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DOUBLE SEQUENCE SPACES OVER n-NORMED SPACES

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Abstract. In this paper, we define some classes of double sequences over n-normed spaces by means of an Orlicz function. We study some relevant algebraic and topological properties. Further some inclusion relations among the classes are also examined.

1. Introduction and preliminaries

By w'' we shall denote the class of all double sequences. The initial works on double sequences is found in Bromwich [2]. Later on it was studied by Hardy [18], Moricz [23], Moricz and Rhoades [24], Tripathy ([32], [31]), Başarir and Sonalcan [1] and many others. Hardy [18] introduced the notion of regular convergence for double sequences. The concept of paranormed sequences was studied by Nakano [25] and Simmons [30] at initial stage. Later on it was studied by many others. The concept of 2-normed spaces was initially developed by Gähler [14] in the mid of 1960's while that of n-normed spaces one can see in Misiak [22]. Since, then many others have studied this concept and obtained various results, see Gunawan ([15], [16]) and Gunawan and Mashadi [17]. The notion of difference sequence spaces was introduced by Kızmaz [20], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [13] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty$, c, c_0 we have sequence spaces

$$Z(\Delta_s^m) = \left\{x = (x_k) \in w : (\Delta_s^m x_k) \in Z\right\},$$

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+sv} \qquad (\text{see [35]}).$$

Taking $s = 1$, we get the spaces which were studied by Et and Çolak [13]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kızmaz [20].

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Similarly, we can define difference operators on double sequence as:

$$\begin{aligned}\Delta a_{ij} &= (a_{ij} - a_{i\,j+1}) - (a_{i+1\,j} - a_{i+1\,j+1})\\ &= a_{ij} - a_{i\,j+1} - a_{i+1\,j} + a_{i+1\,j+1}\,.\end{aligned}$$

An Orlicz function $M\colon [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. Lindenstrauss and Tzafriri [21] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k)\in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\,, \quad \text{for some} \quad \rho>0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|(x_k)\| = \inf\Big\{\rho>0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}\,.$$

Also, it was shown in [21] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p\ge 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L>1$ there exists a constant $K(L)$ such that $M(Lu)\le K(L)M(u)$ for all values of $u\ge 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)\,dt$$

where η is known as the kernel of M, is right differentiable for $t\ge 0$, $\eta(0)=0$, $\eta(t)>0$, η is non-decreasing and $\eta(t)\to\infty$ as $t\to\infty$. Throughout, a double sequence is denoted by $ar = \langle a_{ij}\rangle$.

A double sequence space E is said to be solid if $\langle \alpha_{ij} a_{ij}\rangle \in E$ whenever $\langle a_{ij}\rangle \in E$ and for all double sequences $\langle \alpha_{ij}\rangle$ of scalars with $|\alpha_{ij}|\le 1$, for all $i,\, j\in\mathbb{N}$.

Let $n\in\mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d\ge n\ge 2$. A real valued function $\|\cdot,\ldots,\cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1,x_2,\ldots,x_n\| = 0$ if and only if $x_1,x_2,\ldots,x_n$ are linearly dependent in X;

(2) $\|x_1,x_2,\ldots,x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1,x_2,\ldots,x_n\| = |\alpha|\ \ \|x_1,x_2,\ldots,x_n\|$ for any $\alpha\in\mathbb{R}$,

(4) $\|x+x',x_2,\ldots,x_n\| \le \|x,x_2,\ldots,x_n\| + \|x',x_2,\ldots,x_n\|$

is called an n-norm on X and the pair $(X,\|\cdot,\ldots,\cdot\|)$ is called a n-normed space over the field $\mathbb{R}$.

For example, we may take $X=\mathbb{R}^n$ being equipped with the n-norm $\|x_1,x_2,\ldots,x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1,x_2,\ldots,x_n$ which may be given explicitly by the formula

$$\|x_1,x_2,\ldots,x_n\|_E = |\det(x_{ij})|\,,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \ldots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max\left\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| : i = 1, 2, \ldots, n\right\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every} \quad z_1, \ldots, z_{n-1} \in X\,.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every} \quad z_1, \ldots, z_{n-1} \in X\,.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. For more details about n-normed spaces (see [3], [5], [6], [8], [11], [12]) and references therein.

Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [36, Theorem 10.4.2, P-183]). For more details about sequence spaces (see [4] [7], [9], [10], [26], [27], [28], [29], [33], [34]) and references therein.

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and a_k, $b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

Let M be an Orlicz function and $p = \langle p_{ij} \rangle$ be a double sequence of strictly positive real numbers and $(X, \|\cdot, \ldots, \cdot\|)$ be a real linear $n-$normed space. Then we define the following classes of sequences:

$$\begin{aligned} &W''\big(M, \Delta, p, \|\cdot, \ldots, \cdot\|\big) \\ &\quad = \Big\{\langle a_{ij} \rangle \in w'' : \lim_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \Big(M\Big(\Big\|\frac{\Delta a_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} = 0\,, \\ &\qquad\qquad \text{for each } \ z_1, \ldots, z_{n-1} \in X, \ \text{ for some } \ \rho > 0 \ \text{ and } \ L > 0\Big\}, \end{aligned}$$

$$W_0''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$$
$$=\Big\{\langle a_{ij}\rangle\in w'':\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}=0\,,$$
$$\text{for each }\; z_1,\ldots,z_{n-1}\in X,\;\;\text{for some }\;\rho>0\Big\}$$

and

$$W_\infty''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$$
$$=\Big\{\langle a_{ij}\rangle\in w'':\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}<\infty\,,$$
$$\text{for some }\;\rho>0\Big\}\,.$$

If we take $p=(p_{ij})=1$, we get

$$W''\big(M,\Delta,\|\cdot,\ldots,\cdot\|\big)$$
$$=\Big\{\langle a_{ij}\rangle\in w'':\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta a_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)=0\,,$$
$$\text{for each }\; z_1,\ldots,z_{n-1}\in X,\;\;\text{for some }\;\rho>0\;\text{ and }\;L>0\Big\}\,,$$

$$W_0''\big(M,\Delta,\|\cdot,\ldots,\cdot\|\big)$$
$$=\Big\{\langle a_{ij}\rangle\in w'':\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)=0\,,$$
$$\text{for each }\; z_1,\ldots,z_{n-1}\in X,\;\;\text{for some }\;\rho>0\Big\}$$

and

$$W_\infty''\big(M,\Delta,\|\cdot,\ldots,\cdot\|\big)$$
$$=\Big\{\langle a_{ij}\rangle\in w'':\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)<\infty\,,$$
$$\text{for some }\;\rho>0\Big\}\,.$$

If we take $M(x)=x$, we get

$$W''\big(\Delta,p,\|\cdot,\ldots,\cdot\|\big)$$
$$=\Big\{\langle a_{ij}\rangle\in w'':\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\Big\|\frac{\Delta a_{ij}-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)^{p_{ij}}=0\,,$$
$$\text{for each }\; z_1,\ldots,z_{n-1}\in X,\;\;\text{for some }\;\rho>0\;\text{ and }\;L>0\Big\}\,,$$

$$W_0''\big(\Delta, p, \|\cdot,\ldots,\cdot\|\big) = \Big\{\langle a_{ij}\rangle \in w'' : \lim_{m,n} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)^{p_{ij}} = 0\,, \text{ for each } z_1,\ldots,z_{n-1}\in X, \text{ for some } \rho>0\Big\}$$

and

$$W_\infty''\big(\Delta, p, \|\cdot,\ldots,\cdot\|\big) = \Big\{\langle a_{ij}\rangle \in w'' : \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)^{p_{ij}} < \infty\,, \text{ for some } \rho>0\Big\}.$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces.

2. Some topological properties

In this section of the paper we study very interesting properties like linearity, paranorm and some attractive inclusion relation between the spaces $W''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$, $W_0''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ and $W_\infty''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$.

Theorem 2.1. *Let M be an Orlicz function and $p=(p_{ij})$ be bounded double sequence of strictly positive real numbers. Then the classes of sequences $W''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$, $W_0''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ and $W_\infty''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ are linear spaces over the field of real numbers $\mathbb{R}$.*

Proof. Let $\langle a_{ij}\rangle, \langle b_{ij}\rangle \in W_\infty''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ and α, $\beta\in\mathbb{R}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big[M\Big(\Big\|\frac{\Delta a_{ij}}{\rho_1}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \quad \text{for some} \quad \rho_1>0$$

and

$$\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big[M\Big(\Big\|\frac{\Delta b_{ij}}{\rho_2}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \quad \text{for some} \quad \rho_2>0\,.$$

Let $\rho_3=\max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot,\ldots,\cdot\|$ is a n-norm on X and M is non-decreasing, convex and so by using inequality (1.1), we have

$$\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta(\alpha\langle a_{ij}\rangle+\beta\langle b_{ij}\rangle)}{\rho_3},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}$$
$$\leq \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\|\frac{\Delta\alpha\langle a_{ij}\rangle}{\rho_3},z_1,\ldots,z_{n-1}\Big\|\Big)$$
$$+\Big(\Big\|\frac{\Delta\beta\langle b_{ij}\rangle}{\rho_3},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}$$
$$\leq K \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2^{p_{ij}}}\Big(M\Big(\Big\|\frac{\Delta\langle a_{ij}\rangle}{\rho_1},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}$$
$$+K \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2^{p_{ij}}}\Big(M\Big(\Big\|\frac{\Delta\langle b_{ij}\rangle}{\rho_2},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}$$
$$\leq K \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta\langle a_{ij}\rangle}{\rho_1},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}$$
$$+K \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M\Big(\Big\|\frac{\Delta\langle b_{ij}\rangle}{\rho_2},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{p_{ij}}$$
$$<\infty.$$

Thus, we have $\alpha\langle a_{ij}\rangle+\beta\langle b_{ij}\rangle \in W''_\infty\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$.
Hence $W''_\infty\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ is a linear space. Similarly, we can prove $W''\big(M,\Delta,p,\|\cdot,\cdots,\cdot\|\big)$ and $W''_0\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ are linear spaces over the field of real numbers $\mathbb{R}$. □

Theorem 2.2. *Let M be an Orlicz function and $p=(p_{ij})$ be bounded double sequence of strictly positive real numbers. The sequence spaces $W''\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$, $W''_0\big(M,\Delta,p\|\cdot,\ldots,\cdot\|\big)$ and $W''_\infty\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$ are paranormed spaces, paranormed by*

$$g(\langle a_{ij}\rangle)=\sup_i|a_{i1}|+\sup_j|a_{1j}|$$
$$+\inf\Big\{\rho^{\frac{p_{ij}}{H}}>0:\sup_{\substack{i,j\\ z_1,\ldots,z_{n-1}\in X}}\Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{\frac{p_{ij}}{H}}\leq 1\Big\},$$

where $H=\max(1,G)$, $G=\sup_{i,j}p_{ij}$.

Proof. Clearly $g(0)=0$, $g(-\langle a_{ij}\rangle)=g(\langle a_{ij}\rangle)$.
Let $\langle a_{ij}\rangle,\ \langle b_{ij}\rangle\in W''_\infty\big(M,\Delta,p,\|\cdot,\ldots,\cdot\|\big)$. Then there exist some $\rho_1,\ \rho_2>0$ such that

$$\sup_{\substack{i,j\\ z_1,\ldots,z_{n-1}\in X}}\Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho_1},z_1,\ldots,z_{n-1}\Big\|\Big)\Big)^{\frac{p_{ij}}{H}}\leq 1$$

and

$$\sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta b_{ij}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \le 1 .$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta a_{ij} + \Delta b_{ij}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \\
&\qquad \le \left(\frac{\rho_1}{\rho_1+\rho_2}\right) \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta a_{ij}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \\
&\qquad\quad + \left(\frac{\rho_2}{\rho_1+\rho_2}\right) \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta b_{ij}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \\
&\qquad \le 1 .
\end{aligned}$$

Now

$$\begin{aligned}
&g\big(\langle a_{ij}\rangle + \langle b_{ij}\rangle\big) = |a_{i1} + b_{i1}| + |a_{1j} + b_{1j}| \\
&\quad + \inf\left\{(\rho_1+\rho_2)^{\frac{p_{ij}}{H}} > 0 : \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta a_{ij} + \Delta b_{ij}}{\rho_1+\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \le 1\right\} \\
&\le |a_{i1}| + |b_{i1}| + \inf\left\{\rho_1^{\frac{p_{ij}}{H}} > 0 : \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta a_{ij}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \le 1\right\} \\
&\quad + |a_{1j}| + |b_{1j}| + \inf\left\{\rho_2^{\frac{p_{ij}}{H}} > 0 : \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta b_{ij}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \le 1\right\} \\
&= g\big(\langle a_{ij}\rangle\big) + g\big(\langle b_{ij}\rangle\big) .
\end{aligned}$$

Let $\lambda \in \mathbb{C}$, then the continuity of the product follows from the following inequality

$$\begin{aligned}
g(\lambda\langle a_{ij}\rangle) &= |\lambda a_{i1}| + |\lambda b_{i1}| \\
&\quad + \inf\left\{\rho^{\frac{p_{ij}}{H}} > 0 : \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta \lambda a_{ij}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \le 1\right\} \\
&= |\lambda||a_{i1}| + |\lambda||b_{i1}| \\
&\quad + \inf\left\{(|\lambda| r)^{\frac{p_{ij}}{H}} > 0 : \sup_{\substack{i,j \\ z_1,\dots,z_{n-1}\in X}} \left(M\left(\left\|\frac{\Delta a_{ij}}{r}, z_1, \dots, z_{n-1}\right\|\right)\right)^{\frac{p_{ij}}{H}} \le 1\right\},
\end{aligned}$$

where $\frac{1}{r} = \frac{|\lambda|}{\rho}$. This completes the proof of the theorem. □

Theorem 2.3. *Let M be an Orlicz function and $p = (p_{ij})$ be bounded double sequence of strictly positive real numbers. The sequence spaces $W''\big(M, \Delta, p, \|\cdot, \dots, \cdot\|\big)$,*

$W_0''\big(M,\Delta,p,\|\cdot,\dots,\cdot\|\big)$ and $W_\infty''\big(M,\Delta,p,\|\cdot,\dots,\cdot\|\big)$ are complete paranormed spaces, paranormed defined by g.

Proof. Let $\langle a_{ij}^s\rangle$ be a Cauchy sequence in $W_\infty''\big(M,\Delta,p,\|\cdot,\dots,\cdot\|\big)$. Then $g(\langle a_{ij}^s - a_{ij}^t\rangle)\to 0$ as $s,\, t\to\infty$. For a given $\epsilon>0$, choose $r>0$ and $x_0>0$ be such that $\frac{\epsilon}{rx_0}>0$ and $M\big(\frac{rx_0}{2}\big)\geq 1$. Now $g\big(\langle a_{ij}^s - a_{ij}^t\rangle\big)\to 0$ as $s,\, t\to\infty$ implies that there exists $m_0\in N$ such that

$$g\big(\langle a_{ij}^s - a_{ij}^t\rangle\big) < \frac{\epsilon}{rx_0} \quad \text{for all} \quad s,t\geq m_0\,.$$

Thus, we have

$$\begin{aligned}
&\sup_i |a_{i1}^s - a_{i1}^t| + \sup_j |a_{1j}^s - a_{1j}^t| \\
&\qquad + \inf\Big\{\rho^{\frac{p_{ij}}{H}} : \sup_{\substack{i,j\\ z_1,\dots,z_{n-1}\in X}} \Big(M\Big(\Big\|\frac{\Delta a_{ij}^s - \Delta_{ij}^t}{\rho}, z_1,\dots,z_{n-1}\Big\|\Big)\Big)^{\frac{p_{ij}}{H}} \leq 1\Big\}
\end{aligned}$$

$$< \frac{\epsilon}{rx_0}\,. \tag{2.1}$$

This shows that $\langle a_{i1}^s\rangle$, $\langle a_{1j}^t\rangle$ are Cauchy sequences of real numbers. As the set of real numbers is complete so there exists real numbers a_{i1}, a_{1j} such that

$$\lim_{s\to\infty} a_{i1}^s = a_{i1}\,, \quad \lim_{s\to\infty} a_{1j}^s = a_{1j}\,.$$

Now from (2.1) we have,

$$\begin{aligned}
&\Big(M\Big(\Big\|\frac{\Delta a_{ij}^s - \Delta a_{ij}^t}{\rho}, z_1,\dots,z_{n-1}\Big\|\Big)\Big) \leq 1 \\
\Longrightarrow &\sup_{i,j}\Big(M\Big(\Big\|\frac{\Delta a_{ij}^s - \Delta a_{ij}^t}{\rho}, z_1,\dots,z_{n-1}\Big\|\Big)\Big) \leq 1 \leq M\Big(\frac{rx_0}{2}\Big) \\
\Longrightarrow &\frac{\|(\Delta a_{ij}^s - \Delta a_{ij}^t), z_1,\dots,z_{n-1}\|}{g(\langle a_{ij}^s - a_{ij}^t\rangle)} \leq \frac{rx_0}{2} \\
\Longrightarrow &\|(\Delta a_{ij}^s - \Delta a_{ij}^t), z_1,\dots,z_{n-1}\| < \frac{rx_0}{2}\cdot\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}\,.
\end{aligned}$$

This implies $\langle\Delta_{ij}^s\rangle$ is a Cauchy sequence of real numbers. Let $\lim_{s\to\infty}\Delta a_{ij}^s = y_{ij}$ for all $i,\, j\in N$. Now $\Delta a_{11}^s = a_{11}^s - a_{12}^s - a_{21}^s + a_{22}^s$ and so

$$\lim_{s\to\infty} a_{22}^s = \lim_{s\to\infty}\big(\Delta a_{11}^s - a_{11}^s + a_{12}^s + a_{21}^s\big) = y_{11} - a_{11} + a_{12} + a_{21}\,.$$

Hence $\lim_{s\to\infty} a_{22}^s$ exists. Proceeding in this way we conclude that $\lim_{s\to\infty} a_{ij}^s$ exists. Using continuity of M, we have

$$\lim_{t\to\infty}\sup_{\substack{i,j\\ z_1,\dots,z_{n-1}\in X}}\Big(M\Big(\Big\|\frac{\Delta a_{ij}^s - \Delta a_{ij}^t}{\rho}, z_1,\dots,z_{n-1}\Big\|\Big)\Big) \leq 1\,.$$

Let $s\geq m_0$, then taking infimum of such ρ's we have $g(\langle a_{ij}^s - a_{ij}\rangle) < \epsilon$. Thus $\langle a_{ij}^s - a_{ij}\rangle \in W_\infty''(M,\Delta,p,\|\cdot,\dots,\cdot\|)$. By linearity of the space $W_\infty''(M,\Delta,p,\|\cdot,\dots,\cdot\|)$

we have $\langle a_{ij}\rangle \in W''_\infty(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$. Hence $W''_\infty(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$ is complete. □

Theorem 2.4. *Let M be an Orlicz function and $p = (p_{ij})$ be bounded double sequence of strictly positive real numbers. Then*
(i) $W''(M,\Delta,p,\|\cdot,\ldots,\cdot\|) \subset W''_\infty(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$
(ii) $W''_0(M,\Delta,p,\|\cdot,\ldots,\cdot\|) \subset W''_\infty(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$.

Proof. The proof is easy so we omit it. □

Theorem 2.5. *Let M be an Orlicz function and $p = (p_{ij})$ be bounded double sequence of strictly positive real numbers. Then the spaces $W''(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$ and $W''_0(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$ are nowhere dense subset of $W''_\infty(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$.*

Proof. The proof is easy so we omit it. □

Theorem 2.6. *Let M be an Orlicz function and $p = (p_{ij})$ be bounded double sequence of strictly positive real numbers. Then the following relation holds:*
(i) *If $0 < \inf p_{ij} \le p_{ij} < 1$, then $W''(M,\Delta,p,\|\cdot,\ldots,\cdot\|) \subseteq W''(M,\Delta,\|\cdot,\ldots,\cdot\|)$,*
(ii) *If $1 < p_{ij} \le \sup p_{ij} < \infty$, then $W''(M,\Delta,\|\cdot,\ldots,\cdot\|) \subseteq W''(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$.*

Proof. (i) Let $\langle a_{ij}\rangle \in W''(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$; since $0 < \inf p_{ij} \le p_{ij} < 1$, we have

$$\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right) \le \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right)^{p_{ij}},$$

and hence $\langle a_{ij}\rangle \in W''(M,\Delta,\|\cdot,\ldots,\cdot\|)$.

(ii) Let $p_{ij} > 1$ for each (ij) and $\sup_{i,j} p_{ij} < \infty$. Let $\langle a_{ij}\rangle \in W''(M,\Delta,\|\cdot,\ldots,\cdot\|)$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right) \le \epsilon < 1,$$

for all $m,n \ge \mathbb{N}$. Since

$$\sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right)^{p_{ij}} \le \sup_{\substack{m,n\\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ij}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right).$$

Hence, $\Delta a_{ij} \in W''(M,\Delta,p,\|\cdot,\ldots,\cdot\|)$ and this completes the proof. □

Theorem 2.7. *Let M_1 and M_2 be Orlicz functions, then we have*

$$W''_\infty(M_1,\Delta,p,\|\cdot,\ldots,\cdot\|) \cap W''_\infty(M_2,\Delta,p,\|\cdot,\ldots,\cdot\|) \subseteq W''_\infty(M_1+M_2,\Delta,p,\|\cdot,\ldots,\cdot\|).$$

Proof. Let $\langle a_{ij}\rangle \in W''_\infty(M_1, \Delta, p, \|\cdot, \ldots, \cdot\|) \cap W''_\infty(M_2, \Delta, p, \|\cdot, \ldots, \cdot\|)$. Then

$$\lim_{mn} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \Big(M_1\Big(\Big\|\frac{\Delta a_{ij} - L}{\rho_1}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} = 0\,, \quad \text{for some} \quad \rho_1 > 0\,,$$
$$\text{for each} \quad z_1, \ldots, z_{n-1} \in X$$

and

$$\lim_{mn} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \Big(M_2\Big(\Big\|\frac{\Delta a_{ij} - L}{\rho_2}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} = 0\,, \quad \text{for some} \quad \rho_2 > 0\,,$$
$$\text{for each} \quad z_1, \ldots, z_{n-1} \in X\,.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}\Big((M_1 + M_2)\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} \\
&\quad = \sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M_1\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) + M_2\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} \\
&\quad \le K\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M_1\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} \\
&\qquad + K\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(M_2\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}}\,.
\end{aligned}$$

□

Theorem 2.8. *The sequence space* $W''_\infty\big(\mathcal{M}', A, p, \|\cdot, \ldots, \cdot\|\big)$ *is solid.*

Proof. Let $\langle a_{ij}\rangle \in W''_\infty(M, \Delta, p, \|\cdot, \ldots, \cdot\|)$, i.e.

$$\sup_{\substack{m,n \\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} < \infty\,.$$

Let (α_{ij}) be double sequence of scalars such that $|\alpha_{ij}| \le 1$ for all $i, j \in \mathbb{N} \times \mathbb{N}$. Then we get

$$\begin{aligned}
&\sup_{\substack{m,n \\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \Big(M\Big(\Big\|\frac{\Delta \alpha_{ij} a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}} \\
&\qquad \le \sup_{\substack{m,n \\ z_1,\ldots,z_{n-1}\in X}} \frac{1}{mn} \sum_{i}^{m}\sum_{j=1}^{n} \Big(M\Big(\Big\|\frac{\Delta a_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)^{p_{ij}}
\end{aligned}$$

and this completes the proof. □

Theorem 2.9. *The sequence space* $W''_\infty(M, \Delta, p, \|\cdot, \ldots, \cdot\|)$ *is monotone.*

Proof. It is obvious. □

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