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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A $p(x)$-KIRCHHOFF TYPE PROBLEM VIA VARIATIONAL TECHNIQUES

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Abstract. This paper discusses the existence and multiplicity of solutions for a class of $p(x)$-Kirchhoff type problems with Dirichlet boundary data of the following form

$$\begin{cases} -\Big(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f(x,u)\,, & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

where Ω is a smooth open subset of $\mathbb{R}^N$ and $p \in C(\overline{\Omega})$ with $N < p^- = \inf_{x\in\Omega} p(x) \le p^+ = \sup_{x\in\Omega} p(x) < +\infty$, a, b are positive constants and $f\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. The proof is based on critical point theory and variable exponent Sobolev space theory.

1. Introduction

In this paper we study

$$(1)\qquad \begin{cases} -\Big(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f(x,u)\,, & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \ge 1$, $p \in C(\overline{\Omega})$ with $N < p^- = \inf_{x\in\Omega} p(x) \le p^+ = \sup_{x\in\Omega} p(x) < +\infty$, a, b are positive constants and $f\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Problem (1) is related for example to vibrations and deformations of plates or tended cords. The operator $-\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big)$ is said to be the $p(x)$-Laplacian, and becomes the p-Laplacian when $p(x) = p$. The variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ is a natural generalization of the classical Sobolev space $W_0^{1,p}(\Omega)$ and related preliminaries concerning $W_0^{1,p(x)}(\Omega)$ and $p(x)$-Laplacian equations will be given in Section 2.

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In this paper we examine existence and multiplicity of solutions of the $p(x)$-Kirchhoff equation associated to problem (1) by applying a minimization principle and the genus theory introduced by Krasnoselskii (see [4], [13]). Problem (1) is related to the stationary problem of a model introduced by Kirchhoff [12]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big)\rho\frac{\partial^2 u}{\partial x^2}=0\,,$$

G. Dai and J. Wei established the existence of infinitely many nonnegative solutions for problem (1) by applying a general variational principle due to B. Ricceri (see Theorem 3.1 and Theorem 3.2 in [8]). In this paper, we first prove the existence of a solution to problem (1), by using a minimization principle with the hypothesis $\limsup_{|u|\to+\infty}\frac{F(x,t)}{|u|^{\theta(x)}}\le a(x)$, where $F(x,u)=\int_0^u f(x,t)\,dt$ and a is a measurable function which belongs to $L^\infty(\Omega)$ and $\theta\in C(\overline{\Omega})$ satisfies $\theta^+=\sup_{\overline{\Omega}}\theta(x)<2p^-$. Our second result gives the existence and multiplicity of solutions using Clarke's theorem under the following assumptions:

- there exists positive constants C_1, $C_2>0$ and a function q measurable on Ω such that $C_1t^{q(x)-1}\le f(x,t)\le C_2t^{q(x)-1}$ for all $t\ge0$ and for all $x\in\Omega$, where $q\in L^\infty(\Omega)$ and $1<q^-=\operatorname{essinf}_\Omega q(x)\le q(x)\le q^+=\operatorname{esssup}_\Omega q(x)<p^-=\operatorname{essinf}_\Omega p(x)$,
- $f(x,-t)=-f(x,t)$ for all $t\in\mathbb{R}$ and for all $x\in\overline{\Omega}$.

These hypotheses are a generalization of the hypotheses introduced in [7].

This paper is organized as follows. In Section 2, we present some necessary preliminaries on variable exponent Lebesgue and Sobolev spaces and we recall some definitions and basic properties of the Krasnoselskii genus. In Section 3, using critical point theory, we establish existence and multiplicity results for problem (1).

2. Preliminaries

Suppose that Ω is a smooth bounded open domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $p\in C(\overline{\Omega})$ satisfies

$$1<p^-\doteq\inf_{x\in\Omega}p(x)\le p^+\doteq\sup_{x\in\Omega}p(x)<+\infty\,.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega)=\Big\{u\colon\Omega\to\mathbb{R}\ \text{ is mesurable, }\int_\Omega|u(x)|^{p(x)}\,dx<+\infty\Big\}$$

endowed with the norm

$$|u|_{p(x)}=\inf\Big\{\lambda>0:\int_\Omega\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,dx\le1\Big\}\,.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\}$$

endowed with the norm $\|u\|_{1,p(x)}=|u|_{p(x)}+|\nabla u|_{p(x)}$.

Denote by $C(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ endowed with the norm $|u|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$. $W_0^{1,p(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 (See [10]). *$L^{p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

Proposition 2.2 (See [10]). *Let $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. For any u, $u_k \in L^{p(x)}(\Omega)$, $k = 1, 2, \dots$, we have*

(1) *For $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho\big(\frac{u}{\lambda}\big) = 1$.*

(2) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.

(3) *If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$.*

(4) *If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p+} \le \rho(u) \le |u|_{p(x)}^{p^-}$.*

(5) $\lim_{k\to+\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to+\infty} \rho(u_k) = 0$.

(6) $\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty \Leftrightarrow \lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 2.3 (See [10]). *If u, $u_k \in L^{p(x)}(\Omega)$, $k = 1, 2, \dots$, then the following statements are equivalent to each other:*

(1) $\lim_{k\to+\infty} |u_k - u|_{p(x)} = 0$ *(i.e. $u_k \to u$ in $L^{p(x)}(\Omega)$).*

(2) $\lim_{k\to+\infty} \rho(u_k - u) = 0$.

(3) *$u_k \to u$ in measure in Ω and* $\lim_{k\to+\infty} \rho(u_k) = \rho(u)$.

Proposition 2.4 (See [9]). *The Poincaré-type inequality holds, that is, there exists a positive constant c_Ω such that*

$$|u|_{p(x)} \le c_\Omega |\nabla u|_{p(x)}\,, \quad \forall u \in W_0^{1,p(x)}(\Omega)\,.$$

Thus $|\nabla u|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. We will use this equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{p(x)}$ for simplicity.

We now recall the Krasnoselskii genus and more information on this subject may be found in ([11], [1], [4], [13]). Let E be a real Banach space. Let us denote by Σ the class of all closed subsets $A \subset E - \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 2.1. Let $A \in \Sigma$. The Krasnoselskii genus $\gamma(A)$ is defined as being the least positive integer n such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^n - \{0\})$. If such an n does not exist we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

Theorem 2.1 (See [11]). *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Note $\gamma(S^{N-1}) = N$. If E is of infinite dimension and separable and S is the unit sphere in E, then $\gamma(S) = +\infty$.

Proposition 2.5 (See [11]). *Let A, $B \in \Sigma$. Then:*

- *if there exists an odd map $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$. Consequently, if there exists an odd homeomorphism $f\colon A \to B$, then $\gamma(A) = \gamma(B)$.*
- *if $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
- *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*

Theorem 2.2 (Minimization principle)**.** *Let X be a real reflexive Banach space. If the functional $J\colon X \to \mathbb{R}$ is weakly lower semi-continuous and coercive (i.e. $\lim_{\|u\|\to+\infty} J(u) = +\infty$), then there exists $u_0 \in X$ such that $J(u_0) = \inf_{u\in X} J(u)$. Moreover, if J is also Gateaux differentiable on X, then $J'(u_0) = 0$.*

Definition 2.2. Let $J \in C^1(X, \mathbb{R})$. If any sequences $(u_n) \subset X$ for which $(J(u_n))$ is bounded and $J'(u_n) \to 0$ when $n \to +\infty$ in X' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (denoted by the (P-S) condition).

We now state a theorem due to Clarke.

Theorem 2.3 (See [5], [15])**.** *Let $J \in C^1(E, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*

- *J is bounded from below and even;*
- *there is a compact set $K \in \Sigma$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

3. Main results

In this section we will discuss the existence of weak solutions of (1).

Definition 3.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1) if and only if

$$\Big(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx = \int_\Omega f(x,u)v\,dx\,,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

The energy functional corresponding to problem (1) is defined as follows,

$$J(u) = a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)^2 - \int_\Omega F(x,u)\,dx$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. It is easy to see that $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and for all u, $v \in W_0^{1,p(x)}(\Omega)$

$$J'(u)\cdot v = \Big(a+b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)-2}\nabla u\nabla v\,dx - \int_\Omega f(x,u)v\,dx\,.$$

Thus the critical points of J are the weak solutions of (1).

Theorem 3.1. *Assume that*

$$\limsup_{|u|\to+\infty} \frac{F(x,u)}{|u|^{\theta(x)}} \le a(x)\,, \tag{H1}$$

where $\theta \in C(\overline{\Omega})$ *with* $\theta^- = \inf_{x\in\overline{\Omega}} \theta(x) > 1$ *and* $a \in L^\infty(\Omega)$*. If* $\theta^+ = \sup_{x\in\overline{\Omega}} \theta(x) < 2p^-$*, then* (1) *has a weak solution.*

Proof. From the continuity of F and assumption (H1) we deduce that there exists a positive constant C such that

$$F(x,u) \le a(x)|u|^{\theta(x)} + C\,, \quad \forall u \in \mathbb{R}\,,\ \forall x \in \Omega\,.$$

We have for $\|u\| > 1$ that

$$\begin{aligned} J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)^2 - \int_\Omega F(x,u)\,dx \\ &\ge \frac{a}{p^+}\rho_p(\nabla u) + \frac{b}{2(p^+)^2}\big(\rho_p(\nabla u)\big)^2 - \int_\Omega (a(x)|u|^{\theta(x)} + C)\,dx\,, \\ &\ge \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - C\ \operatorname{meas}(\Omega) - |a|_{L^\infty}\int_\Omega |u|^{\theta(x)}\,dx\,. \end{aligned}$$

Since $p^- > N$, the embedding $W_0^{1,p(x)} \hookrightarrow C(\overline{\Omega})$ is continuous and we see that

$$\int_\Omega |u|^{\theta(x)}\,dx \le \int_\Omega |u|_\infty^{\theta(x)}\,dx \le C_0\int_\Omega \|u\|^{\theta(x)} \le C_0\|u\|^{\theta^+}\,,$$

where $C_0 = \max_{x\in\overline{\Omega}} C^{\theta(x)}$ and C is the constant of the embedding, and this implies that

$$J(u) \ge \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - C\ \operatorname{meas}(\Omega) - |a|_{L^\infty}C_0\|u\|^{\theta^+}\,.$$

Since $\theta^+ < 2p^-$ then J is coercive. Now J is also weakly lower semicontinuous (see Theorem 3.2.9 in [9]), so we see that J has a global minimum point $u \in W_0^{1,p(x)}(\Omega)$, which is a weak solution to problem (1). ☐

Example 3.1. We consider $\Omega = (0,1)$ and $f(x,t) = a(x)\theta(x)|t|^{\theta(x)-2}t$, for all $t \in \mathbb{R}$, and we put for $x \in \Omega$, $a(x) = \sin x$, $\theta(x) = x + \frac{3}{2}$ and $p(x) = x^2 + 2$. Our problem becomes

$$\begin{cases} -\Big(a + b\int_0^1 \frac{1}{x^2+2}|u'|^{x^2+2}\,dx\Big)\big(|u'|^{x^2}u'\big)' = \frac{\sin x}{x^2+2}|u|^{-\frac{2x^2+3}{x^2+2}}u, \quad \text{in} \quad (0,1) \\ u(0) = u(1) = 0\,, \end{cases}$$

and it admits at least one weak solution.

Remark 3.1. To obtain a nontrivial solution, we may assume that $f(x_0,0) \neq 0$ for some $x_0 \in [0,1]$.

Theorem 3.2. *Assume that:*

(H2) *there exist positive constants* C_1, $C_2 > 0$ *and a function* q *measurable on* Ω *such that* $C_1 t^{q(x)-1} \leq f(x,t) \leq C_2 t^{q(x)-1}$ *for all* $t \geq 0$ *and for all* $x \in \Omega$, *where* $q \in L^{\infty}(\Omega)$ *and* $1 < q^- = \operatorname{essinf}_{x\in\Omega} q(x) \leq q(x) \leq q^+ = \operatorname{esssup}_{x\in\Omega} q(x) < p^- = \inf_{x\in\Omega} p(x)$,

(H3) $f(x,-t) = -f(x,t)$ *for all* $t \in \mathbb{R}$ *and for all* $x \in \overline{\Omega}$.

Then (1) *has infinitely many weak solutions.*

For the proof of Theorem 3.2 we will need the following steps.
Step 1: J is bounded from below.
Indeed, for any $u \in W_0^{1,p(x)}(\Omega)$, we have

$$J(u) \geq \frac{a}{p^+}\int_{\Omega} |\nabla u|^{p(x)}\, dx + \frac{b}{2(p^+)^2}\Big(\int_{\Omega} |\nabla u|^{p(x)}\, dx\Big)^2 - \frac{C_2}{q^-}\int_{\Omega} |\nabla u|^{q(x)}\, dx\,.$$

Let $\rho_p(u) = \int_{\Omega} |u|^{p(x)}\, dx$ and $\rho_q(u) = \int_{\Omega} |u|^{q(x)}\, dx$. We have the following four cases:

(i) If $\rho_p(u) > 1$ and $\rho_q(u) < 1$, then by Proposition 2.2, we obtain

$$J(u) \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \frac{C_2}{q^-}|u|_{q(x)}^{q^-}\,.$$

Since $q(x) \leq p(x)$ for all $x \in \Omega$,

$$J(u) \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \frac{C_2 C_3}{q^-}|u|_{p(x)}^{q^-}\,,$$

where C_3 is the constant of the continuous embedding of $L^{p(x)}$ in $L^{q(x)}$. Using the Poincaré inequality, we have

$$J(u) \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \frac{C_2 C_3 c_{\Omega}^{q^-}}{q^-}\|u\|_{p(x)}^{q^-}\,.$$

We note that $2p^- > q^-$, so J is bounded from below.

(ii) If $\rho_p(u) > 1$ and $\rho_q(u) > 1$ then,

$$J(u) \geq \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \frac{C_2 C_3 c_{\Omega}^{q^-}}{q^-}\|u\|_{p(x)}^{q^+}\,.$$

We note that $2p^- > q^+$, so J is bounded from below.

(iii) If $\rho_p(u) < 1$ and $\rho_q(u) < 1$ then,

$$J(u) \geq \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2(p^+)^2}\|u\|^{2p^+} - \frac{C_2 C_3 c_{\Omega}^{q^-}}{q^-}\|u\|_{p(x)}^{q^-}\,.$$

(iv) If $\rho_p(u) < 1$ and $\rho_q(u) > 1$ then,

$$J(u) \geq \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2(p^+)^2}\|u\|^{2p^+} - \frac{C_2 C_3 c_{\Omega}^{q^-}}{q^-}\|u\|_{p(x)}^{q^+}.$$

Since $2p^+ > q^-$ and $2p^+ > q^+$ in (iii) and (iv) successively then J is bounded from below.

Step 2: J satisfies the (P-S) condition.

Indeed, let (u_n) be a Palais-Smale sequence for J. Thus there exists a positive constant C such that $J(u_n) \le C$. Arguing as above, we obtain for all $u \in W_0^{1,p(x)}(\Omega)$, the following cases:

(1) If $\rho_p(u) < 1$ and $\rho_q(u) < 1$ then,

$$J(u) \ge \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2(p^+)^2}\|u\|^{2p^+} - \frac{C_2 C_3 c_\Omega^{q^-}}{q^-}\|u\|_{p(x)}^{q^-}.$$

(2) If $\rho_p(u) < 1$ and $\rho_q(u) > 1$ then,

$$J(u) \ge \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2(p^+)^2}\|u\|^{2p^+} - \frac{C_2 C_3 c_\Omega^{q^-}}{q^-}\|u\|_{p(x)}^{q^+}.$$

(3) If $\rho_p(u) > 1$ and $\rho_q(u) < 1$ then,

$$J(u) \ge \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - \frac{C_2 C_3 c_\Omega^{q^-}}{q^-}\|u\|_{p(x)}^{q^-}.$$

(4) If $\rho_p(u) > 1$ and $\rho_q(u) > 1$ then,

$$J(u) \ge \frac{a}{p^-}\|u\|^{p^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^-} - \frac{C_2 C_3 c_\Omega^{q^-}}{q^-}\|u\|_{p(x)}^{q^+}.$$

Since $q^+ < p^-$, in all cases we now deduce that the sequence (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, passing to a subsequence if necessary, there exists $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. Since a, $b > 0$ we get

$$a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx > a > 0\,.$$

Consider the sequence

$$K_n = J'(u_n)u_n + \int_\Omega f(x,u_n)u_n\,dx - J'(u_n)u - \int_\Omega f(x,u_n)u\,dx\,.$$

From the Lebesgue dominated convergence theorem and the Sobolev embedding, we have that

$$\int_\Omega f(x,u_n)u_n\,dx \to \int_\Omega f(x,u)u\,dx\,, \quad \int_\Omega f(x,u_n)u\,dx \to \int_\Omega f(x,u)u\,dx\,,$$

so we have that $K_n \to 0$ and it can be seen that

$$\begin{aligned} K_n = {} & \Big(a + b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)} \\ & - \Big(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla u\,dx\,. \end{aligned}$$

Let

$$L_n = -\Big(a + b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u_n \nabla u\,dx$$

$$
\begin{aligned}
&+\Big(a+b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u|^{p(x)}\,dx\\
&=-\Big(a+b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\Big[\int_\Omega |\nabla u|^{p(x)-2}\nabla u_n\nabla u\,dx-\int_\Omega |\nabla u|^{p(x)}\,dx\Big].
\end{aligned}
$$

From the weak convergence of (u_n), we have that $L_n \to 0$. Hence,

$$
\begin{aligned}
K_n+L_n &= \Big(a+b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\Big[\int_\Omega |\nabla u_n|^{p(x)}-\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla u\,dx\\
&\qquad -\int_\Omega |\nabla u|^{p(x)-2}\nabla u_n\nabla u\,dx+\int_\Omega |\nabla u|^{p(x)}\,dx\Big]\\
&=\Big(a+b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\\
&\qquad\times\int_\Omega \big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\big)(\nabla u_n-\nabla u)\,dx\,.
\end{aligned}
$$

We can generalize the elementary inequalities from [14] to the variable exponent case and we obtain

$$
(|x|^{p(\cdot)-2}x-|y|^{p(\cdot)-2}y)(x-y)\ge C_{p(\cdot)}|x-y|^{p(\cdot)} \quad \text{if} \quad p(\cdot)\ge 2\,,
$$

where $C_{p(\cdot)} \ge \min\big\{1,\frac{1}{2^{p^+-2}}\big\}$, and

$$
(|x|^{p(\cdot)-2}x-|y|^{p(\cdot)-2}y)(x-y)\ge \frac{C_{p(\cdot)}|x-y|^2}{(|x|+|y|)^{2-p(\cdot)}} \quad \text{if} \quad 1<p(\cdot)<2\,.
$$

We obtain that,

$$
K_n+L_n \ge aC_{p(\cdot)}\rho_{p(\cdot)}\big(\nabla(u_n-u)\big)\,,
$$

and as $C_{p(\cdot)}$ is dominated by a constant we deduce that $\rho_p(\nabla(u_n-u))$ converges to 0 as $n\to+\infty$. By Proposition 2.3, we conclude that $\|u_n-u\|\to 0$ in $W_0^{1,p(x)}(\Omega)$.

Proof of Theorem 3.2. We notice that $W_0^{1,p^+}(\Omega) \subset W_0^{1,p(x)}(\Omega)$. Consider $\{e_1,e_2,\dots\}$, a Schauder basis of the space $W_0^{1,p^+}(\Omega)$ (see [17]), and for each $k\in\mathbb{N}$, consider X_k, the subspace of $W_0^{1,p^+}(\Omega)$ generated by k vectors $\{e_1,e_2,\dots,e_k\}$. Clearly X_k is subspace of $W_0^{1,p(x)}(\Omega)$. So we notice that $X_k\subset L^{q(x)}(\Omega)$ because $X_k\subset W_0^{1,p^+}(\Omega)\subset L^{q(x)}$. Thus, the norms $\|\cdot\|$ and $|\cdot|_{q(x)}$ are equivalent on X_k because X_k is a finite dimension space. Consequently, there exists a positive constant C_k such that

$$
-|u|_{q(x)} \le -C_k\|u\|\,, \quad \text{for all} \quad u\in X_k\,.
$$

Thus we have

$$
J(u)\le \frac{a}{p^-}\rho_p(\nabla u)+\frac{b}{2(p^-)^2}\big(\rho_p(\nabla u)\big)^2-\frac{C_1}{q^+}\rho_q(u)\,.
$$

- If $\rho_p(\nabla u) < 1$ and $\rho_q(u) < 1$, then

$$\begin{aligned} J(u) &\leq \frac{a}{p^-}\|u\|^{p^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^-} - \frac{C_1}{q^+}|u|_{q(x)}^{q^+} \\ &\leq \frac{a}{p^-}\|u\|^{p^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^-} - \frac{C_1}{q^+}C_k\|u\|^{q^+} \\ &= \|u\|^{q^+}\Big[\frac{a}{p^-}\|u\|^{p^- - q^+} + \frac{b}{2(p^-)^2}\|u\|^{2p^- - q^+} - \frac{C_1}{q^+}C_k\Big]. \end{aligned}$$

We choose $R > 0$ small enough such that

$$\frac{a}{p^-}R^{p^- - q^+} + \frac{b}{2(p^-)^2}R^{2p^- - q^+} < \frac{C_1}{q^+}C_k\,.$$

Thus, for $0 < r < R$, we consider the set $K = \{u \in X_k : \|u\| = r\}$. For all $u \in K$, we have

$$\begin{aligned} J(u) &\leq r^{q^+}\Big[\frac{a}{p^-}r^{p^- - q^+} + \frac{b}{2(p^-)^2}r^{2p^- - q^+} - \frac{C_1}{q^+}C_k\Big] \\ &< R^{q^+}\Big[\frac{a}{p^-}R^{p^- - q^+} + \frac{b}{2(p^-)^2}R^{2p^- - q^+} - \frac{C_1}{q^+}C_k\Big] \\ &< 0 = J(0)\,. \end{aligned}$$

We can apply similar reasoning to the other cases since:

- If $\rho_p(\nabla u) < 1$ and $\rho_q(u) > 1$, then

$$J(u) \leq \|u\|^{q^-}\Big[\frac{a}{p^-}\|u\|^{p^- - q^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^- - q^-} - \frac{C_1}{q^+}C_k\Big].$$

- If $\rho_p(\nabla u) > 1$ and $\rho_q(u) < 1$, then

$$J(u) \leq \|u\|^{q^+}\Big[\frac{a}{p^-}\|u\|^{p^+ - q^+} + \frac{b}{2(p^-)^2}\|u\|^{2p^+ - q^+} - \frac{C_1}{q^+}C_k\Big].$$

- If $\rho_p(\nabla u) > 1$ and $\rho_q(u) > 1$, then

$$J(u) \leq \|u\|^{q^-}\Big[\frac{a}{p^-}\|u\|^{p^+ - q^-} + \frac{b}{2(p^-)^2}\|u\|^{2p^+ - q^-} - \frac{C_1}{q^+}C_k\Big].$$

We can be considered the odd homeomorphism $h : K \to S^{k-1}$ defined by $h(u) = (\alpha_1, \alpha_2, ..., \alpha_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem 2.1 and proposition 2.5 we conclude that $\gamma(K) = k$. thanks to theorem 2.3, J has at least k pairs of different critical points. Since k is arbitrary, we obtain infinitely many critical points of J. □

Example 3.2. We consider $\Omega = (0,1)$ and let $f(x,t) = C_1 t^{q(x)-1}$ if $t \geq 0$ and $f(x,t) = -C_1(-t)^{q(x)-1}$ if $t < 0$, where $p(x) = x^2 + 2$ and $q(x) = \frac{1}{x^2+2} + 1$, for all $x \in \Omega$. Then, the problem

$$\begin{cases} -\Big(a + b\int_\Omega \frac{1}{x^2+2}|u'|^{x^2+2}\,dx\Big)\big(|u'|^{x^2}u'\big)' = \operatorname{sgn}(u)C_1|u|^{\frac{1}{x^2+2}}\,, \quad \text{in} \quad (0,1) \\ u(0) = u(1) = 0\,, \end{cases}$$

has infinitely many solutions.

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