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HARDY-ROGERS-TYPE FIXED POINT THEOREMS FOR α-GF-CONTRACTIONS

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Abstract. The aim of this paper is to introduce some new fixed point results of Hardy-Rogers-type for α-η-GF-contraction in a complete metric space. We extend the concept of F-contraction into an α-η-GF-contraction of Hardy-Rogers-type. An example has been constructed to demonstrate the novelty of our results.

1. Introduction

The Banach contraction principle [3] is one of the earliest and most important resluts in fixed point theory. Because of its importance and simplicity, a lot of authors have improved generalized and extended the Banach contraction principle in the literature (see [1–24]) and the references therein.

In [21] Samet et al. introduced a concept of α-ψ-contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapınar et al. [16], refined the notion and obtained various fixed point results. Hussain et al. [11], extended the concept of α-admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [1] introduced pairs of α-admissible mappings satisfying new sufficient contractive conditions different from those in [11, 21], and proved fixed point and common fixed point theorems. Lately, Salimi et al. [20], modified the concept of α-ψ- contractive mappings and established fixed point results. Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set af all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

Definition 1 ([21])**.** Let $T\colon X \to X$ and $\alpha\colon X \times X \to [0, +\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Definition 2 ([20])**.** Let $T\colon X \to X$ and $\alpha, \eta\colon X \times X \to [0, +\infty)$ two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

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If $\eta(x,y)=1$, then above definition reduces to Definition 1. If $\alpha(x,y)=1$, then T is called an η-subadmissible mapping.

Definition 3 ([13])**.** Let (X,d) be a metric space. Let $T\colon X\to X$ and $\alpha,\ \eta\colon X\times X\to[0,+\infty)$ be two functions. We say that T is α-η-continuous mapping on (X,d) if for given $x\in X$, and sequence $\{x_n\}$ with

$$x_n\to x \quad \text{as} \quad n\to\infty,\ \alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1}) \\ \text{for all} \quad n\in\mathbb{N}\Rightarrow Tx_n\to Tx\,.$$

In [6] Edelstein proved the following version of the Banach contraction principle.

Theorem 4 ([6])**.** *Let (X,d) be a metric space and $T\colon X\to X$ be a self mapping. Assume that*

$$d(Tx,Ty)<d(x,y)\,,\quad \textit{holds for all}\quad x,y\in X\quad \textit{with}\quad x\neq y\,.$$

Then T has a unique fixed point in X.

In [24] Wardowski introduced a new type of contractions called F-contractions and proved fixed point theorems concerning F-contractions as a generalization of the Banach contraction principle as follows.

Definition 5 ([24])**.** Let (X,d) be a metric space. A mapping $T\colon X\to X$ is said to be an F-contraction if there exists $\tau>0$ such that

$$\forall x,y\in X\,,\quad d(Tx,Ty)>0\Rightarrow\tau+F\big(d(Tx,Ty)\big)\leq F\big(d(x,y)\big)\,, \tag{1.1}$$

where $F\colon\mathbb{R}_+\to\mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $x,\ y\in\mathbb{R}_+$ such that $x<y$, $F(x)<F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if

$$\lim_{n\to\infty}F(\alpha_n)=-\infty\,;$$

(F3) There exists $k\in(0,1)$ such that $\lim\alpha\to0^+\alpha^kF(\alpha)=0$.

We denote by $\digamma$, the set of all functions satisfying the conditions (F1)–(F3).

Example 6 ([24])**.** Let $F\colon\mathbb{R}_+\to\mathbb{R}$ be given by the formula $F(\alpha)=\ln\alpha$. It is clear that F satisfied (F1)–(F2)–(F3) for any $k\in(0,1)$. Each mapping $T\colon X\to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx,Ty)\leq e^{-\tau}d(x,y)\,,\quad \text{for all}\quad x,y\in X\,,\ Tx\neq Ty\,.$$

It is clear that for $x,\ y\in X$ such that $Tx=Ty$ the inequality $d(Tx,Ty)\leq e^{-\tau}d(x,y)$, also holds, i.e. T is a Banach contraction.

Example 7 ([24])**.** If $F(r)=\ln r+r$, $r>0$ then F satisfies (F1)–(F3) and the condition (1.1) is of the form

$$\frac{d(Tx,Ty)}{d(x,y)}\leq e^{d(Tx,Ty)-d(x,y)}\leq e^{-\tau}\,,\quad \text{for all}\quad x,y\in X\,,\ Tx\neq Ty\,.$$

Remark 8. From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

Theorem 9 ([24]). *Let (X, d) be a complete metric space and let $T\colon X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

In [5] Cosentino et al. presented some fixed point results for F-contraction of Hardy-Rogers-type for self-mappings on complete metric spaces.

Definition 10 ([5]). Let (X, d) be a metric space. a mapping $T\colon X \longrightarrow X$ is called an F-contraction of Hardy-Rogers-type if there exists $F \in \digamma$ and $\tau > 0$ such that

$$\tau + F\big(d(Tx, Ty)\big) \leq \\ F\big(\kappa d\,(x, y) + \beta d\,(x, Tx) + \gamma d\,(y, Ty) + \delta d\,(x, Ty) + Ld(y, Tx)\big)\,,$$

for all x, $y \in X$ with $d\,(Tx, Ty) > 0$, where κ, β, γ, δ, $L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

Theorem 11 ([5]). *Let (X, d) be a complete metric space and let $T\colon X \longrightarrow X$. Assume there exists $F \in \digamma$ and $\tau > 0$ such that T is an F-contraction of Hardy-Rogers-type, that is*

$$\tau + F\big(d(Tx, Ty)\big) \leq \\ F\big(\kappa d\,(x, y) + \beta d\,(x, Tx) + \gamma d\,(y, Ty) + \delta d\,(x, Ty) + Ld(y, Tx)\big)\,,$$

for all x, $y \in X$ with $d\,(Tx, Ty) > 0$, where κ, β, γ, δ, $L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point. Moreover, if $\kappa + \delta + L \leq 1$, then the fixed point of T is unique.

Hussain et al. [11] introduced a family of functions as follows.

Let Δ_G denotes the set of all functions $G\colon \mathbb{R}^{+4} \to \mathbb{R}^+$ satisfying:

(G) for all t_1, t_2, t_3, $t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$ there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Example 12 ([14]). If $G(t_1, t_2, t_3, t_4) = \tau e^{v \min\{t_1, t_2, t_3, t_4\}}$ where $v \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Definition 13 ([14]). Let (X, d) be a metric space and T be a self mapping on X. Also suppose that α, $\eta\colon X \times X \to [0, +\infty)$ be two functions. We say that T is α-η-GF-contraction if for x, $y \in X$, with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have

$$G\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)\,,$$

where $G \in \Delta_G$ and $F \in \Delta_{\mathcal{F}}$.

On the other hand Secelean [22] proved the following lemma and replaced condition (F2 by an equivalent but a more simple condition (F2$'$).

Lemma 14 ([22]). *Let $F\colon \mathbb{R}^{+} \longrightarrow \mathbb{R}$ be an increasing map and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:*
(a) *if* $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$ *then* $\lim\limits_{n\to\infty} \alpha_n = 0$*;*
(b) *if* $\inf F = -\infty$ *and* $\lim\limits_{n\to\infty} \alpha_n = 0$*, then* $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$.

He replaced the following condition.
(F2′) $\inf F = -\infty$
or, also, by
(F2″) there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$.
Recently Piri [19] replaced the following condition (F3′) instead of the condition (F3) in Definition 5.
(F3′) F is continuous on $(0,\infty)$.
We denote by $\Delta_{\mathcal{F}}$ the set of all functions satisfying the conditions (F1), (F2′) and (F3′).
For $p \geq 1$, $F(\alpha) = -\frac{1}{\alpha^{P}}$ satisfies in (F1) and (F2) but it does not apply in (F3) while satisfy conditions (F1), (F2) and (F3′). Also, $a > 1$, $t \in \left(0, \frac{1}{a}\right)$, $F(\alpha) = \frac{-1}{(\alpha+[\alpha])^{t}}$, where $[\alpha]$ denotes the integral part of α, satisfies the condition (F1) and (F2) but it does not satisfy (F3′) , while it satisfies the condition (F3) for any $k \in \left(\frac{1}{a}, 1\right)$. Therefore $\digamma \cap \Delta_{\mathcal{F}} = \emptyset$.

Theorem 15 ([19]). *Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \Delta_{\mathcal{F}}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))\ .$$

Then T has a unique fixed point $x^{*} \in X$ and for every $x \in X$ the sequence $\{T^{n}x\}_{n=1}^{\infty}$ converges to x^{*}.

Definition 16. Let (X,d) be a metric space and T be a self mapping on X. Also suppose that α, $\eta\colon X \times X \to [0,+\infty)$ be two functions. We say that T is an α-η-GF-contraction of Hardy-Rogers-type if for x, $y \in X$, with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$ we have

$$\begin{aligned} (1.2)\quad & G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \\ & \leq F\big(\kappa d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + L d(y,Tx)\big)\,, \end{aligned}$$

where $G \in \Delta_G$, $F \in \Delta_{\mathcal{F}}$, κ, β, γ, δ, $L \geq 0$, $\kappa + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

2. Main result

In this paper, we establish new some fixed point theorems for α-η-GF-contraction of Hardy-Rogers-type in a complete metric space.

Theorem 17. *Let (X,d) be a complete metric space. Let T be a self mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *T is an α-η-GF-contraction of Hardy-Rogers-type;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$*;*

(iv) T *is* α-η*-continuous.*

Then T *has a fixed point in* X*. Moreover,* T *has a unique fixed point when* $\alpha(x, y) \geq \eta(x, x)$ *for all* x*,* $y \in \mathrm{Fix}(T)$ *and* $\kappa + \delta + L \leq 1$*.*

Proof. Let x_0 in X, such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this process, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now since, T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process, we have

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)\,, \quad \text{for all} \quad n \in \mathbb{N}\,. \tag{2.1}$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, there is nothing to prove. So, we assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0\,, \quad \forall n \in \mathbb{N}\,. \tag{2.2}$$

Since, T is an α-η-GF-contraction of Hardy-Rogers-type, for any $n \in \mathbb{N}$, we have

$$G\begin{pmatrix} d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}) \end{pmatrix}$$

$$+F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\begin{pmatrix} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) \\ +\delta d\left(x_{n-1}, Tx_n\right) + L d\left(x_n, Tx_{n-1}\right) \end{pmatrix}$$

which implies

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) + F(d(Tx_{n-1}, Tx_n))$$

$$\leq F\begin{pmatrix} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) \\ +\delta d\left(x_{n-1}, Tx_n\right) + L d(x_n, Tx_{n-1} \end{pmatrix}. \tag{2.3}$$

Now since, $d(x_{n-1}, x_n) \cdot d(x_n, x_{n+1}) \cdot d(x_{n-1}, x_{n+1}) \cdot 0 = 0$, so from (G) there exists $\tau > 0$ such that,

$$G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) = \tau\,.$$

Therefore

$$\begin{aligned} F\big(d(x_n, x_{n+1})\big) &= F\left(d\left(Tx_{n-1}, Tx_n\right)\right) \\ &\leq F\begin{pmatrix} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, Tx_{n-1}\right) + \gamma d\left(x_n, Tx_n\right) \\ +\delta d\left(x_{n-1}, Tx_n\right) + L d\left(x_n, Tx_{n-1}\right) \end{pmatrix} - \tau \\ &= F\begin{pmatrix} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, x_n\right) + \gamma d\left(x_n, x_{n+1}\right) \\ +\delta d\left(x_{n-1}, x_{n+1}\right) + L d\left(x_n, x_n\right) \end{pmatrix} - \tau \\ &\leq F\begin{pmatrix} \kappa d\left(x_{n-1}, x_n\right) + \beta d\left(x_{n-1}, x_n\right) + \gamma d\left(x_n, x_{n+1}\right) \\ +\delta d\left(x_{n-1}, x_n\right) + \delta d\left(x_n, x_{n+1}\right) \end{pmatrix} - \tau \\ &= F\left(\left(\kappa + \beta + \delta\right) d\left(x_{n-1}, x_n\right) + \left(\gamma + \delta\right) d\left(x_n, x_{n+1}\right)\right) - \tau \end{aligned}$$

Since F is strictly increasing, we deduce

$$d\left(x_n,x_{n+1}\right)<\left(\kappa+\beta+\delta\right)d\left(x_{n-1},x_n\right)+\left(\gamma+\delta\right)d\left(x_n,x_{n+1}\right)\,.$$

This implies

$$\left(1-\gamma-\delta\right)d\left(x_n,x_{n+1}\right)<\left(\kappa+\beta+\delta\right)d\left(x_{n-1},x_n\right)\qquad\text{for all}\quad n\in\mathbb{N}\,.$$

From $\kappa+\beta+\gamma+2\delta=1$ and $\gamma\neq 1$, we deduce that $1-\gamma-\delta>0$ and so

$$d\left(x_n,x_{n+1}\right)<\frac{\left(\kappa+\beta+\delta\right)}{\left(1-\gamma-\delta\right)}d\left(x_{n-1},x_n\right)=d\left(x_{n-1},x_n\right)\ \text{for all}\ n\in\mathbb{N}\,.$$

Consequently

$$F\big(d(x_n,x_{n+1})\big)\leq F\big(d(x_{n-1},x_n)\big)-\tau\,. \tag{2.4}$$

Continuing this process, we get

$$\begin{aligned}F\big(d(x_n,x_{n+1})\big)&\leq F\big(d(x_{n-1},x_n)\big)-\tau\\&=F\big(d(Tx_{n-2},Tx_{n-1})\big)-\tau\\&\leq F\big(d(x_{n-2},x_{n-1})\big)-2\tau\\&=F\big(d(Tx_{n-3},Tx_{n-2})\big)-2\tau\\&\leq F\big(d(x_{n-3},x_{n-2})\big)-3\tau\\&\ \ \vdots\\&\leq F\big(d(x_0,x_1)\big)-n\tau\,.\end{aligned}$$

This implies that

$$F\big(d(x_n,x_{n+1})\big)\leq F\big(d(x_0,x_1)\big)-n\tau\,. \tag{2.5}$$

And so $\lim\limits_{n\to\infty}F\left(d\left(Tx_{n-1},Tx_n\right)\right)=-\infty$, which together with (F2′) and Lemma 14 gives that

$$\lim_{n\to\infty}d(x_n,Tx_n)=0\,. \tag{2.6}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence. Arguing by contradiction, we have that there exists $\epsilon>0$ and sequence $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n)>q(n)>n\,,\quad d(x_{p(n)},x_{q(n)})\geq\epsilon\,,\quad d(x_{p(n)-1},x_{q(n)})<\epsilon\quad\forall n\in\mathbb{N}\,. \tag{2.7}$$

So, we have

$$\epsilon\leq d(x_{p(n)},x_{q(n)})\leq d(x_{p(n)},x_{p(n)-1})+d(x_{p(n)-1},x_{q(n)})$$

$$\leq d(x_{p(n)},x_{p(n)-1})+\epsilon=d(x_{p(n)-1},Tx_{p(n)-1})+\epsilon\,. \tag{2.8}$$

Letting $n\longrightarrow\infty$ in (2.8) and using (2.6), we obtain

$$\lim_{n\longrightarrow\infty}d(x_{p(n)},x_{q(n)})=\epsilon\,. \tag{2.9}$$

Also, from (2.6) there exists a natural number $n_1 \in \mathbb{N}$ such that

$$d(x_{p(n)}, Tx_{p(n)}) < \frac{\epsilon}{4} \quad \text{and} \quad d(x_{q(n)}, Tx_{q(n)}) < \frac{\epsilon}{4}, \quad \forall\, n \geq n_1 . \tag{2.10}$$

Next, we claim that

$$d(Tx_{p(n)}, Tx_{q(n)}) = d(x_{p(n)+1}, x_{q(n)+1}) > 0 \quad \forall\, n \geq n_1 . \tag{2.11}$$

Arguing by contradiction, there exists $m \geq n_1$ such that

$$d(x_{p(m)+1}, x_{q(m)+1}) = 0 . \tag{2.12}$$

It follows from (2.7), (2.10) and (2.12) that

$$\begin{aligned}\epsilon &\leq d(x_{p(m)}, x_{q(m)}) \leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)}) \\ &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, x_{q(m)}) \\ &= d(x_{p(m)}, Tx_{p(m)}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)}, Tx_{q(m)}) \\ &< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} .\end{aligned}$$

This contradiction establishes the relation (2.11) it follows from (2.11) and (1.2) that

$$G\begin{pmatrix} d\left(x_{p(n)}, Tx_{p(n)}\right), d\left(x_{q(n)}, Tx_{q(n)}\right), \\ d\left(x_{p(n)}, Tx_{q(n)}\right), d\left(x_{q(n)}, Tx_{p(n)}\right) \end{pmatrix} + F\big(d\left(Tx_{P(n)}, Tx_{q(n)}\right)\big)$$

$$\leq F\begin{pmatrix} \kappa d\left(x_{p(n)}, x_{q(n)}\right) + \beta d\left(x_{p(n)}, Tx_{p(n)}\right) + \gamma d\left(x_{q(n)}, Tx_{q(n)}\right) \\ +\delta d\left(x_{p(n)}, Tx_{q(n)}\right) + L d\left(x_{q(n)}, Tx_{p(n)}\right) \end{pmatrix} \qquad \forall\, n \geq n_1 ,$$

which implies,

$$G\begin{pmatrix} d\left(x_{p(n)}, x_{p(n)+1}\right), d\left(x_{q(n)}, x_{q(n)+1}\right), \\ d\left(x_{p(n)}, x_{q(n)+1}\right), d\left(x_{q(n)}, x_{p(n)+1}\right) \end{pmatrix} + F\big(d\left(x_{P(n)+1}, x_{q(n)+1}\right)\big)$$

$$\leq F\begin{pmatrix} \kappa d\left(x_{p(n)}, x_{q(n)}\right) + \beta d\left(x_{p(n)}, x_{p(n)+1}\right) + \gamma d\left(x_{q(n)}, x_{q(n)+1}\right) \\ +\delta d\left(x_{p(n)}, x_{q(n)+1}\right) + L d\left(x_{q(n)}, x_{p(n)+1}\right) \end{pmatrix} .$$

Now since, $0 \cdot d\left(x_{q(n)}, Tx_{q(n)}\right) \cdot d\left(x_{p(n)}, Tx_{q(n)}\right) \cdot d\left(x_{q(n)}, Tx_{p(n)}\right) = 0$, so from (G) there exists $\tau > 0$ such that,

$$G\big(0, d\left(x_{q(n)}, Tx_{q(n)}\right), d\left(x_{p(n)}, Tx_{q(n)}\right), d\left(x_{q(n)}, Tx_{p(n)}\right)\big) = \tau .$$

Therefore,

$$\begin{aligned}\tau + F\big(d\left(Tx_{P(n)}, Tx_{q(n)}\right)\big) \\ \leq F\begin{pmatrix} \kappa d\left(x_{p(n)}, x_{q(n)}\right) + \beta d\left(x_{p(n)}, Tx_{p(n)}\right) + \gamma d\left(x_{q(n)}, Tx_{q(n)}\right) \\ +\delta d\left(x_{p(n)}, Tx_{q(n)}\right) + L d\left(x_{q(n)}, Tx_{p(n)}\right) \end{pmatrix}\end{aligned} \tag{2.13}$$

So from (F3′), (2.6), (2.9) and (2.13), we have

$$\tau + F(\epsilon) \leq F\left((\kappa + \delta + L)\,\epsilon\right) = F(\epsilon) .$$

This contradiction show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x in X. Since T is an α-η-continuous and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, then $x_{n+1} = Tx_n \to Tx$ as $n \to \infty$. That is, $x = Tx$. Hence x is a fixed point of T. Let x, $y \in \mathrm{Fix}\,(T)$ where $x \neq y$, then from

$$\begin{aligned} &G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \\ &\leq F\big(\kappa d\,(x,y) + \beta d\,(x,Tx) + \gamma d\,(y,Ty) + \delta d\,(x,Ty) + Ld(y,Tx)\big) \\ &= F\big(\,(\kappa+\delta+L)\,d\,(x,y)\,\big)\,. \end{aligned}$$

Which is a contradiction, if $\kappa + \delta + L \leq 1$ and hence $x = y$. □

Theorem 18. *Let (X,d) be a complete metric space. Let T be a self mapping satisfying the following assertions:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *T is an α-η-GF-contraction of Hardy-Rogers-type;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$ then either*

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \quad \text{or} \quad \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all x, $y \in \mathrm{Fix}(T)$ and $\kappa + \delta + L \leq 1$.

Proof. As similar lines of the Theorem 17, we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad \text{and} \quad x_n \to x \quad \text{as} \quad n \to \infty\,.$$

Since, by (iv), either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \quad \text{or} \quad \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)\,,$$

holds for all $n \in \mathbb{N}$. This implies

$$\alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x_{n+2}) \quad \text{or} \quad \alpha(x_{n+2}, x) \geq \eta(x_{n+2}, x_{n+3})\,.$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x)$$

and from (1.2), we deduce that

$$\begin{aligned} &G\big(d(x_{n_k}, Tx_{n_k}), d(x,Tx), d(x_{n_k}, Tx), d(x, Tx_{n_k})\big) + F\big(d(Tx_{n_k}, Tx)\big) \\ &\leq F\big(\kappa d(x_{n_k}, x) + \beta d(x_{n_k}, Tx_{n_k}) + \gamma d(x,Tx) + \delta d(x_{n_k}, Tx) + Ld(x, Tx_{n_k})\big)\,. \end{aligned}$$

This implies

$$\begin{aligned} (2.14) \quad &F\big(d(Tx_{n_k}, Tx)\big) \\ &\leq F\big(\kappa d\,(x_{n_k}, x) + \beta d\,(x_{n_k}, x_{n_k+1}) + \gamma d\,(x,Tx) + \delta d\,(x_{n_k}, Tx) + Ld(x, x_{n_k+1})\big)\,. \end{aligned}$$

From (F1) we have

$$\begin{aligned} (2.15) \quad &d(x_{n_k+1}, Tx) \\ &< \kappa d\,(x_{n_k}, x) + \beta d\,(x_{n_k}, x_{n_k+1}) + \gamma d\,(x,Tx) + \delta d\,(x_{n_k}, Tx) + Ld(x, x_{n_k+1})\,. \end{aligned}$$

By taking the limit as $k \to \infty$ in (2.15), we obtain

$$d(x,Tx) < (\gamma+\delta)\, d(x,Tx) < d(x,Tx)\,. \tag{2.16}$$

Which is implies that $d(x,Tx)=0$, implies x is a fixed point of T. Uniqueness follows similarly as in Theorem 17. □

Theorem 19. *Let T be a continuous selfmapping on a complete metric space X. If for x, $y \in X$ with $d(x,Tx) \le d(x,y)$ and $d(Tx,Ty)>0$, we have*

$$\begin{aligned} &G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \\ &\le F\big(\kappa d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + L d(y,Tx)\big)\,, \end{aligned}$$

where $G \in \Delta_G$, $F \in \Delta_{\mathcal{F}}$, κ, β, γ, δ, $L \ge 0$, $\kappa+\beta+\gamma+2\delta = 1$ and $\gamma \ne 1$. Then T has a fixed point in X.

Proof. Let us define α, $\eta\colon X\times X \to [0,+\infty)$ by

$$\alpha(x,y) = d(x,y) \quad \text{and} \quad \eta(x,y) = d(x,y) \quad \text{for all} \quad x,y \in X\,.$$

Now, $d(x,y) \le d(x,y)$ for all x, $y \in X$, so $\alpha(x,y) \ge \eta(x,y)$ for all $x,y \in X$. That is, conditions (i) and (iii) of Theorem 17 hold true. Since T is continuous, so T is α-η-continuous. Let $\eta(x,Tx) \le \alpha(x,y)$ and $d(Tx,Ty) > 0$, we have $d(x,Tx) \le d(x,y)$ with $d(Tx,Ty)>0$, then

$$\begin{aligned} &G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \\ &\le F\big(\kappa d\,(x,y) + \beta d\,(x,Tx) + \gamma d\,(y,Ty) + \delta d\,(x,Ty) + L d(y,Tx)\big)\,. \end{aligned}$$

That is, T is an α-η-GF-contraction mapping of Hardy-Rogers-type. Hence, all conditions of Theorem 17 satisfied and T has a fixed point. □

Corollary 20. *Let T be a continuous selfmapping on a complete metric space X. If for x, $y \in X$ with $d(x,Tx) \le d(x,y)$ and $d(Tx,Ty)>0$, we have*

$$\tau + F\big(d(Tx,Ty)\big) \le F\big(\kappa d\,(x,y) + \beta d\,(x,Tx) + \gamma d\,(y,Ty) + \delta d\,(x,Ty) + L d(y,Tx)\big)\,,$$

where $\tau>0$, κ, β, γ, δ, $L \ge 0$, $\kappa+\beta+\gamma+2\delta=1$ and $\gamma\ne 1$ and $F \in \Delta_{\mathcal{F}}$. Then T has a fixed point in X.

Corollary 21. *Let T be a continuous selfmapping on a complete metric space X. If for x, $y \in X$ with $d(x,Tx) \le d(x,y)$ and $d(Tx,Ty)>0$, we have*

$$\begin{aligned} &\tau e^{\upsilon \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}} + F\big(d(Tx,Ty)\big) \\ &\le F\big(\,(\kappa d\,(x,y) + \beta d\,(x,Tx) + \gamma d\,(y,Ty) + \delta d\,(x,Ty) + L d(y,Tx)\,\big)\,, \end{aligned}$$

where $\tau>0$, κ, β, γ, δ, L, $\upsilon \ge 0$, $\kappa+\beta+\gamma+2\delta=1$, $\gamma\ne 1$ and $F\in\Delta_{\mathcal{F}}$. Then T has a fixed point in X.

Example 22. Let $S_n = \frac{n(n+1)(n+2)}{3}$, $n\in\mathbb{N}$, $X = \{S_n : n \in \mathbb{N}\}$ and $d\,(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define the mapping $T\colon X \longrightarrow X$, by $T\,(S_1) = S_1$ and $T\,(S_n) = S_{n-1}$, for all $n>1$ and $\alpha\,(x,y) = 1$ for all $x \in X$, $\eta\,(x,Tx) = \frac{1}{2}$ for all $x\in X$, $G\,(t_1,t_2,t_3,t_4) = \tau$ where $\tau = \frac{7}{2} > 0$. Since $\lim\limits_{n\to\infty} \frac{d(T(S_n),T(S_1))}{d(S_n,S_1)} = \lim\limits_{n\to\infty} \frac{S_{n-1}-2}{S_n-2} = \frac{(n-1)n(n+1)-6}{n(n+1)(n+2)-6} = 1$, T is not Banach contraction. On the other hand taking $F(r) = \frac{-1}{r} + r \in \Delta_{\mathcal{F}}$, we obtain the result that T is an α-η-GF-contraction of Hardy-Rogers-type with $\kappa = \beta = \frac{1}{3}$, $\gamma = \frac{1}{6}$, $\delta = \frac{1}{12}$ and $L = \frac{7}{12}$. To see this, let us consider the following calculation. We conclude the following three cases:

Case 1: For every $m \in \mathbb{N}$, $m > n = 1$, then $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, we have

$$|T(S_m) - T(S_1)| = |S_1 - T(S_m)| = |S_{m-1} - S_1| = 2 \times 3 + 3 \times 4 + \cdots + (m-1)\,m\,,$$
$$|S_m - S_1| = 2 \times 3 + 3 \times 4 + \cdots + m\,(m+1)\,,$$
$$|S_m - T(S_m)| = |S_m - S_{m-1}| = m\,(m+1)\,,$$
$$|S_1 - T(S_1)| = |S_1 - S_1| = 0\,.$$

Since $m > 1$ and

$$\frac{-1}{2 \times 3 + \cdots + (m-1)\,m} < \frac{-1}{\begin{bmatrix} \frac{1}{3}(2 \times 3 + \cdots + m(m+1)) + \frac{1}{3}m(m+1) \\ + \frac{1}{12}(2 \times 3 + \cdots + m(m+1)) + \frac{7}{12}(2 \times 3 + \cdots + (m-1)m) \end{bmatrix}}\,.$$

We have

$$\frac{7}{2} - \frac{1}{2 \times 3 + 3 \times 4 + \cdots + (m-1)m} + [2 \times 3 + 3 \times 4 + \cdots + (m-1)m]$$
$$< \frac{7}{2} - \frac{1}{\begin{bmatrix} \frac{1}{3}(2 \times 3 + \cdots + m(m+1)) + \frac{1}{3}m(m+1) \\ + \frac{1}{12}(2 \times 3 + \cdots + m(m+1)) + \frac{7}{12}(2 \times 3 + \cdots + (m-1)m) \end{bmatrix}}$$
$$+ [2 \times 3 + 3 \times 4 + \cdots + (m-1)m]$$
$$\leq - \frac{1}{\begin{bmatrix} \frac{1}{3}(2 \times 3 + \cdots + m(m+1)) + \frac{1}{3}m(m+1) \\ + \frac{1}{12}(2 \times 3 + \cdots + m(m+1)) + \frac{7}{12}(2 \times 3 + \cdots + (m-1)m) \end{bmatrix}}$$
$$+ \begin{bmatrix} \frac{1}{3}(2 \times 3 + \cdots + m(m+1)) + \frac{1}{3}m(m+1) \\ + \frac{1}{12}(2 \times 3 + \cdots + m(m+1)) + \frac{7}{12}(2 \times 3 + \cdots + (m-1)m) \end{bmatrix}.$$

So, we get

$$\frac{7}{2} - \frac{1}{|T(S_m) - T(S_1)|} + \left|T(S_m) - T(S_1)\right|$$
$$< - \frac{1}{\frac{1}{3}|S_m - S_1| + \frac{1}{3}|S_m - T(S_m)| + \frac{1}{6}|S_1 - T(S_1)| + \frac{1}{12}|S_m - T(S_1)| + \frac{7}{12}|S_1 - T(S_m)|}$$
$$+ \left[\frac{1}{3}|S_m - S_1| + \frac{1}{3}|S_m - T(S_m)| + \frac{1}{6}|S_1 - T(S_1)| + \frac{1}{12}|S_m - T(S_1)| + \frac{7}{12}|S_1 - T(S_m)|\right].$$

Case 2: For $1 \leq m < n$, similar to Case 1.

Case 3: For $m > n > 1$, then $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, we have

$$\begin{aligned}
|T(S_m) - T(S_n)| &= n \times (n+1) + (n+1)(n+2) + \cdots + (m-1)m\,, \\
|S_m - S_n| &= (n+1)(n+2) + (n+2)(n+3) + \cdots + m(m+1)\ , \\
|S_m - T(S_m)| &= |S_m - S_{m-1}| = m(m+1)\ , \\
|S_n - T(S_n)| &= |S_n - S_{n-1}| = n(n+1)\ , \\
|S_m - T(S_n)| &= |S_m - S_{n-1}| = n(n+1) + \cdots + m(m+1)\ , \\
|S_n - T(S_m)| &= |S_n - S_{m-1}| = (n+1)(n+2) + \cdots + (m-1)m\,.
\end{aligned}$$

Since $m > n > 1$, and

$$\begin{aligned}
&\frac{-1}{n \times (n+1) + (n+1)(n+2) + \cdots + (m-1)m} \\
&< \frac{-1}{\begin{bmatrix} \frac{1}{3}((n+1)(n+2) + ... + m(m+1)) + \frac{1}{3}m(m+1) + \frac{1}{6}n(n+1) \\ + \frac{1}{12}(n(n+1) + \cdots + (m-1)m) + \frac{7}{12}((n+1)(n+2) + \cdots + (m-1)m) \end{bmatrix}}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{7}{2} - \frac{1}{n \times (n+1) + (n+1)(n+2) + \cdots + (m-1)m} \\
&+ \left[n \times (n+1) + (n+1)(n+2) + \cdots + (m-1)m\right] \\
&< \frac{7}{2} - \frac{1}{\begin{bmatrix} \frac{1}{3}((n+1)(n+2) + \cdots + m(m+1)) + \frac{1}{3}m(m+1) + \frac{1}{6}n(n+1) \\ + \frac{1}{12}(n(n+1) + \cdots + (m-1)m) + \frac{7}{12}((n+1)(n+2) + \cdots + (m-1)m) \end{bmatrix}} \\
&+ [n \times (n+1) + (n+1)(n+2) + \cdots + (m-1)m] \\
&\leq -\frac{1}{\begin{bmatrix} \frac{1}{3}((n+1)(n+2) + \cdots + m(m+1)) + \frac{1}{3}m(m+1) + \frac{1}{6}n(n+1) \\ + \frac{1}{12}(n(n+1) + \cdots + (m-1)m) + \frac{7}{12}((n+1)(n+2) + \cdots + (m-1)m) \end{bmatrix}} \\
&+ \begin{bmatrix} \frac{1}{3}((n+1)(n+2) + \cdots + m(m+1)) + \frac{1}{3}m(m+1) + \frac{1}{6}n(n+1) \\ + \frac{1}{12}(n(n+1) + \cdots + (m-1)m) + \frac{7}{12}((n+1)(n+2) + \cdots + (m-1)m) \end{bmatrix}.
\end{aligned}$$

So, we get

$$\begin{aligned}
&\frac{7}{2} - \frac{1}{|T(S_m) - T(S_n)|} + |T(S_m) - T(S_n)| \\
&< -\frac{1}{\frac{1}{3}|S_m - S_n| + \frac{1}{3}|S_m - T(S_m)| + \frac{1}{6}|S_n - T(S_n)| + \frac{1}{12}|S_m - T(S_n)| + \frac{7}{12}|S_n - T(S_m)|}
\end{aligned}$$

$$+\left[\frac{1}{3}|S_m - S_n| + \frac{1}{3}|S_m - T(S_m)| + \frac{1}{6}|S_n - T(S_n)| + \frac{1}{12}|S_m - T(S_n)| + \frac{7}{12}|S_n - T(S_m)|\right].$$

Therefore

$$\begin{aligned}\frac{7}{2} &+ F\big(d(T(S_m), T(S_n))\big)\\ &\leq F\Big(\frac{1}{3}d(S_m, S_n) + \frac{1}{3}d\big(S_m, T(S_m)\big) + \frac{1}{6}d\big(S_n, T(S_n)\big)\\ &\quad + \frac{1}{12}d\big(S_m, T(S_n)\big) + \frac{7}{12}d\big(S_n, T(S_m)\big)\Big).\end{aligned}$$

for all m, $n \in \mathbb{N}$. Hence all condition of theorems are satisfied, T has a fixed point.

Let $(X, d, \preceq)$ be a partially ordered metric space. Let $T\colon X \to X$ is such that for x, $y \in X$, with $x \preceq y$ implies $Tx \preceq Ty$, then the mapping T is said to be non-decreasing. We derive following important result in partially ordered metric spaces.

Theorem 23. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) T is nondecreasing and ordered GF-contraction of Hardy-Rogers-type;;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(iii) either for a given $x \in X$ and sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ we have $Tx_n \to Tx$
or if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$ then either

$$Tx_n \preceq x \text{ or } T^2x_n \preceq x$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

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