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RIEMANNIAN FOLIATIONS WITH PARALLEL OR HARMONIC BASIC FORMS

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Abstract. In this paper, we consider a Riemannian foliation that admits a nontrivial parallel or harmonic basic form. We estimate the norm of the O'Neill tensor in terms of the curvature data of the whole manifold. Some examples are then given.

1. Introduction

In [3], J. F. Grosjean obtained some non-existence results on minimal submanifolds carrying parallel or harmonic forms. Indeed, given a Riemannian manifold (M^m, g) admitting a parallel p-form and let (N^n, h) be a Riemannian manifold satisfying a certain curvature pinching condition depending on m and p, he proved that there is no minimal immersion from M into N. His proof is based on computing the curvature term (which is zero in this case) in the Bochner-Weitzenböck formula and using the Gauss formula relating the curvatures of M and N. As a consequence, he deduced various rigidity results when N is the hyperbolic space $\mathbb{H}^n$, the Riemannian product $\mathbb{H}^r \times \mathbb{S}^s$ or the complex hyperbolic space $\mathbb{CH}^n$.

In the same spirit, he proved that for any compact manifold (M^m, g) carrying a harmonic p-form (or a non-zero pth betti number $b_p(M)$) and isometrically immersed into a Riemannian manifold (N^n, h), there exists at least a point x of M so that (see also [1])

$$\frac{m}{\sqrt{p}}\Big(\frac{p-1}{p}\Big)|B(x)||H(x)| \geq k(x) - \Big(\frac{p-1}{p}\Big)((m-1)\bar{K}_1 + \bar{\rho}_1)(x)\,,$$

and

$$m\Big(\frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}}\Big)|B(x)||H(x)| \geq \mathrm{Scal}^M(x) - (m-2)[(m-1)\bar{K}_1 + \bar{\rho}_1](x),$$

where $|B(x)|$, $H(x)$, $k(x)$ and $\bar{\rho}_1(x)$ denote respectively the norm of the second fundamental form B, the mean curvature of the immersion, the smallest eigenvalue of the Ricci curvature of M, the largest eigenvalue of the curvature operator of

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(N^n, h) and $\bar{K}_1(x)$ is the largest sectional curvature of N. These inequalities come from a lower bound of the curvature term (which is non-positive at the point x) in the Bochner-Weitzenböck formula. Thus, if the manifold (M^m, g) is minimally immersed into (N^n, h) and satisfying the pinching condition

$$\min_M(\mathrm{Scal}^M) > (m-2)\big((m-1)\max_N(\bar{K}_1) + \max_N(\bar{\rho}_1)\big)\,,$$

then (M^m, g) is a homology sphere (see also [6]).

In this paper, we investigate the study of foliated manifolds that admits a nontrivial particular form. In fact, we consider a Riemannian manifold (M, g) equipped with a Riemannian foliation $\mathcal{F}$, which roughly speaking, is the decomposition of M into submanifolds (called leaves) given by local Riemannian submersions to a base manifold. We assume that the manifold M admits a parallel (resp. harmonic) basic p-form , with respect to the connection defined in Section 2. This corresponds locally to the existence of such a form on the base manifold of the submersions. When shifting the study from immersions to submersions, many objects are replaced by their dual. In particular, the O'Neill tensor [7] plays the role of the second fundamental form and thus, we aim to estimate the norm of the O'Neill tensor in terms of different curvature data of the manifold M. The main tool is to use the transverse Bochner-Weitzenböck formula for foliations [4]. Recall that this tensor completely determines the geometry of the foliation. Indeed, it vanishes if and only if the normal bundle of the foliation is integrable.

The paper is organized as follows. In Section 2, we recall some well-known facts on differential forms and review some preliminaries on Riemannian foliations. In Section 3, we treat the case where the manifold admits a parallel basic form. We compute the curvature term in the transverse Bochner-Weitzenböck formula and relate it to the curvature of the manifold M using the O'Neill formulas. We then deduce a lower bound estimate for the O'Neill tensor (see Thm. 3.3 for $p > 1$ and Cor. 3.2 for a rigidity result when $p = 1$). In the last section, we study the case where there exists a harmonic basic form. As before, we deduce a new estimate of the O'Neill tensor (see Thm. 4.2).

2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension n and ∇^M be the Levi-Civita connection associated with the metric g. In all the paper, we make the following notations for the curvatures: $R^M(X,Y) = \nabla^M_{[X,Y]} - [\nabla^M_X, \nabla^M_Y]$ and $R^M_{XYZW} = g(R^M(X,Y)Z,W)$ for any $X, Y, Z, W \in \Gamma(TM)$. We will denote respectively by $K^M_0(x)$ and $K^M_1(x)$ the smallest and the largest sectional curvature and by $\rho^M_0(x)$ and $\rho^M_1(x)$ the smallest and largest eigenvalue of the curvature operator $\rho^M(X \wedge Y, Z \wedge W) = g(R^M(X,Y)Z,W)$ at a point $x \in M$. Thus, we have the following inequalities

$$\rho^M_0(x) \le K^M_0(x) \le K^M_1(x) \le \rho^M_1(x)\,. \tag{2.1}$$

Now, let us recall some definitions on forms. The inner product of any two p-forms α and β is defined as

$$\langle\alpha,\beta\rangle = \frac{1}{p!}\sum_{1\le i_1,\dots,i_p\le n}\alpha(e_{i_1},e_{i_2},\dots,e_{i_p})\beta(e_{i_1},e_{i_2},\dots,e_{i_p}),$$

where $\{e_1,\dots,e_n\}$ is an orthonormal frame of TM. The interior product of a p-form α with a vector field X is a $(p-1)$-form defined by

$$(X\lrcorner\alpha)(X_1,\dots,X_{p-1}) = \alpha(X,X_1,\dots,X_{p-1}).$$

More generally, the interior product of α with s vector fields $X_1,X_2,\dots,X_s$ is a $(p-s)$-form which is defined as the following

$$((X_1\wedge\cdots\wedge X_s)\lrcorner\alpha)(Y_1,\dots,Y_{p-s}) = \alpha(X_s,\dots,X_1,Y_1,\dots,Y_{p-s}).$$

As a consequence from the definition, we get the rule $X\lrcorner(\omega\wedge\theta) = (X\lrcorner\omega)\wedge\theta + (-1)^p\omega\wedge(X\lrcorner\theta)$, where p is the degree of ω. If the manifold is orientable, the Hodge operator $*$ defined on a p-form α satisfies the following property:

$$X\lrcorner(*\alpha) = (-1)^p * (X^*\wedge\alpha). \tag{2.2}$$

Assume now that (M^n,g) is endowed with a Riemannian foliation $\mathcal{F}$ of codimension q. That means $\mathcal{F}$ is given by an integrable subbundle L of TM of rank $n-q$ such that the metric g satisfies the holonomy-invariance condition on the normal vector bundle $Q = TM/L$; that is $\mathcal{L}_X g|_Q = 0$ for all $X\in\Gamma(L)$, where $\mathcal{L}$ denotes the Lie derivative [9]. We call g a bundle-like metric. This latter condition gives rise to a transverse Levi-Civita connection on Q defined by [10]

$$\nabla_X Y = \begin{cases}\pi[X,Y], & \text{if} \quad X\in\Gamma(L)\ ,\\ \pi(\nabla^M_X Y), & \text{if} \quad X\in\Gamma(Q),\end{cases}$$

where $\pi\colon TM\to Q$ is the projection. A fundamental property of the connection ∇ is that it is flat along the leaves, that is $X\lrcorner R^\nabla = 0$ for any $X\in\Gamma(L)$. Thus, we can associate to ∇ all the curvature data such as the transverse Ricci curvature Ric^∇ and transverse scalar curvature Scal^∇. A basic form α on M is a differential form which depends locally on the transverse variables; that is satisfying the rules $X\lrcorner\alpha = 0$ and $X\lrcorner d\alpha = 0$ for any $X\in\Gamma(L)$. It is easy to see that the exterior derivative d preserves the set of basic forms, and its restriction to this set will be denoted by d_b. We let δ_b the formal adjoint of d_b with respect to the L^2-product. Then we have

$$d_b = \sum_{i=1}^q e_i\wedge\nabla_{e_i}, \qquad \delta_b = -\sum_{i=1}^q e_i\lrcorner\nabla_{e_i} + \kappa_b\lrcorner,$$

where $\{e_i\}_{i=1,\dots,q}$ is a local orthonormal frame of Q and κ_b is the basic component of the mean curvature field of the foliation $\kappa = \sum\limits_{s=1}^{n-q}\pi(\nabla^M_{V_s}V_s)$. Here $\{V_s\}_{s=1,\dots,n-q}$ is a local orthonormal frame of L. The basic Laplacian is defined as $\Delta_b = d_b\delta_b + \delta_b d_b$.

Recall that when the foliation is transversally orientable, the basic Hodge operator $*_b$ is defined on the set of basic p-forms as being

$$*_b\alpha = (-1)^{(n-q)(q-p)} * (\alpha \wedge \chi_{\mathcal{F}})\,,$$

where $\chi_{\mathcal{F}}$ is the volume form of the leaves. The operator $*_b$ preserves the basic forms and satisfies the same property as (2.2). In [4], the authors define a new twisted exterior derivative $\tilde{d}_b := d_b - \frac{1}{2}\kappa_b\wedge$ and prove that the associated twisted Laplacian $\tilde{\Delta}_b := \tilde{d}_b\tilde{\delta}_b + \tilde{\delta}_b\tilde{d}_b$ commutes with the basic Hodge operator. In particular, this shows that the twisted cohomology group (i.e. the one associated with $\tilde{d}_b$) satisfies the Poincaré duality. Here $\tilde{\delta}_b := \delta_b - \frac{1}{2}\kappa_b\lrcorner$ denotes the L^2-adjoint of $\tilde{d}_b$. Moreover, they state the transverse Bochner-Weitzenböck formula for $\tilde{\Delta}_b$

$$\tilde{\Delta}_b\alpha = \nabla^*\nabla\alpha + \frac{1}{4}|\kappa_b|^2\alpha + R(\alpha)\,,$$

where $R(\alpha) = -\sum\limits_{j=1}^{q} e_j^* \wedge (e_i\lrcorner R^\nabla(e_i,e_j)\alpha)$. As for ordinary manifolds [3], the scalar product of $R(\alpha)$ by α gives after the use of the first Bianchi identity that

$$\begin{aligned} \big\langle R(\alpha),\alpha\big\rangle = &\sum_{1\leq i,j\leq q} \mathrm{Ric}^\nabla_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle \\ &-\frac{1}{2}\sum_{1\leq i,j,k,l\leq q} R^\nabla_{ijkl}\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle\,. \end{aligned} \tag{2.3}$$

On the other hand, the geometry of a Riemannian foliation can be interpreted in terms of the so-called the O'Neill tensor [7]. It is a 2-tensor field given for all X, $Y \in \Gamma(TM)$ by

$$A_XY = \pi^\perp\big(\nabla^M_{\pi(X)}\pi(Y)\big) + \pi\big(\nabla^M_{\pi(X)}\pi^\perp(Y)\big)\,,$$

where $\pi^\perp$ denotes the projection of TM onto L. By the bundle-like condition, the O'Neill tensor is a skew-symmetric tensor with respect to the vector fields Y, $Z \in \Gamma(Q)$ and it is equal to $A_YZ = \frac{1}{2}\pi^\perp([Y,Z])$ and for any $V \in \Gamma(L)$ we have $g(A_YV,Z) = -g(V,A_YZ)$. Thus we deduce that the normal bundle is integrable if and only if the O'Neill tensor vanishes. If moreover the bundle L is totally geodesic, the foliation is isometric to a local product.

We point out that the curvature of M can be related to the one on the normal bundle Q via the O'Neill tensor by the formula [7]

$$R^M_{XYZW} = R^Q_{XYZW} - 2g(A_XY,A_ZW) + g(A_YZ,A_XW) + g(A_ZX,A_YW)\,, \tag{2.4}$$

where X, Y, Z, W are vector fields in $\Gamma(Q)$. One can easily see by (2.4) that the norm of the O'Neill tensor $|A|^2 := \sum\limits_{\substack{1\leq i\leq q\\ 1\leq s\leq n-q}} |A_{e_i}V_s|^2$ can be bounded at any point by

$$\mathrm{Scal}^\nabla - q(q-1)K_1^M \leq 3|A|^2 \leq \mathrm{Scal}^\nabla - q(q-1)K_0^M\,.$$

In particular, if the transversal scalar curvature does not belong to the interval $[q(q-1)K_0^M, q(q-1)K_1^M]$, the normal bundle cannot be integrable.

3. Foliations with parallel basic forms

In this section, we discuss the case where a Riemannian manifold endowed with a Riemannian foliation admits a parallel basic form. That is a basic p-form α satisfying $\nabla\alpha = 0$.

Proposition 3.1. *Let $(M, g, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension q. Assume that there exists a nontrivial parallel basic p-form α. Then we have*

$$\begin{aligned} 0 \leq &- \sum_{1\leq i,j,l\leq q} R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \frac{1}{2}\sum_{1\leq i,j,k,l\leq q} R^M_{ijkl}\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\rangle \\ &+ \sum_{s=1}^{n-q}\left\{\Big|\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i)\lrcorner\alpha\Big|^2 - 2\sum_{i=1}^{q}|A_{e_i}V_s\lrcorner\alpha|^2\right\}, \end{aligned} \tag{3.1}$$

where $\{e_i\}_{i=1,\dots,q}$ and $\{V_s\}_{s=1,\dots,n-q}$ are local orthonormal frames of Q and L, respectively.

Proof. From Equation (2.4), we have the following formulas

$$R^{\nabla}_{ijkl} = R^M_{ijkl} + 2g(A_{e_i}e_j, A_{e_k}e_l) - g(A_{e_j}e_k, A_{e_i}e_l) - g(A_{e_k}e_i, A_{e_j}e_l), \tag{3.2}$$

and that,

$$\mathrm{Ric}^{\nabla}_{ij} = \sum_{l=1}^{q}\left\{R^M_{lilj} + 2g(A_{e_l}e_i, A_{e_l}e_j) - g(A_{e_i}e_l, A_{e_l}e_j) - g(A_{e_l}e_l, A_{e_i}e_j)\right\}. \tag{3.3}$$

The existence of a parallel basic form α implies that $\langle R(\alpha), \alpha\rangle = 0$. Thus plugging these last two equations into (2.3), we get that

$$\begin{aligned} 0 = &\sum_{1\leq i,j,l\leq q} R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + 3g(A_{e_l}e_i, A_{e_l}e_j)\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle \\ &\times \sum_{1\leq i,j,k,l\leq q}\Big\{ -\frac{1}{2}R^M_{ijkl}\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle \\ &- g(A_{e_i}e_j, A_{e_k}e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle \\ &+ \frac{1}{2}g(A_{e_j}e_k, A_{e_i}e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, e_l\wedge e_k\lrcorner\alpha\big\rangle \\ &+ \frac{1}{2}g(A_{e_k}e_i, A_{e_j}e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle\Big\}. \end{aligned} \tag{3.4}$$

The last two summations in the above equality are in fact equal. Indeed, using that the O'Neill tensor is antisymmetric, we find

$$\begin{aligned} &\sum_{1\leq i,j,k,l\leq q} g(A_{e_k}e_i, A_{e_j}e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle \\ &\qquad = -\sum_{1\leq i,j,k,l\leq q} g(A_{e_i}e_k, A_{e_j}e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle \end{aligned}$$

$$= - \sum_{1\leq i,j,k,l\leq q} g(A_{e_i}e_l, A_{e_j}e_k)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_k\wedge e_l)\lrcorner\alpha\big\rangle$$

$$= \sum_{1\leq i,j,k,l\leq q} g(A_{e_i}e_l, A_{e_j}e_k)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle\,.$$

On the other hand, we have

$$\sum_{1\leq i,j,k,l\leq q} g(A_{e_i}e_j, A_{e_k}e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle$$

$$= \sum_{\substack{1\leq i,j,k,l\leq q\\ 1\leq s\leq n-q}} g(A_{e_i}e_j, V_s)g(A_{e_k}e_l, V_s)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle$$

$$= \sum_{\substack{1\leq i,j,k,l\leq q\\ 1\leq s\leq n-q}} g(A_{e_i}V_s, e_j)g(A_{e_k}V_s, e_l)\big\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle$$

$$= \sum_{s=1}^{n-q}\Big\langle \Big(\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i)\Big)\lrcorner\alpha, \Big(\sum_{k=1}^{q}(A_{e_k}V_s\wedge e_k)\Big)\lrcorner\alpha\Big\rangle$$

$$= \sum_{s=1}^{n-q}\Big|\Big(\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i)\Big)\lrcorner\alpha\Big|^2\,. \tag{3.5}$$

Also we have that

$$\sum_{1\leq i,j,l\leq q} g(A_{e_l}e_i, A_{e_l}e_j)\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle = \sum_{\substack{1\leq i,j,l\leq q\\ 1\leq s\leq n-q}} g(A_{e_l}e_i, V_s)g(A_{e_l}e_j, V_s)\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle$$

$$= \sum_{\substack{1\leq i,j,l\leq q\\ 1\leq s\leq n-q}} g(A_{e_l}V_s, e_i)g(A_{e_l}V_s, e_j)\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle$$

$$= \sum_{\substack{1\leq l\leq q\\ 1\leq s\leq n-q}} \langle A_{e_l}V_s\lrcorner\alpha, A_{e_l}V_s\lrcorner\alpha\rangle = \sum_{\substack{1\leq l\leq q\\ 1\leq s\leq n-q}} |A_{e_l}V_s\lrcorner\alpha|^2\,. \tag{3.6}$$

In order to estimate the last term in (3.4), we introduce the p-tensor

$$\mathcal{B}^+(\alpha)(X_1,\dots,X_p) = \sum_{i=1}^{q}(e_i\lrcorner\alpha\wedge A_{e_i})(X_1,\dots,X_p)\,,$$

for any $X_1,\dots,X_p\in\Gamma(Q)$. We now proceed the computation as in [3]. The norm of the tensor $\mathcal{B}^+(\alpha)$ is equal to

$$|\mathcal{B}^+(\alpha)|^2 = \frac{1}{p!}\sum_{1\leq i_1,\dots,i_p,i,j\leq q}\big\langle (e_i\lrcorner\alpha\wedge A_{e_i})_{i_1,\dots,i_p}, (e_j\lrcorner\alpha\wedge A_{e_j})_{i_1,\dots,i_p}\big\rangle$$

$$= \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p,i,j\leq q\\ r,t}} (-1)^{r+t} g(A_{e_i}e_{i_r}, A_{e_j}e_{i_t})\alpha_{ii_1,\dots,\hat{i}_r,\dots,i_p}\alpha_{ji_1,\dots,\hat{i}_t,\dots,i_p}$$

$$
\begin{aligned}
&= \frac{1}{p!}\sum_{\substack{1\le i_1,\dots,i_p,i,j\le q\\ r=t}} g(A_{e_i}e_{i_r},A_{e_j}e_{i_r})\alpha_{ii_1,\dots,\hat{i}_r,\dots,i_p}\alpha_{ji_1,\dots,\hat{i}_r,\dots,i_p}\\
&\quad+\frac{1}{p!}\sum_{\substack{1\le i_1,\dots,i_p,i,j\le q\\ r<t}} (-1)^{r+t} g(A_{e_i}e_{i_r},A_{e_j}e_{i_t})\alpha_{ii_1,\dots,\hat{i}_r,\dots,i_p}\alpha_{ji_1,\dots,\hat{i}_t,\dots,i_p}\\
&\quad+\frac{1}{p!}\sum_{\substack{1\le i_1,\cdots,i_p,i,j\le q\\ r>t}} (-1)^{r+t} g(A_{e_i}e_{i_r},A_{e_j}e_{i_t})\alpha_{ii_1,\dots,\hat{i}_r,\dots,i_p}\alpha_{ji_1,\dots,\hat{i}_t,\dots,i_p}\\
&= \frac{1}{(p-1)!}\sum_{1\le i_1,\dots,i_{p-1},i,j,k\le q} g(A_{e_i}e_k,A_{e_j}e_k)\alpha_{ii_1,\dots,i_{p-1}}\alpha_{ji_1,\dots,i_{p-1}}\\
&\quad-\frac{2}{p!}\sum_{\substack{1\le i_1,\dots,i_p,i,j\le q\\ r<t}} g(A_{e_i}e_{i_r},A_{e_j}e_{i_t})\alpha_{ii_ti_1,\dots,\hat{i}_r,\dots,\hat{i}_t,\dots,i_p}\alpha_{ji_ri_1,\dots,\hat{i}_r,\dots,\hat{i}_t,\dots,i_p}.
\end{aligned}
$$

Since we can choose $\frac{p(p-1)}{2}$ numbers r,t with $r<t$ from the set $\{1,\cdots,p\}$, the last equality can be reduced to

$$
\begin{aligned}
|\mathcal{B}^+(\alpha)|^2 &= \frac{1}{(p-1)!}\sum_{1\le i_1,\cdots,i_{p-1},i,j,k\le q} g(A_{e_i}e_k,A_{e_j}e_k)\alpha_{ii_1,\dots,i_{p-1}}\alpha_{ji_1,\dots,i_{p-1}}\\
&\quad-\frac{1}{(p-2)!}\sum_{1\le i_1,\dots,i_{p-2},i,j,k,l\le q} g(A_{e_i}e_k,A_{e_j}e_l)\alpha_{ili_1,\dots,i_{p-2}}\alpha_{jki_1,\dots,i_{p-2}}\\
&= \sum_{1\le i,j,k\le q} g(A_{e_i}e_k,A_{e_j}e_k)\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle\\
&\quad-\sum_{1\le i,j,k,l\le q} g(A_{e_i}e_k,A_{e_j}e_l)\big\langle(e_l\wedge e_i)\lrcorner\alpha,(e_k\wedge e_j)\lrcorner\alpha\big\rangle\\
&= \sum_{1\le i,j,k\le q} g(A_{e_k}e_i,A_{e_k}e_j)\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle\\
&\quad+\sum_{1\le i,j,k,l\le q} g(A_{e_i}e_l,A_{e_j}e_k)\langle(e_j\wedge e_i)\lrcorner\alpha,(e_l\wedge e_k)\lrcorner\alpha\rangle\,. \qquad (3.7)
\end{aligned}
$$

Returning back to the Equation (3.4) and after plugging Equations (3.5), (3.6) and (3.7), we get the following

$$
\begin{aligned}
0 &= \sum_{1\le i,j,l\le q} R^M_{lilj}\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle-\frac{1}{2}\sum_{1\le i,j,k,l\le q} R^M_{ijkl}\big\langle(e_j\wedge e_i)\lrcorner\alpha,(e_l\wedge e_k)\lrcorner\alpha\big\rangle\\
&\quad+|\mathcal{B}^+(\alpha)|^2+2\sum_{\substack{1\le l\le q\\ 1\le s\le n-q}} |A_{e_l}V_s\lrcorner\alpha|^2-\sum_{s=1}^{n-q}\Big|\Big(\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i)\Big)\lrcorner\alpha\Big|^2.
\end{aligned}
$$

Finally using the fact that $|\mathcal{B}^+(\alpha)|^2\ge 0$, we deduce the desired inequality. $\square$

For $p = 1$, we find by (3.1) that the lowest sectional curvature K_0^M should be non-positive. Hence we have

Corollary 3.2. *Let $(M, g, \mathcal{F})$ be a Riemannian manifold with positive sectional curvature and endowed with a Riemannian foliation $\mathcal{F}$. Then M does not admit a parallel basic 1-form.*

In the following, we will treat the case $p \geq 2$. For that, we aim to estimate each term in inequality (3.1). As in [3], we define the basic 2-form $\theta^{i_1,\dots,i_{p-2}} = \frac{1}{2}\sum\limits_{1\leq i,j\leq q} \alpha_{iji_1,\dots,i_{p-2}} e_i \wedge e_j$. Thus the second term of inequality (3.1) can be bounded from above by

$$\begin{aligned}\frac{1}{2}\sum_{1\leq i,j,k,l\leq q} & R^M_{ijkl}\left\langle (e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\right\rangle \\ &= \frac{2}{(p-2)!}\sum_{1\leq i_1,\cdots,i_{p-2}\leq q} \rho^M(\theta^{i_1,\dots,i_{p-2}}, \theta^{i_1,\dots,i_{p-2}}) \\ &\leq \frac{2}{(p-2)!}\rho_1^M \sum_{1\leq i_1,\dots,i_{p-2}\leq q} |\theta^{i_1,\dots,i_{p-2}}|^2 = p(p-1)\rho_1^M|\alpha|^2 \,. \end{aligned} \tag{3.8}$$

Using the Cauchy-Schwarz inequality and the fact that $|v\wedge w\lrcorner\alpha| \leq |v|\,|w\lrcorner\alpha|$ for any vectors v, w, the last term in (3.1) is bounded by

$$\Big|\sum_{i=1}^q (A_{e_i}V_s\wedge e_i)\lrcorner\alpha\Big|^2 \leq q\sum_{i=1}^q |(A_{e_i}V_s\wedge e_i)\lrcorner\alpha|^2 \leq q\sum_{i=1}^q |A_{e_i}V_s\lrcorner\alpha|^2 \,. \tag{3.9}$$

Now we state our main result:

Theorem 3.3. *Let $(M, g, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension $q \geq 4$. Assume that the manifold admits a nontrivial parallel basic p-form α with $2 \leq p \leq q-2$. Then we have*

$$(q-2)|A|^2 \geq K_0^M q(q-1) - \big(p(p-1) + (q-p)(q-p-1)\big)\rho_1^M \,.$$

Proof. Plugging the estimates in (3.8) and (3.9) into inequality (3.1), we get that

$$\begin{aligned} 0 \leq & -\sum_{1\leq i,j,l\leq q} R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + p(p-1)\rho_1^M|\alpha|^2 \\ & + (q-2)\sum_{\substack{1\leq i\leq q\\ 1\leq s\leq n-q}} |A_{e_i}V_s\lrcorner\alpha|^2 \,. \end{aligned} \tag{3.10}$$

Since α is a parallel basic p-form, the $(q-p)$-form $*_b\alpha$ is also parallel. Thus replacing α by $*_b\alpha$ in (3.10), we find the inequality

$$\begin{aligned} 0 \leq & -\sum_{1\leq i,j,l\leq q} R^M_{lilj}\big\langle e_i\lrcorner(*_b\alpha), e_j\lrcorner(*_b\alpha)\big\rangle + (q-p)(q-p-1)\rho_1^M|\alpha|^2 \\ & + (q-2)\sum_{\substack{1\leq i\leq q\\ 1\leq s\leq n-q}} |A_{e_i}V_s\wedge\alpha|^2 \,. \end{aligned} \tag{3.11}$$

In the last term of (3.11), we use the equality 2.2 for the basic Hodge operator. Now the sum of inequalities (3.10) and (3.11) gives the desired inequality after the use of

$$\begin{aligned}
&\sum_{1\leq i,j,l\leq q} R^M_{lilj}\big(\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \langle e_i\lrcorner(*_b\alpha), e_j\lrcorner(*_b\alpha)\rangle\big)\\
&\qquad = \sum_{1\leq i,j,l\leq q} R^M_{lilj}\big(\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \langle e_i\wedge\alpha, e_j\wedge\alpha\rangle\big)\\
&\qquad = \sum_{1\leq i,j,l\leq q} R^M_{lilj}\big(\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \langle e_j\lrcorner(e_i\wedge\alpha), \alpha\rangle\big)\\
&\qquad = \sum_{1\leq i,j,l\leq q} R^M_{lilj}\big(\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \delta_{ij}|\alpha|^2 - \langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle\big)\\
&\qquad = \sum_{1\leq i,l\leq q} R^M_{lili}|\alpha|^2\,,
\end{aligned}\tag{3.12}$$

which is greater than $K_0^M q(q-1)|\alpha|^2$. □

We point out that the theorem is of interest only if

$$K_0^M q(q-1) - \big(p(p-1) + (q-p)(q-p-1)\big)\rho_1^M > 0\,,$$

which means by (2.1) that the manifold M is of positive sectional curvature.

Example. Let us consider the round sphere $\mathbb{S}^{2m-1}$ equiped with the standard metric of constant curvature 1. We denote by $\mathcal{F}$ the 1-dimensional Riemannian fibers given by the action [2]

$$e^{2i\pi t}(z_1,\dots,z_m) = (e^{2i\pi\theta_1 t}z_1, e^{2i\pi\theta_2 t}z_2,\dots,e^{2i\pi\theta_m t}z_m)\,,$$

with $0<\theta_1\leq\theta_2\leq\cdots\leq\theta_m\leq 1$. These foliations are Seifert fibrations (i.e. the fibers are compact) if and only if all $\theta_i's$ are rational and the Hopf fibration corresponds to the case where $\theta_1=\theta_2=\cdots=\theta_m=1$. In the following, we will compute the O'Neill tensor of the foliation $\mathcal{F}$ and study the optimality of the estimate in Theorem 3.3. Without loss of generality, we can assume that $\theta_1=1$. The vector X that generates $\mathcal{F}$ is given by

$$X = (iz_1, i\theta_2 z_2,\dots, i\theta_m z_m)\,.$$

For an integer $l\in\{1,\dots,m-1\}$ and $p\in\{1,\dots,m-2\}$, we define the vector fields Y_l and W_p on the tangent space of $\mathbb{S}^{2m-1}$ by the following [5]

$$Y_l = (0,\dots,0,-(\sum_{k=l+1}^m |z_k|^2)z_l, |z_l|^2 z_{l+1},\dots,|z_l|^2 z_m)\,,$$

and,

$$W_p = (0,\dots,0,-(\sum_{k=p+1}^m \theta_k^2|z_k|^2)iz_p, \theta_p\theta_{p+1}|z_p|^2 iz_{p+1},\dots,\theta_p\theta_m|z_p|^2 iz_m)\,.$$

We also denote by W_{m-1} the vector field on $T\mathbb{S}^{2m-1}$ by

$$W_{m-1} = (0, \dots, 0, -\theta_m |z_m|^2 i z_{m-1}, \theta_{m-1}|z_{m-1}|^2 i z_m)\,.$$

It is easy to see that the set $\{X, Y_l, W_p, W_{m-1}\}$ is an orthogonal frame of the tangent space of the sphere for any l and p. Recall now that given an orthonormal frame $\{X/|X|, e_i = Z_i/|Z_i|\}$ of the tangent space of the round sphere for $i = 1, \dots, 2m-2$, the norm of O'Neill tensor can be computed as follows

$$|A|^2 = \sum_{i,j} |A_{e_i} e_j|^2 = \frac{1}{4}\sum_{i,j} |\pi^\perp([e_i, e_j])|^2 = \frac{1}{2|X|^2}\sum_{i<j}\frac{1}{|Z_i|^2|Z_j|^2}|([Z_i, Z_j], X)|^2\,,$$

where $\pi^\perp\colon TM \to \mathbb{R}X$ is the projection. On the one hand, a straightforward computation of the norms yields to

$$|X|^2 = |z_1|^2 + \sum_{k=2}^m \theta_k^2 |z_k|^2\,.$$

Moreover, for any l and p, we have

$$|Y_l|^2 = |z_l|^2 \Big(\sum_{k=l+1}^m |z_k|^2\Big)\Big(\sum_{k=l}^m |z_k|^2\Big)\,,$$

$$|W_p|^2 = |z_p|^2 \Big(\sum_{t=p+1}^m \theta_t^2|z_t|^2\Big)\Big(\sum_{s=p}^m \theta_s^2 |z_s|^2\Big)\,.$$

Also we find that,

$$|W_{m-1}|^2 = (\theta_m^2 |z_m|^2 + \theta_{m-1}^2 |z_{m-1}|^2)|z_m|^2|z_{m-1}|^2\,.$$

On the other hand, the computation of the Lie brackets yields for any l to

$$([Y_l, W_l], X) = -2\theta_l |z_l|^2 \Big(\sum_{s=l+1}^m \theta_s^2 |z_s|^2\Big)\Big(\sum_{k=l}^m |z_k|^2\Big)\,,$$

and for $l > p$,

$$\big([Y_l, W_p], X\big) = 2|z_l|^2 \theta_p |z_p|^2 \sum_{k=l+1}^m (\theta_l^2 - \theta_k^2)|z_k|^2\,.$$

Also, we have that

$$\big([Y_{m-1}, W_{m-1}], X\big) = -2|z_{m-1}|^2|z_m|^2\theta_{m-1}\theta_m\big(|z_{m-1}|^2 + |z_m|^2\big)\,.$$

The other Lie brackets are all equal to zero. Thus the O'Neill tensor is equal to

$$|A|^2 = \frac{2}{|X|^2}\Big\{\frac{\theta_{m-1}^2\theta_m^2(|z_{m-1}|^2 + |z_m|^2)}{\theta_{m-1}^2|z_{m-1}|^2 + \theta_m^2|z_m|^2)} + \sum_{j=1}^{m-2}\frac{\theta_j^2(\sum_{s=j+1}^m \theta_s^2|z_s|^2)(\sum_{k=j}^m |z_k|^2)}{(\sum_{s=j}^m \theta_s^2|z_s|^2)(\sum_{k=j+1}^m |z_k|^2)}$$

$$+ \sum_{j=1}^{m-2}\sum_{i=j+1}^{m-1}\frac{|z_i|^2|z_j|^2(\sum_{k=i+1}^m(\theta_i^2 - \theta_k^2)|z_k|^2)^2}{(\sum_{t=j+1}^m \theta_t^2|z_t|^2)(\sum_{s=j}^m \theta_s^2|z_s|^2)(\sum_{k=i+1}^m |z_k|^2)(\sum_{k=i}^m |z_k|^2)}\Big\}\,.$$

We will now prove that the norm is constant if only if all $\theta_i' s$ are equal to 1. Indeed, if we evaluate this norm when it corresponds to the cases where $|z_m| \to 1$, $|z_i| \to 0$ $i \neq m$ and $|z_{m-1}| \to 1$, $|z_i| \to 0$, $i \neq m-1$, we find after identifying that $\theta_{m-1} = \theta_m = \theta$. The value of the O'Neill tensor corresponding to the case $|z_1|^2 = |z_m|^2 \to \frac{1}{2}$, $|z_i| \to 0$, $2 \leq i \leq m-1$ gives that $\theta = 1$. The same computation can be done successively to prove that $\theta_i' s$ are equal to one for $i \neq m, m-1$ when considering the case $|z_l|^2 = |z_m|^2 \to \frac{1}{2}$, $|z_i| \to 0$. Comparing the lower bound of the inequality in Theorem 3.3 with the norm of the O'Neill tensor which is equal to $2(m-1)$, we find that the optimality is realized for $\mathbb{S}^5$.

Next, we will get another pinching condition which doesn't require the positivity of the sectional curvature. We have

Theorem 3.4. *Under the same condition as in Theorem 3.3, we have*

$$(q-2)|A|^2 \geq \mathrm{Scal}^M - K_1^M(n-q)(n+q-1) - (p(p-1) + (q-p)(q-p-1))\rho_1^M \,.$$

Proof. The proof is a direct consequence from the fact that

$$\sum_{1 \leq i,l \leq q} R^M_{lili} \geq \mathrm{Scal}_M - K_1^M(n-q)(n+q-1)\,.$$

$\square$

The inequality in Theorem 3.4 is of interest if

$$K_1^M(n-q)(n+q-1) + (p(p-1) + (q-p)(q-p-1))\rho_1^M \leq \mathrm{Scal}^M$$

which with the use of $\mathrm{Scal}^M \leq K_1^M n(n-1)$ gives that $K_1^M > 0$.

Remark 1. The computations in Proposition 3.1, Theorems 3.3 and 3.4 are local. Therefore, if there is one point at which the estimates do not hold, then there is no locally defined parallel p-form for any Riemannian foliation near that point.

4. Foliations with harmonic basic forms

In this section, we study the case where a Riemannian manifold endowed with a Riemannian foliation admits a basic harmonic form. That is a basic p-form α such that $\Delta_b \alpha = 0$.

Proposition 4.1. *Let $(M, g, \mathcal{F})$ be a Riemannian manifold endowing with a Riemannian foliation. Then, we have*

$$\begin{aligned} 2\langle R(\alpha), \alpha\rangle \geq & -\frac{p-7}{3} \sum_{1 \leq i,j \leq q} \mathrm{Ric}^{\nabla}_{ij} \langle e_i \lrcorner \alpha, e_j \lrcorner \alpha\rangle + \frac{p-1}{3} \sum_{1 \leq i,j,l \leq q} R^M_{lilj} \langle e_i \lrcorner \alpha, e_j \lrcorner \alpha\rangle \\ & - \sum_{1 \leq i,j,k,l \leq q} R^M_{ijkl} \langle (e_j \wedge e_i) \lrcorner \alpha, (e_l \wedge e_k) \lrcorner \alpha\rangle \\ & - \sum_{s=1}^{n-q} \Big\{ \sum_{i=1}^q |A_{e_i} V_s \lrcorner \alpha|^2 + 2 |\sum_{i=1}^q (A_{e_i} V_s \wedge e_i) \lrcorner \alpha|^2 \Big\}, \end{aligned} \tag{4.1}$$

for any basic p-form α.

Proof. For $p = 1$, the inequality is clearly satisfied by (2.3). In order to prove the inequality for $p \geq 2$, we introduce as in [3] the operator

$$\mathcal{B}^-\alpha = \frac{1}{(p-2)!}\sum_{i,i_1,\ldots,i_{p-2}}\big((e_i\wedge e_{i_1}\wedge\cdots\wedge e_{i_{p-2}})\lrcorner\alpha\wedge A_{e_i}\big)\otimes e_{i_1}^*\wedge\cdots\wedge e_{i_{p-2}}^*\,.$$

The norm of the tensor $\mathcal{B}^-\alpha$ is being defined as the sum

$$|\mathcal{B}^-\alpha|^2 = \frac{1}{(p-2)!}\sum_{k,l,i_1,\ldots,i_{p-2}}|(\mathcal{B}^-\alpha)_{kli_1\ldots i_{p-2}}|^2\,.$$

Therefore, we compute

$$\begin{aligned}
\frac{(p-2)!}{2}|\mathcal{B}^-\alpha|^2 &= \frac{1}{2}\sum_{k,l,i_1,\ldots,i_{p-2}}|(\mathcal{B}^-\alpha)_{kli_1\cdots i_{p-2}}|^2\\
&= \frac{1}{2}\sum_{\substack{i_1,\ldots,i_{p-2}\\ i,j,k,l}}\big\langle(((e_i\wedge e_{i_1}\wedge\cdots\wedge e_{i_{p-2}})\lrcorner\alpha\wedge A_{e_i})_{kl},((e_j\wedge e_{i_1}\wedge\cdots\wedge e_{i_{p-2}})\lrcorner\alpha\wedge A_{e_j})_{kl}\big\rangle\\
&= \frac{1}{2}\sum_{\substack{i_1,\ldots,i_{p-2}\\ i,j,k,l}}\langle\alpha_{ii_1\ldots i_{p-2}k}A_{e_i}e_l-\alpha_{ii_1\cdots i_{p-2}l}A_{e_i}e_k,\alpha_{ji_1\ldots i_{p-2}k}A_{e_j}e_l-\alpha_{ji_1\ldots i_{p-2}l}A_{e_j}e_k\rangle\\
&= \sum_{\substack{i_1,\ldots,i_{p-2}\\ i,j,k,l}}\alpha_{iki_1\ldots i_{p-2}}\alpha_{jki_1\ldots i_{p-2}}g(A_{e_i}e_l,A_{e_j}e_l)-\alpha_{iki_1\cdots i_{p-2}}\alpha_{jli_1\ldots i_{p-2}}g(A_{e_i}e_l,A_{e_j}e_k)\\
&= (p-1)!\sum_{1\leq i,j,l\leq q}\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle g(A_{e_i}e_l,A_{e_j}e_l)\\
&\quad-(p-2)!\sum_{1\leq i,j,k,l\leq q}\big\langle(e_k\wedge e_i)\lrcorner\alpha,(e_l\wedge e_j)\lrcorner\alpha\big\rangle g(A_{e_i}e_l,A_{e_j}e_k)\,.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
\frac{1}{2}|\mathcal{B}^-\alpha|^2 &= (p-1)\sum_{1\leq i,j,l\leq q}\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle g(A_{e_i}e_l,A_{e_j}e_l)\\
&\quad-\sum_{1\leq i,j,k,l\leq q}\langle(e_j\wedge e_i)\lrcorner\alpha,(e_l\wedge e_k)\lrcorner\alpha\rangle g(A_{e_i}e_l,A_{e_k}e_j)\,.
\end{aligned}$$

Plugging now Equations (3.2) and (3.3) into the above one, we find that

$$\begin{aligned}
\frac{1}{2}|\mathcal{B}^-\alpha|^2 &= \frac{p-1}{3}\Big\{\sum_{1\leq i,j\leq q}\mathrm{Ric}^{\nabla}_{ij}\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle-\sum_{1\leq i,j,l\leq q}R^M_{lilj}\langle e_i\lrcorner\alpha,e_j\lrcorner\alpha\rangle\Big\}\\
&\quad+\sum_{1\leq i,j,k,l\leq q}\big\{-R^{\nabla}_{ijkl}+R^M_{ijkl}+2g(A_{e_i}e_j,A_{e_k}e_l)
\end{aligned}$$

$$
\begin{aligned}
&- g(A_{e_k}e_i, A_{e_j}e_l)\Big\}\big\langle(e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\big\rangle\\
=&\frac{p-1}{3}\Big\{\sum_{1\leq i,j\leq q}\mathrm{Ric}^{\nabla}_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle - \sum_{1\leq i,j,l\leq q}R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle\Big\}\\
&+2\langle R(\alpha),\alpha\rangle - 2\sum_{1\leq i,j\leq q}\mathrm{Ric}^{\nabla}_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle\\
&+\sum_{1\leq i,j,k,l\leq q}R^M_{ijkl}\langle(e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\rangle\\
&+2\sum_{s=1}^{n-q}|(\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i))\lrcorner\alpha|^2\\
&-\sum_{1\leq i,j,k,l\leq q}g(A_{e_k}e_i, A_{e_j}e_l)\langle(e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\rangle\,.
\end{aligned}
$$

We used in the last equality Equations (2.3) and (3.5). Interchanging now the indice k to l and vice versa in the last expression, we get

$$
\begin{aligned}
\frac{1}{2}|\mathcal{B}^-\alpha|^2 =& \frac{p-1}{3}\Big\{\sum_{1\leq i,j\leq q}\mathrm{Ric}^{\nabla}_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle - \sum_{1\leq i,j,l\leq q}R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle\Big\}\\
&+2\langle R(\alpha),\alpha\rangle - 2\sum_{1\leq i,j\leq q}\mathrm{Ric}^{\nabla}_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \sum_{1\leq i,j,k,l\leq q}R^M_{ijkl}\langle(e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\rangle\\
&+2\sum_{s=1}^{n-q}\Big|\Big(\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i)\Big)\lrcorner\alpha\Big|^2 - \sum_{1\leq i,j,k,l\leq q}g(A_{e_i}e_l, A_{e_j}e_k)\langle(e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\rangle\,.
\end{aligned}
$$

Finally, after the use of Equation (3.7), we find with the help of (3.6) that

$$
\begin{aligned}
\frac{1}{2}|\mathcal{B}^-\alpha|^2 + |\mathcal{B}^+\alpha|^2 =& \frac{p-7}{3}\sum_{1\leq i,j\leq q}\mathrm{Ric}^{\nabla}_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle - \frac{p-1}{3}\sum_{1\leq i,j,l\leq q}R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle\\
&+2\langle R(\alpha),\alpha\rangle + \sum_{1\leq i,j,k,l\leq q}R^M_{ijkl}\langle(e_j\wedge e_i)\lrcorner\alpha, (e_l\wedge e_k)\lrcorner\alpha\rangle\\
&+\sum_{s=1}^{n-q}\Big\{\sum_{i=1}^{q}|A_{e_i}V_s\lrcorner\alpha|^2 + 2\Big|\Big(\sum_{i=1}^{q}(A_{e_i}V_s\wedge e_i)\Big)\lrcorner\alpha\Big|^2\Big\}.
\end{aligned}
$$

Since the l.h.s. of the equality above is non-negative, we finish the proof of the proposition. □

We now investigate the case where the form α is a harmonic basic form. We have

Theorem 4.2. *Let $(M, g, \mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation of codimension q. Assume that the manifold admits a* *harmonic basic p-form, there exists at least a point $x \in M$ such that*

$$\begin{aligned}(2q+1)|A|^2(x) \geq \; & -\frac{p-7}{3}\mathrm{Scal}^\nabla(x) + \Big(\frac{p-1}{3}\Big)q(q-1)K_0^M(x) \\ & - 2\big(p\,(p-1) + (q-p)(q-p-1)\big)\rho_1^M(x)\,,\end{aligned}$$

where $2 \leq p \leq q-2$.

Proof. As in the proof of Theorem 3.3, we use Inequality (3.9) in order to deduce that

$$\begin{aligned}2\big\langle R(\alpha),\alpha\big\rangle \geq & -\frac{p-7}{3}\sum_{1\leq i,j\leq q}\mathrm{Ric}^\nabla_{ij}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle + \frac{p-1}{3}\sum_{1\leq i,j,l\leq q}R^M_{lilj}\langle e_i\lrcorner\alpha, e_j\lrcorner\alpha\rangle \\ & - \sum_{1\leq i,j,k,l\leq q}R^M_{ijkl}\big\langle(e_j\wedge e_i)\lrcorner\alpha,(e_l\wedge e_k)\lrcorner\alpha\big\rangle - (2q+1)\sum_{\substack{1\leq i\leq q\\1\leq s\leq n-q}}|A_{e_i}V_s\lrcorner\alpha|^2\end{aligned}$$

for any basic p-form α. Applying the above inequality for the $(q-p)$-form $*_b\alpha$ and then summing the two equations, we get by using (3.12) and (3.8)

$$\begin{aligned}2\big(\langle R(\alpha),\alpha\rangle + \langle R(*_b\alpha),*_b\alpha\rangle\big) \geq & -\frac{p-7}{3}\,\mathrm{Scal}^\nabla|\alpha|^2 + \frac{p-1}{3}q(q-1)K_0^M|\alpha|^2 \\ & - 2\big(p\,(p-1)+(q-p)(q-p-1)\big)\rho_1^M|\alpha|^2 \\ & - (2q+1)|A|^2|\alpha|^2\,.\end{aligned}$$

If the basic form α is now harmonic, i.e. $d_b\alpha = \delta_b\alpha = 0$, then the twisted derivative is equal to $\tilde{d}_b\alpha = -\frac{1}{2}\kappa\wedge\alpha$ and its adjoint is $\tilde{\delta}_b\alpha = -\frac{1}{2}\kappa\lrcorner\alpha$. Thus

$$\int_M\langle\tilde{\Delta}_b\alpha,\alpha\rangle v_g = \int_M(|\tilde{d}_b\alpha|^2 + |\tilde{\delta}_b\alpha|^2)v_g = \frac{1}{4}\int_M|\kappa|^2|\alpha|^2v_g\,.$$

This implies by the transverse Bochner-Weitzenböck formula, that $\int_M\langle R(\alpha),\alpha\rangle v_g \leq 0$. Since the basic Hodge operator commutes with the twisted Laplacian [4], the same inequality holds for $*_b\alpha$. Thus, we get the required inequality. $\square$

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