,\Phi,\Delta)=\tau\Phi$.

Example 8. $\tau_8(\Gamma,\Lambda,\Phi,\Delta):=\mathrm{id}_{HY}\otimes C^1_2\tau\Phi$.

Example 9. $\tau_9(\Gamma,\Lambda,\Phi,\Delta):=C^1_2\tau\Phi\otimes\mathrm{id}_{HY}$.

Example 10. $\tau_{10}(\Gamma,\Lambda,\Phi,\Delta):=\mathrm{id}_{VY}\otimes C^1_2\tau\Phi$.

Example 11. $\tau_{11}(\Gamma,\Lambda,\Phi,\Delta) := C_2^1\tau\Phi\otimes \mathrm{id}_{VY}$.

By Section 3 (Corollary 1), if Λ is torsion free, then (eventually modulo signum) $\tau\Phi = \mathcal{T}or^{V^*\otimes V^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$, the $V^*Y\otimes V^*Y\otimes VY$-component of $\mathcal{T}or(\Gamma,\Lambda,\Phi,\Delta)$ in the Γ-decomposition.

In general, a Lie derivative of an arbitrary map $g\colon N\to N_1$ with respect to vector fields $\xi\colon N\to TN$ and $\eta\colon N_1\to TN_1$ is the map

$$\mathcal{L}_{(\xi,\eta)}g = Tg\circ\xi - \eta\circ g\colon N\to TN_1\,.$$

If we have another fibred manifold $Z\to M$ with general connection Ω and a base preserving morphism $f\colon Y\to Z$, then the covariant derivative $D_{\Gamma,\Omega}f\colon Y\to VZ\otimes T^*M$ is defined by

$$(D_{\Gamma,\Omega}f)(\xi) := \mathcal{L}_{(\Gamma\xi,\Omega\xi)}f\,.$$

Consider $\Phi\colon Y\times_M E\to VY$. According to [5, p.55], Γ induces a general connection $\mathcal{V}\Gamma$ on $VY\to M$. Further we construct the product connection $\Gamma\times\Delta$ on $Y\times_M E$. Then $D_{\Gamma\times\Delta,\mathcal{V}\Gamma}\Phi\colon Y\times_M E\to VVY\otimes T^*M$. The values lie in a sub-bundle characterized by $V\pi=0$, where $\pi\colon VY\to Y$ is the bundle projection. This sub-bundle coincides with $VY\times_Y VY$. The covariant differential $D_{(\Gamma,\Delta)}\Phi\colon Y\times_M E\to VY\otimes T^*M$ is the second component of $D_{\Gamma\times\Delta,\mathcal{V}\Gamma}\Phi$. (This construction of the covariant differential was proposed by I. Kolář in [4].)

We can consider the covariant differential as the corresponding map $D_{(\Gamma,\Delta)}\Phi\colon (Y\times_M E)\times_M TM\to VY$. Then we define the modified covariant differential $\tilde{D}_{(\Gamma,\Delta)}\Phi\colon Y\to V^*Y\otimes T^*Y\otimes VY$ by

$$(\tilde{D}_{(\Gamma,\Delta)}\Phi)(y)(X_1,X_2) := D_{(\Gamma,\Delta)}\Phi(\Phi^{-1}(X_1),Tp_Y(X_2))\in V_yY\,,$$

$X_1\in V_yY$, $X_2\in T_yY$. We can treat it as the tensor field of type $T^*\otimes T^*\otimes T$ on Y (the other parts of it in the decomposition we define to be 0). Thus we have the following tensor field of type $T^*\otimes T^*\otimes T$ on Y canonically induced by $(\Gamma,\Lambda,\Phi,\Delta)$.

Example 12. $\tau_{12}(\Gamma,\Lambda,\Phi,\Delta) := \tilde{D}_{(\Gamma,\Delta)}\Phi$.

By Section 3 (Corollary 2), if Λ is torsion free, then (eventually modulo signum) $\tau_{12}(\Gamma,\Lambda,\Phi,\Delta) = \mathcal{T}or^{V^*\otimes H^*\otimes V}(\Gamma,\Lambda,\Phi,\Delta)$, the $V^*Y\otimes(H^\Gamma Y)^*\otimes VY$-part of $\mathcal{T}or(\Gamma,\Lambda,\Phi,\Delta)$ in the Γ-decomposition.

3. Estimation of dimension of the vector space of natural operators

Let $x^1,\dots,x^m$ be the usual coordinates on $\mathbb{R}^m$. Let $\mathbb{R}^{m,n}$ be the trivial bundle over $\mathbb{R}^m$ with the standard fiber $\mathbb{R}^n$ and $x^1,\dots,x^m,y^1,\dots,y^n$ be the usual fiber coordinates on $\mathbb{R}^{m,n}$. Let $\mathbb{R}^{m,n}$ be also the trivial vector bundle over $\mathbb{R}^m$ and $x^1,\dots,x^m,v^1,\dots,v^n$ be the usual vector bundle coordinates on $\mathbb{R}^{m,n}$.

Let

$$(1)\qquad \Gamma^o=\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}\,,\ \Lambda^o=(0)\,,\ \Phi^o=\sum_{p=1}^n v^p\frac{\partial}{\partial y^p}\,,\ \Delta^o=\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}$$

be the trivial general connection on $\mathbb{R}^{m,n}$, the torsion free flat classical linear connection on $\mathbb{R}^m$, the canonical parallelism on $\mathbb{R}^{m,n}$ and the trivial linear connection on $\mathbb{R}^{m,n}$, respectively.

In this section we study a natural operator A in the sense of Definition 2.

From Corollary 19.8 in [5], we get immediately the following proposition.

Proposition 1. *Let* $p_Y\colon Y \to M$ *be an* $\mathcal{FM}_{m,n}$*-object and* $p_E\colon E \to M$ *be a* $\mathcal{VB}_{m,n}$*-object,* $y \in Y_x$, $x \in M$. *Let* $(\Gamma, \Lambda, \Phi, \Delta) \in Con(Y) \times Con^o_{clas}(M) \times Par(Y \times_M E) \times Con_{lin}(E)$. *There exists a finite number* $r = r(\Gamma, \Lambda, \Phi, \Delta, y)$ *such that for any* $(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1) \in Con(Y) \times Con^o_{clas}(M) \times Par(Y \times_M E) \times Con_{lin}(E)$ *we have the following implication*

$$(j^r_y\Gamma_1 = j^r_y\Gamma\,,\ j^r_x\Lambda_1 = j^r_x\Lambda\,,\ j^r_y\Phi_1 = j^r_y\Phi\,,\ j^r_x\Delta_1 = j^r_x\Delta) \Rightarrow A(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)(y) = A(\Gamma, \Lambda, \Phi, \Delta)(y)\,.$$

It is clear that A is determined by the values

$$A(\Gamma, \Lambda, \Phi, \Delta)(y) \in T^*_yY \otimes T^*_yY \otimes T_yY$$

for fibred manifolds $p_Y\colon Y \to M$ with m-dimensional bases and n-dimensional fibres, vector bundles $p_E\colon E \to M$ with n-dimensional fibres, general connections Γ on $p_Y\colon Y \to M$, torsion free classical linear connections Λ on M, vertical parallelisms $\Phi\colon Y \times_M E \to VY$, linear connections Δ on $p_E\colon E \to M$ and $y \in Y_x$, $x \in M$.

Using the invariance of A with respect to (respective) fibred manifold charts and vector bundle charts and Proposition 1, we can assume $E = Y = \mathbb{R}^{m,n}$, $y = (0,0)$,

$$\Gamma = \Gamma^o + \sum F^p_{j;\alpha\beta} x^\alpha y^\beta dx^j \otimes \frac{\partial}{\partial y^p}\,, \tag{2}$$

where the sum is over all m-tuples α and all n-tuples β of non-negative integers and $j = 1, \dots, m$ and $p = 1, \dots, n$ with $1 \le |\alpha| + |\beta| \le K$ (i.e. we can assume $F^p_{j;(0)(0)} = 0$),

$$\Lambda = \Big(\sum \Lambda^i_{jk;\gamma} x^\gamma\Big)_{i,j,k=1,\dots,m}\,,\ \Lambda^i_{jk;\gamma} = \Lambda^i_{kj;\gamma}\,, \tag{3}$$

where the sums are over all m-tuples γ of non-negative integers with $1 \le |\gamma| \le K$ (i.e. we can assume $\Lambda^i_{jk;(0)} = 0$),

$$\Phi = \Phi^o + \sum a^s_{q;\delta\sigma} x^\delta y^\sigma v^q \frac{\partial}{\partial y^s}\,, \tag{4}$$

where the sum is over all m-tuples δ and all n-tuples σ of non-negative integers and $s, q = 1, \dots, n$ with $1 \le |\delta| + |\sigma| \le K$ (i.e. we can assume $a^s_{q;(0)(0)} = 0$) (we remark that such Φ can not be defined globally (it may not be a diffeomorphism $\mathbb{R}^{m,n} \times_{\mathbb{R}^m} \mathbb{R}^{m,n} \tilde{=} V\mathbb{R}^{m,n}$ but it is defined locally on some neighborhood of $(0,0)$ (it is a diffeomorphism $U \times_{\underline{U}} \mathbb{R}^{m,n}_{|\underline{U}} \tilde{=} VU$)),

$$\Delta = \Delta^o + \sum \Delta^p_{jq;\rho} x^\rho v^q dx^j \otimes \frac{\partial}{\partial v^p}\,, \tag{5}$$

where the sum is over all m-tuples ρ of non-negative integers and $j = 1, \ldots, m$ and $p, q = 1, \ldots, n$ with $0 \le |\rho| \le K$, where K is an arbitrary positive integer.

Given a positive integer K we define a smooth (as A is regular) map $A_K \colon \mathbb{R}^{n(K)} \to \mathbb{R}^q = T^*_{(0,0)}\mathbb{R}^{m,n} \otimes T^*_{(0,0)}\mathbb{R}^{m,n} \otimes T_{(0,0)}\mathbb{R}^{m,n}$ by

$$A_K((F^p_{j;\alpha\beta}), (\Lambda^i_{jk;\gamma}), (a^s_{q;\delta\sigma}), (\Delta^i_{jq;\rho})) := A(\Gamma, \Lambda, \Phi, \Delta)(0,0)\,, \tag{6}$$

where Γ, Λ, Φ, Δ are as in (2)–(5).

Clearly, A is determined by the collection of all A_K, $K = 1, 2, \ldots$.

Using the invariance of A with respect to $(\varphi_t \times \phi_t, \varphi_t \times \phi_t)$, $\varphi_t = t\,\mathrm{id}_{\mathbb{R}^m}$, $\phi_t = t\,\mathrm{id}_{\mathbb{R}^n}$, $t > 0$, we get the homogeneous condition

$$\begin{aligned} &tA_K((F^p_{j;\alpha\beta}), (\Lambda^i_{jk;\gamma}), (a^s_{q;\delta\sigma}), (\Delta^p_{jq;\rho})) \\ &\qquad = A_K((t^{|\alpha|+|\beta|}F^p_{j;\alpha\beta}), (t^{|\gamma|+1}\Lambda^i_{jk;\gamma}), (t^{|\delta|+|\sigma|}a^s_{q;\delta\sigma}), (t^{|\rho|+1}\Delta^p_{jq;\rho}))\,. \end{aligned}$$

By the homogeneous function theorem, from this homogeneity condition we obtain.

Lemma 1. *A_K is independent of $F^p_{j;\alpha\beta}$ with $|\alpha|+|\beta| \ge 2$, A is independent of $\Lambda^i_{jk;\gamma}$ with $|\gamma| \ge 1$, A_K is independent of $a^s_{q;\delta\sigma}$ with $|\delta| + |\sigma| \ge 2$ and A_K is independent of $\Delta^p_{jq;\rho}$ with $|\rho| \ge 1$. Even, A_K is a linear combination with real coefficients of $\Delta^i_{jk;(0)(0)}$ and $F^p_{j;\alpha\beta}$, $a^s_{q;\delta\sigma}$ with $|\alpha| + |\beta| = 1$, $|\delta| + |\sigma| = 1$, $i, j, k = 1, \ldots, m$, $p, q, s = 1, \ldots, n$.*

In particular, $A_K(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$.

Even, we have proved the following fact.

Proposition 2. *Any natural operator A in the sense of Definition 2 is of order not more than* 1.

Using these facts, we prove the following lemma.

Lemma 2. *Let $m \ge 2$ and $n \ge 2$. A natural operator A in the sense of Definition 2 is fully determined by the collection of values*

$$A^1 := A(\Gamma^o + x^2 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)(0,0)\,, \tag{7}$$

$$A^2 := A(\Gamma^o + y^1 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)(0,0)\,, \tag{8}$$

$$A^3 := A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^1 \frac{\partial}{\partial y^1}, \Delta^o)(0,0)\,, \tag{9}$$

where Γ^o, Λ^o, Φ^o, Δ^o are defined in (1).

Proof. (a) We are going to observe that the value $A(\Gamma^o + x^{j_o}dx^{i_o} \otimes \frac{\partial}{\partial y^{p_o}}, \Lambda^o, \Phi^o, \Delta^o)(0,0)$ is determined by A^1.

If $i_o = j_o$, by the invariance of A with respect to

$$\big((x^1, \ldots, x^m, y^1, \ldots, y^{p_o} + \frac{1}{2}(x^{i_o})^2, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n)\big)\,,$$

from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$ we get $A(\Gamma^o + x^{i_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{p_o}}, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$.

If $i_o \neq j_o$, there exists a respective permutation of coordinates sending $\Gamma^o + x^2 dx^1 \otimes \frac{\partial}{\partial y^1}$ into $\Gamma^o + x^{j_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{p_o}}$ and preserving Λ^o, Φ^o, Δ^o. Then using the invariance of A with respect to this permutation, we end the observation.

(b) We are going to observe that A^2 determines the value $A(\Gamma^o + y^{q_o} dx^{i_o} \otimes \frac{\partial}{\partial y^{p_o}}, \Lambda^o, \Phi^o, \Delta^o)(0,0)$.

By the invariance of A with respect to

$$(f, g) = \big((x^1, \dots, x^m, y^1 + y^2, y^2, \dots, y^m), (x^1, \dots, x^m, v^1 + v^2, v^2, \dots, v^n)\big)$$

we see $A(\Gamma^o + (y^1 - y^2) dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)(0,0)$ is the image of A^2 by (f, g), and then it is determined by A^2. Therefore $A(\Gamma^o + y^2 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)(0,0) = A^2 - A(\Gamma^o + (y^1 - y^2) dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)(0,0)$ is determined by A^2.

Now, using the invariance of A with a respective permutation of coordinates, we end the observation in this case.

(c) We are going to observe that A^3 determines the value $A(\Gamma^o, \Lambda^o, \Phi^o + v^{q_o} y^{s_o} \frac{\partial}{\partial y^{p_o}}, \Delta^o)(0,0)$.

If $p \neq 1$, then

$$\big((x^1, \dots, x^m, y^1 + y^p, y^2, \dots, y^n), (x^1, \dots, x^m, v^1 + v^p, v^2, \dots, v^n)\big)$$

preserves $\Gamma^o, \Lambda^o, \Delta^o$ and sends $\Phi^o + v^2 y^1 \frac{\partial}{\partial y^1}$ into $\Phi^o + v^2 (y^1 - y^p) \frac{\partial}{\partial y^1}$. Then (similarly as in the case (b) of the proof) $A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^p \frac{\partial}{\partial y^1}, \Delta^o)(0,0)$ is determined by A^3. In particular $A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^2 \frac{\partial}{\partial y^1}, \Delta^o)(0,0)$ and $A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^3 \frac{\partial}{\partial y^1}, \Delta^o)(0,0)$ are determined by A^3.

By the invariance of A with respect to

$$\big((x^1, \dots, x^m, y^1 + y^1 y^2, y^3, \dots, y^n), (x^1, \dots, x^m, v^1, \dots, v^n)\big)$$

from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$ we get $A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^1 \frac{\partial}{\partial y^1} + v^1 y^2 \frac{\partial}{\partial y^1}, \Delta^o)(0,0) = 0$ (because Φ^o is mapped into $\Phi^o + v^2 y^1 \frac{\partial}{\partial y^1} + v^1 y^2 \frac{\partial}{\partial y^1} + \dots$, where the dots have the 1-jet equal to 0, and Γ^o, Λ^o, Δ^o and A are preserved, and A is of order not more than 1), i.e. $A(\Gamma^o, \Lambda^o, \Phi^o + v^1 y^2 \frac{\partial}{\partial y^1}, \Delta^o)(0,0)$ is determined by A^3 (it is $-A^3$). By the invariance of A with respect to

$$\big((x^1, \dots, x^m, y^1 + \frac{1}{2}(y^1)^2, y^2, \dots, y^n), (x^1, \dots, x^m, v^1, \dots, v^n)\big),$$

from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o) = 0$ we get $A(\Gamma^o, \Lambda^o, \Phi^o + v^1 y^1 \frac{\partial}{\partial y^1}, \Delta^o)(0,0) = 0$.

Now, using the invariance of A with respect to a respective permutation of coordinates, we end the observation.

(d) Let us denote

$$A^4 := A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o + v^1 dx^1 \otimes \frac{\partial}{\partial v^1})(0,0)\,. \tag{10}$$

We are going to observe that $A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o+v^{q_o}dx^{i_o}\otimes\frac{\partial}{\partial v^{p_o}})(0,0)$ is determined by A^4.

Using the invariance of A with respect to

$$(f,g)=\big((x^1,\dots,x^m,y^1+y^2,y^2,\dots,y^n),(x^1,\dots,x^m,v^1+v^2,v^2,\dots,v^n)\big)$$

we deduce that $A'=A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o+(v^1-v^2)dx^1\otimes\frac{\partial}{\partial v^1})(0,0)$ is determined by A^4 (it is image of A^4 by (f,g)). So, $A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o+v^2dx^1\frac{\partial}{\partial v^1})(0,0)$ is determined by A^4 (it is A^4-A').

Now, using the invariance of A with respect to a respective permutation of coordinates, we end the observation.

(e) We are going to observe that A^4 determines the value $A(\Gamma^o,\Lambda^o,\Phi^o+x^{i_o}v^{q_o}\frac{\partial}{\partial v^{p_o}},\Delta^o)(0,0)$.

Using the invariance of A with respect to

$$\big((x^1,\dots,x^m,y^1,\dots,y^n),(x^1,\dots,x^m,v^1,\dots,v^{p_o}+x^{i_o}v^{q_o},\dots,v^n)\big)\,,$$

since $(x^1,\dots,x^m,v^1,\dots,v^{p_o}+x^{i_o}v^{q_o},\dots,v^n)^{-1}=(x^1,\dots,x^m,v^1,\dots,v^{p_o}-x^{i_o}v^{q_o}+\tilde\varphi(x^{i_o})v^{q_o},\dots,v^n)$ with $j^1_0\tilde\varphi=0$ (if $p_o\neq q_o$, $\tilde\varphi=0$), from $A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o)(0,0)=0$ we get

$$A\Big(\Gamma^o,\Lambda^o,\Phi^o-x^{i_o}v^{q_o}\frac{\partial}{\partial y^{p_o}},\Delta^o+v^{q_o}dx^{i_o}\otimes\frac{\partial}{\partial v^{p_o}}\Big)(0,0)=0\,,$$

i.e. $A(\Gamma^o,\Lambda^o,\Phi^o+x^{i_o}v^{q_o}\frac{\partial}{\partial y^{p_o}},\Delta^o)(0,0)=A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o+v^{q_o}dx^{i_o}\otimes\frac{\partial}{\partial v^{q_o}})(0,0)$ is determined by A^4 because of the part (d) of the proof. In particular (for $i_o=1\,,\ p_o=1\,,\ q_o=1$), we proved

$$A^4=A\Big(\Gamma^o,\Lambda^o,\Phi^o+x^1v^1\frac{\partial}{\partial y^1},\Delta^o\Big)(0,0)\,. \tag{11}$$

(g) We are going to prove that A^4 is determined by A^2.

Using the invariance of A with respect to

$$\big((x^1,\dots,x^m,y^1+x^1y^1,y^2,\dots,y^n),(x^1,x^2,\dots,x^m,v^1,\dots,v^n)\big)$$

from $A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o)=0$ we get

$$A\Big(\Gamma^o+y^1dx^1\otimes\frac{\partial}{\partial y^1},\Lambda^o,\Phi^o+x^1v^1\frac{\partial}{\partial y^1},\Delta^o\Big)(0,0)=0\,.$$

Hence $A^4=-A^2$ because of (11).

The proof of Lemma 2 is complete. □

Now, we prove the following lemma.

Lemma 3. *Let $m\geq 2$ and $n\geq 3$. Let A^1, A^2, A^3 be the values (7)–(9) from Lemma 2. There are real numbers $a_1,\dots,a_{12}$ such that*

$$A^1=a_1\Big(d_{(0,0)}x^2\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial y^1}_{|(0,0)}-d_{(0,0)}x^1\otimes d_{(0,0)}x^2\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\Big)\,, \tag{12}$$

$$
\begin{aligned}
A^2 = {} & a_2 \sum_{p=1}^{n} d_{(0,0)}x^1 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\
& + a_3 \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\
& + a_4 d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \\
& + a_5 d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \\
& + a_6 \sum_{i=1}^{m} d_{(0,0)}x^1 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\
& + a_7 \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} ,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
A^3 = {} & a_8 \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\
& + a_9 \sum_{p=1}^{n} d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\
& + a_{10} \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\
& + a_{11} \sum_{i=1}^{m} d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\
& + a_{12} \Big(d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \\
& - d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \Big) .
\end{aligned} \tag{14}
$$

Proof. a. By the invariance of A with respect to

$$\big((t^1x^1, \dots, t^m x^m, \tau^1 y^1, \dots, \tau^n y^n), (t^1x^1, \dots, t^m x^m, \tau^1 v^1, \dots, \tau^n v^n)\big) \tag{15}$$

for $t^1 > 0, \dots, t^m > 0$, $\tau^1 > 0, \dots, \tau^n > 0$ we get immediately

$$A^1 = b_1 d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \frac{\partial}{\partial y^1}_{|(0,0)} + b_2 d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \frac{\partial}{\partial y^1}_{|(0,0)}$$

for some real numbers b_1, b_2. But by the invariance of A with respect to

$$\big((x^1,\dots,x^m,y^1+x^1x^2,y^3,\dots,y^n),(x^1,\dots,x^m,v^1,\dots,v^n)\big)$$

from $A(\Gamma^o,\Lambda^o,\Phi^o,\Delta^o)(0,0)=0$ we get

$$A\Big(\Gamma^o+x^2dx^1\otimes\frac{\partial}{\partial y^1}+x^1dx^2\otimes\frac{\partial}{\partial y^1},\Lambda^o,\Phi^o,\Delta^o\Big)(0,0)=0\,.$$

Therefore $b_1=-b_2$. We define $a_1:=b_1=-b_2$. That is why, formula (12) holds.

b. By the invariance of A with respect to (15) we get immediately

$$\begin{aligned}A^2=&\sum_{p=1}^n b_p d_{(0,0)}y^p\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial y^p}{}_{|(0,0)}\\&+\sum_{p=1}^n c_p d_{(0,0)}x^1\otimes d_{(0,0)}y^p\otimes\frac{\partial}{\partial y^p}{}_{|(0,0)}\\&+\sum_{i=1}^m d_i d_{(0,0)}x^1\otimes d_{(0,0)}x^i\otimes\frac{\partial}{\partial x^i}{}_{|(0,0)}\\&+\sum_{i=1}^m e_i d_{(0,0)}x^i\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial x^i}{}_{|(0,0)}\,.\end{aligned}$$

Next, by the invariance of A with respect to respective permutation of coordinates, we deduce $b_2=\dots=b_n$, $c_2=\dots=c_n$, $d_2=\dots=d_m$, $e_2=\dots=e_m$. Then

$$\begin{aligned}A^2=&\,a_2\sum_{p=1}^n d_{(0,0)}x^1\otimes d_{(0,0)}y^p\otimes\frac{\partial}{\partial y^p}{}_{|(0,0)}\\&+a_3\sum_{p=1}^n d_{(0,0)}y^p\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial y^p}{}_{|(0,0)}\\&+a_4 d_{(0,0)}x^1\otimes d_{(0,0)}y^1\otimes\frac{\partial}{\partial y^1}{}_{|(0,0)}\\&+a_5 d_{(0,0)}y^1\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial y^1}{}_{|(0,0)}\\&+a_6\sum_{i=1}^m d_{(0,0)}x^1\otimes d_{(0,0)}x^i\otimes\frac{\partial}{\partial x^i}{}_{|(0,0)}\\&+a_7\sum_{i=1}^m d_{(0,0)}x^i\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial x^i}{}_{|(0,0)}\\&+b d_{(0,0)}x^1\otimes d_{(0,0)}x^1\otimes\frac{\partial}{\partial x^1}{}_{|(0,0)}\,.\end{aligned}$$

Then by the invariance of A with respect to

$$\big((x^1,x^2+x^1,x^3,\dots,x^m,y^1,\dots,y^n),(x^1,x^2+x^1,x^3,\dots,x^m,v^1,\dots,v^n)\big)$$

from the last equality we get $b = 0$. That is why, formula (13) is true.

c. By the invariance of A with respect to (15) we get immediately

$$\begin{aligned} A^3 = &\sum_{p=1}^{n} b_p d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\ &+ \sum_{p=1}^{n} c_p d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\ &+ \sum_{i=1}^{m} d_i d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\ &+ \sum_{i=1}^{m} e_i d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} . \end{aligned}$$

Then by the invariance of A with respect to respective permutation of coordinates, we deduce $b_3 = \cdots = b_n$, $c_3 = \cdots = c_n$, $d_1 = \cdots = d_m$ and $e_1 = \cdots = e_m$. Then

$$\begin{aligned} A^3 = &\lambda_1 d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \\ &+ \lambda_2 d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \\ &+ \lambda_3 d_{(0,0)}y^2 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^2}_{|(0,0)} \\ &+ \lambda_4 \sum_{p=3}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\ &+ \lambda_5 \sum_{p=3}^{n} d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} \\ &+ \lambda_6 \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} \\ &+ \lambda_7 \sum_{i=1}^{m} d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} . \end{aligned} \tag{16}$$

Then by the invariance of A with respect to

$$\big((x^1, \dots, x^m, y^1 - y^2, \dots, y^n), (x^1, \dots, x^m, v^1 - v^2, \dots, v^n)\big)$$

from (16), we deduce

$$\begin{aligned}
&A^3 + A(\Gamma^o \Lambda^o, \Phi^o + v^2 y^2 \frac{\partial}{\partial y^1}, \Delta^o)(0,0)\\
&= A^3 + \lambda_1 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\\
&\quad + \lambda_2 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\\
&\quad - \lambda_3 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} .
\end{aligned}$$

On the other hand, by the invariance of A with respect to

$$\big((x^1, \dots, x^m, y^1 + \frac{1}{2}(y^2)^2, y^2, \dots, y^n), (x^1, \dots, x^m, v^1, \dots, v^n)\big)$$

from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$, we obtain

$$A\Big(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^2 \frac{\partial}{\partial y^1}, \Delta^o\Big)(0,0) = 0 .$$

So, $\lambda_1 + \lambda_2 - \lambda_3 = 0$.

From the invariance of A with respect to

$$\big((x^1, \dots, x^m, y^1 - y^3, y^2, \dots, y^n), (x^1, \dots, x^m, v^1 - v^3, v^2, \dots, v^n)\big)$$

(we assume $n \geq 3$) from (16) we get (after cancelling A^3)

$$\begin{aligned}
A\Big(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^3 \frac{\partial}{\partial y^1}, \Delta^o\Big)(0,0) &= (\lambda_1 - \lambda_5) d_{(0,0)} y^2 \otimes d_{(0,0)} y^3 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\\
&\quad + (\lambda_2 - \lambda_4) d_{(0,0)} y^3 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} .
\end{aligned}$$

Then by the invariance of A with respect to the switching (y^2 and y^3) and (v^2 and v^3) we get

$$\begin{aligned}
A\Big(\Gamma^o, \Lambda^o, \Phi^o + v^3 y^2 \frac{\partial}{\partial y^1}\Big)(0,0) &= (\lambda_1 - \lambda_5) d_{(0,0)} y^3 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\\
&\quad + (\lambda_2 - \lambda_4) d_{(0,0)} y^2 \otimes d_{(0,0)} y^3 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} .
\end{aligned}$$

On the other hand by the invariance of A with respect to

$$\big((x^1, \dots, x^m, y^1 + y^2 y^3, y^2, \dots, y^n), (x^1, \dots, x^m, v^1, \dots, v^n)\big)$$

from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$ we get

$$A\Big(\Gamma^o, \Lambda^o, \Phi^o + y^3 v^2 \frac{\partial}{\partial y^1} + y^2 v^3 \frac{\partial}{\partial y^1}, \Delta^o\Big)(0,0) = 0 .$$

So, $\lambda_1 - \lambda_5 = -(\lambda_2 - \lambda_4)$.

That is why, formula (14) holds.

The proof of the lemma is complete. □

From Lemma 3 it follows immediately the following proposition.

Proposition 3. *If $m \geq 2$ and $n \geq 3$, the dimension of the vector space of all natural operators in the sense of Definition 2 is of the dimension not more than* 12.

4. Linear independence of natural operators from Examples 1–12

We prove the following proposition.

Proposition 4. *Let $m \geq 2$ and $n \geq 2$. The natural operators τ_i ($i = 1, \dots, 12$) in the sense of Definition 2 from Examples 1–12 are linearly independent.*

Proof. By Lemma 2, it is sufficient to study the values (7)–(9) for $A = \tau_i$, $i = 1, \dots, 12$. To compute these values, we use Proposition 1 in [4].

a. *The case* $\Psi = (\Gamma^o + x^2 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)$.

In this case, we have (in the notation of Proposition 1 in [4]) $F_1^1(x,y) = x^2$ and other $F_i^p(x,y) = 0$, $\Lambda_{ij}^k = 0$, $\frac{\partial a_s^p}{\partial x^j} = 0$, $\frac{\partial a_s^p}{\partial y^q} = 0$, $\Delta_{sj}^r = 0$. Then (by Proposition 1 in [4]) $d\eta^1 = \xi^1 dx^2$ and other $d\eta^p = 0$, and $d\xi^i = 0$. Then (modulo signum)

$$\mathcal{T}or(\Psi)(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} .$$

Hence (modulo signum)

$$\tau_1(\Psi)(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$

and $\tau_i(\Psi)(0,0) = 0$ for $i = 2, \dots, 6$.

By the coordinate expression of the torsion tensor of vertical parallelism in Section 3 of [4], $\tau\Phi^o(0,0) = 0$. Then $\tau_i(\Psi)(0,0) = 0$ for $i = 7, \dots, 11$.

By the coordinate expression of the covariant differential in Section 4 in [4], we have $\tau_{12}(0,0)(\Psi) = 0$.

b. *The case* $\Psi = (\Gamma^o + y^1 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^0, \Delta^o)$.

Now, by Proposition 1 in [4], $d\eta^1 = \xi^1 dy^1$ and other $d\eta^p = 0$, and $d\xi^i = 0$. Then (modulo signum)

$$\mathcal{T}or(\Psi)(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} .$$

Then $\tau_1(\Psi)(0,0)=0$ and (modulo signum)

$$\tau_2(\Psi)(0,0)=d_{(0,0)}x^1\otimes d_{(0,0)}y^1\otimes \frac{\partial}{\partial y^1}_{|(0,0)},$$

$$\tau_3(\Psi)(0,0)=\sum_{i=1}^m d_{(0,0)}x^i\otimes d_{(0,0)}x^1\otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_4(\Psi)(0,0)=\sum_{i=1}^m d_{(0,0)}x^1\otimes d_{(0,0)}x^i\otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_5(\Psi)(0,0)=\sum_{p=1}^n d_{(0,0)}x^1\otimes d_{(0,0)}y^p\otimes \frac{\partial}{\partial y^p}_{|(0,0)},$$

$$\tau_6(\Psi)(0,0)=\sum_{p=1}^n d_{(0,0)}y^p\otimes d_{(0,0)}x^1\otimes \frac{\partial}{\partial y^p}_{|(0,0)}.$$

Since $\tau\Phi^o(0,0)=0$ (see the case a of the proof), $\tau_i(\Psi)(0,0)=0$ for $i=7,\dots,11$.

By the coordinate expression of the covariant differential,

$$\tau_{12}(\Psi)(0,0)=-d_{(0,0)}y^1\otimes d_{(0,0)}x^1\otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

c. *The case* $\Psi=(\Gamma^o,\Lambda^o,\Phi^o+v^2y^1\frac{\partial}{\partial y^1},\Delta^o)$.

By Proposition 1 in [4], $d\eta^1=\eta^2dy^1$ and $d\eta^p=0$ for other p, and $d\xi^i=0$. Then (modulo signum)

$$\mathcal{T}or(\Psi)(0,0)=d_{(0,0)}y^2\otimes d_{(0,0)}y^1\otimes \frac{\partial}{\partial y^1}_{|(0,0)}-d_{(0,0)}y^1\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

Then $\tau_i(\Psi)(0,0)=0$ for $i=1,\dots,6$.

By Section 3 of [4], one can compute

$$\tau\Big(\Phi^o+v^2y^1\frac{\partial}{\partial y^1}\Big)(0,0)=d_{(0,0)}y^2\otimes d_{(0,0)}y^1\otimes \frac{\partial}{\partial y^1}_{|(0,0)}-d_{(0,0)}y^1\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

Then

$$\tau_7(\Psi)(0,0)=d_{(0,0)}y^2\otimes d_{(0,0)}y^1\otimes \frac{\partial}{\partial y^1}_{|(0,0)}-d_{(0,0)}y^1\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)},$$

$$\tau_8(\Psi)(0,0)=\sum_{i=1}^m d_{(0,0)}x^i\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_9(\Psi)(0,0) = \sum_{i=1}^{m} d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} ,$$

$$\tau_{10}(\Psi)(0,0) = \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} ,$$

$$\tau_{11}(\Psi)(0,0) = \sum_{p=1}^{n} d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} .$$

By the coordinate expression of the covariant differential, $\tau_{12}(\Psi)(0,0) = 0$.

Now, it is easily seen that the natural operators $\tau_1, \dots, \tau_{12}$ are linearly independent. The proof of Proposition 4 is complete. □

Else, using Lemma 2, from the proof of Proposition 3 we have the following facts.

Corollary 1. *If* Λ *is torsion free, then* (*eventually modulo signum*)

$$\tau\Phi = \mathcal{T}or^{V^*\otimes V^*\otimes V}(\Gamma, \Lambda, \Phi, \Delta) ,$$

where $\mathcal{T}or^{V^*\otimes V^*\otimes V}(\Gamma, \Lambda, \Phi, \Delta)$ *is the* $V^*Y \otimes V^*Y \otimes VY$*-part of* $\mathcal{T}or(\Gamma, \Lambda, \Phi, \Delta)$ *in the* Γ*-decomposition of Section 2.*

Corollary 2. *If* Λ *is torsion free, then* (*eventually modulo signum*)

$$\tau_{12}(\Gamma, \Lambda, \Phi, \Delta) = \mathcal{T}or^{V^*\otimes H^*\otimes V}(\Gamma, \Lambda, \Phi, \Delta) ,$$

where $\mathcal{T}or^{V^*\otimes H^*\otimes V}(\Gamma, \Lambda, \Phi, \Delta)$ *is the* $V^*Y \otimes (H^\Gamma Y)^* \otimes VY$*-part of* $\mathcal{T}or(\Gamma, \Lambda, \Phi, \Delta)$ *in the* Γ*-decomposition of Section 2.*

5. The main result

From Propositions 3 and 4 it follows the main theorem of the paper.

Theorem 1. *Let* $m \geq 2$ *and* $n \geq 3$*. Any natural operator* A *in the sense of Definition 1 is of the form*

$$A_{Y,E}(\Gamma, \Lambda, \Phi, \Delta) = (\Gamma, \Lambda, \Phi, \Delta) + \sum_{i=1}^{12} \lambda_i \tau_i(\Gamma, \Lambda, \Phi, \Delta)$$

for some (*uniquely determined by* A) *real numbers* λ_i*, where* τ_i *are the operators described in Examples 1–12 and* $(\Gamma, \Lambda, \Phi, \Delta)$ *is the connection constructed by I. Kolář in* [4].

References

[1] Doupovec, M., Mikulski, W. M., *Reduction theorems for principal and classical connections*, Acta Math. Sinica **26** (1) (2010), 169–184.

[2] Gancarzewicz, J., *Horizontal lifts of linear connections to the natural vector bundles*, Differential geometry (Santiago de Compostela, 1984), vol. 131, Pitman, Boston, MA, 1985, pp. 318–341.

[3] Janyška, J., Vondra, J., *Natural principal connections on the principal gauge prolongation of a principal bundle*, Rep. Math. Phys. **64** (3) (2009), 395–415.

[4] Kolář, I., *Induced connections on total spaces of fibred bundles*, Int. J. Geom. Methods Mod. Phys. (2010), 705–711.

[5] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

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