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# INTEGRABLE SOLUTIONS FOR IMPLICIT FRACTIONAL ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper we study the existence of integrable solutions for initial value problem for implicit fractional order functional differential equations with infinite delay. Our results are based on Schauder type fixed point theorem and the Banach contraction principle fixed point theorem.

## 1. Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 15, 20, 21, 24]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas et al. [1, 2], Kilbas et al. [18], Lakshmikantham et al. [19], and the papers by Agarwal et al. [3, 4], Belarbi et al. [6], Benchohra et al. [7, 8, 9], El-Sayed and Abd El-Salam [11], and the references therein.

To our knowledge, the literature on integrable solutions for fractional differential equations is very limited. El-Sayed and Hashem [12] studies the existence of integrable and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered $L^p$-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

Motivated by the above papers, in this paper we deal with the existence of solutions for initial value problem (IVP for short), for implicit fractional order functional differential equations with infinite delay

$$ {}^cD^{\alpha}y(t) = f(t, y_t, {}^cD^{\alpha}y_t)\,, \qquad t \in J := [0, b] \tag{1}$$

$$ y(t) = \phi(t)\,, \qquad t \in (-\infty, 0], \tag{2}$$

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where ${}^cD^\alpha$ is the Caputo fractional derivative, and $f\colon J\times\mathcal{B}\times\mathcal{B}\to\mathbb{R}$ is a given function satisfying some assumptions that will be specified later, and $\mathcal{B}$ is called a phase space that will be defined later (see Section 2). For any function $y$ defined on $(-\infty,b]$ and any $t\in J$, we denote by $y_t$ the element of $\mathcal{B}$ defined by $y_t(\theta)=y(t+\theta)$, $\theta\in(-\infty,0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time $t$.

In the literature devoted to equations with finite delay, the state space is usually the space of all continuous function on $[-r,0]$, $r>0$ and $\alpha=1$ endowed with the uniform norm topology; see the book of Hale and Lunel [14]. When the delay is infinite, the selection of the state $\mathcal{B}$ (i.e. phase space) plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [13] (see also Kappel and Schappacher [17] and Schumacher [23]). For a detailed discussion on this topic we refer the reader to the book by Hino et al. [16].

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on the Banach contraction principle (Theorem 3.1) and the second one on Schauder type fixed point theorem (Theorem 3.2). An example is given in Section 4 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t\in J\}\,,$$

where $|\cdot|$ denotes a suitable complete norm on $\mathbb{R}$.

Let $L^1(J,\mathbb{R})$ denotes the class of Lebesgue integrable functions on the interval $J$, with the norm

$$\|u\|_{L_1} = \int_0^b |u(t)|\,dt\,.$$

**Definition 2.1** ([18, 22])**.** The fractional (arbitrary) order integral of the function $h\in L^1([a,b],\mathbb{R}_+)$ of order $\alpha\in\mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,ds\,,$$

where $\Gamma(\cdot)$ is the gamma function. When $a=0$, we write $I^\alpha h(t)=h(t)*\varphi_\alpha(t)$, where $\varphi_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$, and $\varphi_\alpha(t)=0$ for $t\le 0$, and $\varphi_\alpha\to\delta(t)$ as $\alpha\to 0$, where $\delta$ is the delta function.

**Definition 2.2** ([18])**.** The Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$ is given by

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds\,,$$

where $n = [\alpha]+1$. If $\alpha \in (0,1]$, then

$$({}^cD^{\alpha}_{a+}h)(t) = I^{1-\alpha}_{a+} \frac{d}{dt} h(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} h(s)\, ds\,.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

**Proposition 2.1.** [18] *Let $\alpha$, $\beta > 0$. Then we have*

(i) $I^{\alpha}\colon L^1(J,\mathbb{R}_+) \to L^1(J,\mathbb{R}_+)$, *and if* $f \in L^1(J,\mathbb{R}_+)$, *then*

$$I^{\alpha}I^{\beta}f(t) = I^{\beta}I^{\alpha}f(t) = I^{\alpha+\beta}f(t)\,.$$

(ii) *If* $f \in L^p(J,\mathbb{R}_+)$, $1 \le p \le +\infty$, *then* $\|I^{\alpha}f\|_{L_p} \le \frac{b^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L_p}$.

(iii) *The fractional integration operator $I^{\alpha}$ is linear.*

The following theorems will be needed.

**Theorem 2.1** (Schauder fixed point theorem [10])**.** *Let $E$ a Banach space and $Q$ be a convex subset of $E$ and $T\colon Q \to Q$ is compact, and continuous map. Then $T$ has at least one fixed point in $Q$.*

**Theorem 2.2** (Kolmogorov compactness criterion [10])**.** *Let $\Omega \subseteq L^p(J,\mathbb{R})$, $1 \le p \le \infty$. If*

(i) *$\Omega$ is bounded in $L^p(J,\mathbb{R})$, and*

(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$,*

*then $\Omega$ is relatively compact in $L^p(J,\mathbb{R})$, where*

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)\, ds\,.$$

In this paper, we assume that the state space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were introduced by Hale and Kato in [13].

$(A_1)$ If $y\colon (-\infty,b] \to \mathbb{R}$, and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

  (i) $y_t$ is in $\mathcal{B}$

  (ii) $\|y_t\|_{\mathcal{B}} \le K(t)\int_0^t |y(s)|ds + M(t)\|y_0\|_{\mathcal{B}}$,

  (iii) $|y(t)| \le H\|y_t\|_{\mathcal{B}}$, where $H \ge 0$ is a constant, $K\colon J \to [0,\infty)$ is continuous, $M\colon [0,\infty) \to [0,\infty)$ is locally bounded and $H$, $K$, $M$, are independent of $y(\cdot)$.

$(A_2)$ For the function $y(\cdot)$ in $(A_1)$, $y_t$ is a $\mathcal{B}$-valued continuous function on $J$.

$(A_3)$ The space $\mathcal{B}$ is complete.

## 3. Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (1) − (2).
Let the space

$$\Omega = \{y\colon (-\infty, b] \to \mathbb{R} : y|_{(-\infty,0]} \in \mathcal{B} \quad \text{and} \quad y|_J \in L^1(J)\}\,.$$

**Definition 3.1.** A function $y \in \Omega$ is said to be a solution of IVP (1)–(2) if $y$ satisfies (1) and (2).

For the existence of solutions for the problem (1)–(2), we need the following auxiliary lemma.

**Lemma 3.1.** *The solution of the IVP* (1)–(2) *can be expressed by the integral equation*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds\,, \qquad t \in J\,, \tag{3}$$

$$y(t) = \phi(t)\,, \qquad t \in (-\infty, 0]\,, \tag{4}$$

*where $x$ is the solution of the functional integral equation*

$$x(t) = f\Big(t, \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_s\, ds, x_t\Big)\,. \tag{5}$$

**Proof.** Let $y$ be solution of (3)–(4), then for $t \in J$ and $t \in (-\infty, 0]$, we have (1) and (2), respectively. □

To present the main result, let us introduce the following assumptions:

- (H1) $f\colon J \times \mathcal{B}^2 \to \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathcal{B}^2$ and continuous in $(u_1, u_2) \in \mathcal{B}^2$, for almost all $t \in J$.
- (H2) There exist constants $k_1$, $k_2 > 0$ such that
$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le k_1 \|x_1 - x_2\|_{\mathcal{B}} + k_2 \|y_1 - y_2\|_{\mathcal{B}}\,,$$
for $t \in J$, and every $x_1$, $x_2$, $y_1$, $y_2 \in \mathcal{B}$.

Our first existence result for the IVP (1)–(2) is based on the Banach contraction principle. Set

$$K_b = \sup\{|K(t)| : t \in J\}\,.$$

**Theorem 3.1.** *Assume that the assumptions* (H1)-(H2) *are satisfied. If*

$$\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^{\alpha}}{\Gamma(\alpha+1)} < 1\,, \tag{6}$$

*then the IVP* (1)–(2) *has a unique solution on the interval* $(-\infty, b]$.

**Proof.** Transform the problem (1)–(2) into a fixed point problem. Consider the operator $N\colon \Omega \to \Omega$ defined by:

$$(Ny)(t) = \begin{cases} \phi(t)\,, & t \in (-\infty, 0]\,;\\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha y_s, y_s)\, ds\,, & t \in J\,. \end{cases}$$

We shall use the Banach contraction principle to prove that $N$ has a fixed point. Let $x(\cdot)\colon (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 0\,, & \text{if } \ t \in J\,; \\ \phi(t)\,, & \text{if } \ t \in (-\infty, 0]\,. \end{cases}$$

Then $x_0 = \phi$. For each $z \in L^1(J, \mathbb{R})$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} z(t)\,, & \text{if } \ t \in J\,; \\ 0\,, & \text{if } \ t \in (-\infty, 0]\,. \end{cases}$$

if $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha y_s, y_s)\, ds\,,$$

we can decompose $y(\cdot)$ as $y(t) = \overline{z}(t) + x(t)$, $0 \le t \le b$, which implies $y_t = \overline{z}_t + x_t$, for every $0 \le t \le b$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)\, ds\,.$$

Set

$$L_0 = \{z \in L^1(J, \mathbb{R}) : z_0 = 0\}\,,$$

and let $\|\cdot\|_b$ be the seminorm in $L_0$ defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \int_0^b |z(t)|\, dt = \int_0^b |z(t)|\, dt\,, \quad z \in L_0\,.$$

$L_0$ is a Banach space with norm $\|\cdot\|_b$. Let the operator $P\colon L_0 \to L_0$ be defined by

$$(7) \qquad (Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)\, ds\,, \quad t \in J\,,$$

That the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving that $P$ has a fixed point. We shall show that $P\colon L_0 \to L_0$ is a contraction map. Indeed, consider $z$, $z^* \in L_0$. Then we have for each $t \in J$

$$\begin{aligned}
&|P(z)(t) - P(z^*)(t)| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s) - f(s, I^\alpha(\overline{z}^*_s + x_s), \overline{z}^*_s + x_s)|\, ds \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [k_1 \|I^\alpha(\overline{z}_s - \overline{z}^*_s)\|_{\mathcal{B}} + k_2 \|\overline{z}_s - \overline{z}^*_s\|_{\mathcal{B}}]\, ds \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b \left[k_1 \|I^\alpha(z(s) - z^*(s))\| + k_2 \|z(s) - z^*(s)\|\right]\, ds \\
&\quad \le \Big(\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 b^\alpha}{\Gamma(\alpha+1)}\Big) \|z - z^*\|_b\,.
\end{aligned}$$

Therefore

$$\|P(z) - P(z^*)\|_b \le \Big(\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^{\alpha}}{\Gamma(\alpha+1)}\Big)\|z - z^*\|_b \,.$$

Consequently by (6) $P$ is a contraction. As a consequence of the Banach contraction principle, we deduce that $P$ has a unique fixed point which is a solution of the problem (1)–(2). □

The following result is based on Schauder fixed point theorem.

**Theorem 3.2.** *Assume that* (H1) *and the following condition hold.*

(H3) *There exist a positive function* $a \in L^1(J)$ *and constants,* $q_i > 0$*;* $i = 1, 2$ *such that:*

$$|f(t, u_1, u_2)| \le |a(t)| + q_1\|u_1\|_{\mathcal{B}} + q_2\|u_2\|_{\mathcal{B}} \,, \quad \forall\ (t, u_1, u_2) \in J \times \mathbb{R}^2 \,.$$

*If*

$$K_b\left(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^{\alpha}}{\Gamma(\alpha+1)}\right) < 1 \,, \tag{8}$$

*then the IVP* (1)–(2) *has at least one solution* $y \in L^1(J, \mathbb{R})$*.*

**Proof.** Let $P\colon L_0 \to L_0$ be defined as in(7), and

$$r = \frac{\frac{b^{\alpha}\|a\|_{L^1}}{\Gamma(\alpha+1)} + M_b\|\phi\|_{\mathcal{B}}(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^{\alpha}}{\Gamma(\alpha+1)})}{1 - K_b(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^{\alpha}}{\Gamma(\alpha+1)})} \,,$$

where $M_b = \sup\{|M(t)| : t \in J\}$, and consider the set

$$B_r := \{z \in L_0, \|z\|_b \le r\} \,.$$

Clearly $B_r$ is nonempty, bounded, convex and closed. We shall show that the operator $P$ satisfies the assumptions of Schauder fixed point theorem. The proof will be given in three steps.

**Step 1**: $P$ is continuous.

Let $z_n$ be a sequence such that $z_n \to z$ in $L_0$. Then

$$\begin{aligned}|(Pz_n)(t) - (Pz)(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^{\alpha}(\overline{z}_{n_s} + x_s), \overline{z}_{n_s} + x_s) \\ - f(s, I^{\alpha}(\overline{z}_s + x_s), \overline{z}_s + x_s)|\, ds\end{aligned}$$

Since $f$ is a continuous function, we have

$$\begin{aligned}&\|P(z_n) - P(z)\|_b \\ &\quad \le \frac{b^{\alpha}}{\Gamma(\alpha+1)} \|f(\cdot, I^{\alpha}(\overline{z}_{n_{(\cdot)}} + x_{(\cdot)}, \overline{z}_{n_{(\cdot)}}) + x_{(\cdot)}) \\ &\qquad - f(\cdot, I^{\alpha}(\overline{z}_{(\cdot)} + x_{(\cdot)}), \overline{z}_{(\cdot)} + x_{(\cdot)})\|_{L_1} \to 0 \\ &\qquad\qquad \text{as } n \to \infty \,.\end{aligned}$$

**Step 2**: $P$ maps $B_r$ into itself.

Let $z \in B_r$. Since $f$ is a continuous functions, we have for each $t \in [0, b]$

$$
\begin{aligned}
|(Pz)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)|\, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [a(t)| + q_1 \|I^\alpha(\overline{z}_s + x_s)\|_{\mathcal{B}} + q_2 \|\overline{z}_s + x_s\|_{\mathcal{B}}]\, ds \\
&\leq \frac{b^\alpha \|a\|_{L_1}}{\Gamma(\alpha+1)} + \Big(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)}\Big)(K_b r + M_b \|\phi\|_{\mathcal{B}}),
\end{aligned}
$$

where

$$\|\overline{z}_s + x_s\|_{\mathcal{B}} \leq \|\overline{z}_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}}\,.$$

Hence $\|(Pz)\|_{L_1} \leq r$. Then $PB_r \subset B_r$.

**Step 3**: $P$ is compact.

We will show that $P$ is compact, this is $PB_r$ is relatively compact. Clearly $PB_r$ is bounded in $L_0$, i.e. condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Pz)_h \to (Pz)$, in $L_0$ for each $z \in B_r$.

Let $z \in B_r$, then we have

$$
\begin{aligned}
&\|(Pz)_h - (Pz)\|_{L^1} \\
&\quad = \int_0^b |(Pz)_h(t) - (Pz)(t)|\, dt \\
&\quad = \int_0^b \Big|\frac{1}{h}\int_t^{t+h} (Pz)(s) ds - (Pz)(t)\Big|\, dt \\
&\quad \leq \int_0^b \Big(\frac{1}{h}\int_t^{t+h} |(Pz)(s) - (Pz)(t)|\, ds\Big)\, dt \\
&\quad \leq \int_0^b \frac{1}{h}\int_t^{t+h} |I^\alpha f\,(s, \overline{z}_s + x_s), \overline{z}_s + x_s) - I^\alpha f\,(t, I^\alpha(\overline{z}_t + x_t), \overline{z}_t + x_t)\,|\, ds\, dt\,.
\end{aligned}
$$

Since $z \in B_r \subset L_0$ and assumption (H3) that implies $f \in L_0$ and by Proposition 2.1, it follows that $I^\alpha f \in L^1(J, \mathbb{R})$, then we have

$$
\frac{1}{h}\int_t^{t+h} \big|I^\alpha f\,((\overline{z}_s + x_s), \overline{z}_s + x_s) - I^\alpha f\,(t, I^\alpha(\overline{z}_t + x_t), \overline{z}_t + x_t)\,\big|\, ds \to 0 \quad \text{as} \quad h \to 0\,, \quad t \in J\,. \tag{9}
$$

Hence

$$(Pz)_h \to (Pz) \quad \text{uniformly as} \quad h \to 0\,.$$

Then by Kolmogorov compactness criterion, $PB_r$ is relatively compact. As a consequence of Schauder's fixed point theorem the IVP (1)–(2) has at least one solution in $B_r$. □

## 4. Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional initial value problem,

$$(10)\quad {}^cD^\alpha y(t) = \frac{ce^{-\gamma t+t}}{(e^t+e^{-t})(1+\|y_t\|+\|{}^cD^\alpha y_t\|)}\,, \qquad t\in J:=[0,b]\,,\ \alpha\in(0,1]\,,$$

$$(11)\qquad y(t)=\phi(t)\,, \qquad t\in(-\infty,0]\,,$$

where $c>1$ is fixed. Let $\gamma$ be a positive real constant and

$$B_\gamma = \{y\in L^1(-\infty,0] : \lim_{\theta\to-\infty} e^{\gamma\theta}y(\theta),\ \text{ exists in } \mathbb{R}\}\,.$$

The norm of $B_\gamma$ is given by

$$\|y\|_\gamma = \int_{-\infty}^0 e^{\gamma\theta}|y(\theta)|\,d\theta\,.$$

Let $y\colon(-\infty,b]\to\mathbb{R}$ be such that $y_0\in B_\gamma$. Then

$$\begin{aligned}\lim_{\theta\to-\infty} e^{\gamma\theta}y_t(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta}y(t+\theta) = \lim_{\theta\to-\infty} e^{\gamma(\theta-t)}y(\theta)\\ &= e^{\gamma t}\lim_{\theta\to-\infty} e^{\gamma\theta}y_0(\theta)<\infty.\end{aligned}$$

Hence $y_t\in B_\gamma$. Finally we prove that

$$\|y_t\|_\gamma \le K(t)\int_0^t |y(s)|\,ds + M(t)\|y_0\|_\gamma\,,$$

where $K=M=1$ and $H=1$. We have

$$|y_t(\theta)| = |y(t+\theta)|\,.$$

If $\theta+t\le 0$, we get

$$|y_t(\theta)| \le \int_{-\infty}^0 |y(s)|\,ds\,.$$

For $t+\theta\ge 0$, then we have

$$|y_t(\theta)| \le \int_0^t |y(s)|\,ds\,.$$

Thus for all $t+\theta\in J$, we get

$$|y_t(\theta)| \le \int_{-\infty}^0 |y(s)|\,ds + \int_0^t |y(s)|\,ds\,.$$

Then

$$\|y_t\|_\gamma \le \|y_0\|_\gamma + \int_0^t |y(s)|\,ds\,.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that $B_\gamma$ is a phase space. Set

$$f(t,y,z) = \frac{e^{-\gamma t+t}}{c(e^t+e^{-t})(1+y+z)}, \qquad (t,x,z) \in J \times B_\gamma \times B_\gamma .$$

For $t \in J$, $y_1$, $y_2$, $z_1$, $z_2 \in B_\gamma$, we have

$$\begin{aligned}
|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \frac{e^{-\gamma t+t}}{c(e^t+e^{-t})}\Big|\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\Big| \\
&= \frac{e^{-\gamma t+t}(|y_1-y_2|+|z_1-z_2|)}{c(e^t+e^{-t})(1+y_1+z_1)(1+y_2+z_2)} \\
&\leq \frac{e^{-\gamma t}\times e^t(|y_1-y_2|+|z_1-z_2|)}{c(e^t+e^{-t})} \\
&\leq \frac{e^{-\gamma t}(\|y_1-y_2\|_\gamma+\|z_1-z_2\|_\gamma)}{c} \\
&\leq \frac{1}{c}\|y_1-y_2\|_\gamma + \frac{1}{c}\|z_1-z_2\|_\gamma .
\end{aligned}$$

Hence the condition (H2) holds. We choose $b$ such that $\frac{K_b b^{2\alpha}}{c\Gamma(2\alpha+1)} + \frac{K_b b^{\alpha}}{c\Gamma(\alpha+1)} < 1$. Since $K_b = 1$, then

$$\frac{b^{2\alpha}}{c\Gamma(2\alpha+1)} + \frac{b^{\alpha}}{c\Gamma(\alpha+1)} < 1 .$$

Then by Theorem 3.1, the problem (10)–(11) has a unique integrable solution on $[-\infty, b]$.

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