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ABELIAN ANALYTIC TORSION AND SYMPLECTIC VOLUME

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Abstract. This article studies the abelian analytic torsion on a closed, oriented, Sasakian three-manifold and identifies this quantity as a specific multiple of the natural unit symplectic volume form on the moduli space of flat abelian connections. This identification computes the analytic torsion explicitly in terms of Seifert data.

1. Introduction

This article studies the abelian analytic torsion on Sasakian three-manifolds. The analytic torsion is a topological invariant that was introduced by D.B. Ray and I.M. Singer [21] as an analytic analogue of the combinatorially defined Reidemeister torsion [22]. It is a well known fact that these two torsions agree, as was independently shown by W. Müller, [15], and J. Cheeger, [7], for unimodular representations. More recently an elegant new proof of this equivalence has been given by M. Braverman [6] using the Witten laplacian [27].

Our main objective in this article is to compute the (square-root of the) analytic torsion explicitly as a natural symplectic volume form on the moduli space of flat abelian connections. This identification is motivated by the work of C. Beasley and E. Witten [3] involving Chern-Simons theory on contact three-manifolds. Recall that A.S. Schwarz [25] has shown that the abelian Chern-Simons partition function is proportional to the analytic torsion and our study is also natural in light of this fact. Our main result, Theorem 9, shows that two mathematically a priori different definitions of the abelian Chern-Simons partitition function derived from [3] are rigorously equivalent. Our main strategy is to use the the work of M. Rumin and N. Seshadri [24] which naturally connects the analytic torsion with contact structures on three-manifolds.

Throughout, X will denote a closed, orientable three-manifold, and $(X, \phi, \xi, \kappa, \mathrm{G})$ will denote X equipped with a *Sasakian* structure. See [5], [9] for standard background on Sasakian and contact geometry. For convenience we recall that a *Sasakian manifold* is a normal contact metric manifold, $(X, \phi, \xi, \kappa, \mathrm{G})$, where

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- $\kappa \in \Omega^1(X)$ is a contact form, i.e. $\kappa \wedge d\kappa \neq 0$, $\xi \in \Gamma(TX)$ is the Reeb vector field,
- $\phi \in \mathrm{End}(TX)$, $\phi(Y) =: JY$ for $Y \in \Gamma(H)$, $\phi(\xi) = 0$ where $J \in \mathrm{End}(H)$ is an almost complex structure on the contact distribution $H := \ker \kappa \subset TX$, and,
- $\mathrm{G} = \kappa \otimes \kappa + d\kappa \circ (\mathbb{I} \otimes \phi)$.

Definition 1. A *Seifert manifold* is a closed orientable three-manifold that admits a locally free $\mathbb{U}(1)$-action.

Remark 2. See [18] for a general definition and classification of Seifert manifolds.

Let Σ denote the base of a Seifert manifold when viewed as the total space of a $\mathbb{U}(1)$-bundle,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & & \downarrow \\ & & \Sigma \end{array}$$

It is well known that the topological isomorphism class of a Seifert manifold X is determined by its Seifert invariants [18],

$$[g, n; (\alpha_1, \beta_1), \ldots, (\alpha_M, \beta_M)], \quad \gcd(\alpha_j, \beta_j) = 1,$$

where g is the genus of Σ. Geometrically, the $\mathbb{U}(1)$ action on X is rotations of the fibres over Σ and the points in the $\mathbb{U}(1)$ fiber over each orbifold point p_j on Σ are fixed by the cyclic subgroup $\mathbb{Z}_{\alpha_j}$ of $\mathbb{U}(1)$. The fundamental group $\pi_1(X)$ is generated by the following elements [18],

$$\begin{aligned} &a_p, b_p\,, && p = 1, \ldots, g, \\ &c_j\,, && j = 1, \ldots, M, \\ &h\,, \end{aligned}$$

which satisfy the relations,

$$\begin{aligned} [a_p, h] = [b_p, h] = [c_j, h] &= 1\,, \\ c_j^{\alpha_j} h^{\beta_j} &= 1\,, \\ \prod_{p=1}^{g} [a_p, b_p] \prod_{j=1}^{M} c_j &= h^n\,. \end{aligned} \tag{1}$$

Geometrically, the generator h is associated to the generic $\mathbb{U}(1)$ fiber over Σ, the generators a_p, b_p come from the $2g$ non-contractible cycles on Σ, and the generators c_j come from the small one cycles in Σ around each of the orbifold points p_j.

Remark 3. Since the analytic torsion is defined with respect to a choice of metric, we naturally work with Sasakian structures. Recall that X admits a Sasakian structure $(X, \phi, \xi, \kappa, \mathrm{G}) \iff$

- [5, Theorem 7.5.1, 7.5.2] X admits a Seifert structure that is the total space of a non-trivial principal $\mathbb{U}(1)$ orbibundle over a Hodge orbifold surface, Σ.

For this article, Seifert structures on X are induced by Sasakian structures.

Let $\mathbb{T}$ denote a compact, connected abelian Lie group of real dimension N, $\mathfrak{t}$ denote its Lie algebra and $\Lambda \subset \mathfrak{t}$ the integral lattice. Let $\operatorname{Tors} H^2(X,\Lambda)$ denote the torsion subgroup of $H^2(X,\Lambda)$. For P a principal $\mathbb{T}$-bundle over X, $\mathcal{A}_P$ is the affine space of connections on P modeled on the vector space of $\mathbb{T}$-invariant horizontal one-forms on P, $(\Omega^1_{\text{hor}}(P,\mathfrak{t}))^{\mathbb{T}} \simeq \Omega^1(X,\mathfrak{t})$. The group of smooth gauge transformations is the group of $\mathbb{T}$ equivariant smooth maps $\mathcal{G} := (\operatorname{Map}^\infty(P,\mathbb{T}))^{\mathbb{T}} \simeq \operatorname{Map}^\infty(X,\mathbb{T})$ and acts on $\mathcal{A}_P$ in the standard way. That is, for $g \in \operatorname{Map}^\infty(P,\mathbb{T})$, and $A \in \mathcal{A}_P$, $A\cdot g := A + g^*\vartheta$, where $\vartheta \in \Omega^1(\mathbb{T},\mathfrak{t})$ denotes the Maurer-Cartan form on $\mathbb{T}$. In order to define the Chern-Simons action, a negative definite symmetric bilinear form on $\mathfrak{t}$ needs to be chosen. Let $B\mathbb{T}$ denote the classifying space of principal $\mathbb{T}$-bundles. Valid choices for such forms $\langle\cdot,\cdot\rangle \in \operatorname{Sym}^2_{\mathbb{T}}(\mathfrak{t}^*)$ are classified by elements of $H^4(B\mathbb{T},\mathbb{Z})$ [8], [4]. Choosing a basis e^α for $H^2(B\mathbb{T},\mathbb{Z})$, an element in $H^4(B\mathbb{T},\mathbb{Z})$ may be written as $M_{\alpha\beta}\, e^\alpha \cup e^\beta$, where $M_{\alpha\beta}$ is an $N\times N$ integral symmetric matrix. For the purposes of this article we choose $\langle\cdot,\cdot\rangle$ corresponding to $M_{\alpha\beta} = -2\mathbb{I}_{\alpha\beta}$, where $\mathbb{I}_{\alpha\beta}$ is the identity matrix. Let W be a compact oriented four-manifold such that $\partial W = X$, which always exists [20]. Extend P to a $\mathbb{T}$-bundle Q over W, which is always possible in our case [4]. Given a form $\alpha \in \Omega^j(P,\mathfrak{t})$, let $\widetilde{\alpha} \in \Omega^j(Q,\mathfrak{t})$ denote the corresponding extension to Q. For a connection $A \in \Omega^1(P,\mathfrak{t})$, denote the curvature form of the extension $\widetilde{A} \in \Omega^1(Q,\mathfrak{t})$ by $F_{\widetilde{A}} \in \Omega^2(W,\mathfrak{t})$.

Definition 4. The *Chern-Simons action* of a $\mathbb{T}$-connection $A \in \mathcal{A}_P$ is defined by,

$$\operatorname{CS}_{X,P}(A) := \frac{1}{4\pi}\int_W \langle F_{\widetilde{A}} \wedge F_{\widetilde{A}}\rangle \mod (2\pi\mathbb{Z}). \tag{2}$$

We also define the following

- $m_X := \frac{N}{2}(\dim H^1(X;\mathbb{R}) - 2\dim H^0(X;\mathbb{R}))$,
- A_P denotes a flat connection on a principal $\mathbb{T}$-bundle P over X,
- $c_1(X) = n + \sum_{j=1}^M \frac{\beta_j}{\alpha_j}$ is the first orbifold Chern number of the Seifert manifold X,
- $s(\alpha,\beta) := \frac{1}{4\alpha}\sum_{j=1}^{\alpha-1}\cot\left(\frac{\pi j}{\alpha}\right)\cot\left(\frac{\pi j\beta}{\alpha}\right) \in \mathbb{Q}$ is the Rademacher-Dedekind sum,
- $\eta_0 = N\left(\frac{c_1(X)}{6} - 2\sum_{j=1}^M s(\alpha_j,\beta_j)\right)$ is the *adiabatic eta-invariant* of the Sasakian manifold $(X,\phi,\xi,\kappa,\mathrm{G})$ [17],
- $\mathcal{M}_X \simeq \coprod_{[P]\in \operatorname{Tors} H^2(X,\Lambda)} \mathbb{T}^{2g}$ denotes the moduli space of flat abelian connections on a closed three-manifold. A particular component of $\mathcal{M}_X$ corresponding to a bundle class $[P] \in \operatorname{Tors} H^2(X,\Lambda)$ is denoted as, $\mathcal{M}_P \simeq H^1(X,\mathfrak{t})/H^1(X,\Lambda) \simeq \mathbb{T}^{2g}$. The number of components of $\mathcal{M}_X$ is computed for Sasakian three-manifolds in the following theorem.

  Theorem 5 ([16, Theorem 8.1], [19])**.** *Given a closed oriented Sasakian three-manifold* $(X,\phi,\xi,\kappa,\mathrm{G})$ *(so that* $c_1(X)\neq 0$*) then,*

  $$\mathcal{M}_X \simeq \mathbb{T}^{2g}\times \operatorname{Tors}(H^2(X,\Lambda)) \simeq \operatorname{Hom}(\pi_1(X),\mathbb{T}),$$

  where, $|\operatorname{Tors} H^2(X,\Lambda)| = |c_1(X)\cdot\prod_{j=1}^M \alpha_j|^N$.

- $\Omega_P := \sum_{1\le i\le g, 1\le j\le N} d\theta_{i,j}\wedge d\bar{\theta}_{i,j}$ is the standard symplectic form on $\mathcal{M}_P$,

- $\omega_P := \frac{\Omega^{gN}}{(gN)!(2\pi)^{2gN}} \in \Omega^{2gN}(\mathcal{M}_P, \mathbb{R})$, and $\omega \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R})$ is the symplectic form such that its restriction to the connected component $\mathcal{M}_P$ is ω_P.
- $K_X = \frac{1}{|c_1(X)\cdot\prod_i \alpha_i|^{N/2}}$,
- $\sqrt{T_X} \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R})$ is the (square-root) of the analytic torsion (see Def. 13 and Remark 17). We also write $\sqrt{T_X} \in \Omega^{2gN}(\mathcal{M}_P, \mathbb{R})$ when restricting $\sqrt{T_X}$ to a connected component $\mathcal{M}_P$.
  – The eta-invariant for the odd signature operator, L^{o}, acting on $\Omega^1(X, \mathfrak{t}) \oplus \Omega^3(X, \mathfrak{t})$, is defined by analytic continuation,

$$\eta(\mathrm{L}^{\mathrm{o}}) := \lim_{s\to 0} \sum_{\lambda \in \mathrm{spec}^*(\mathrm{L}^{\mathrm{o}})} \mathrm{sgn}(\lambda)|\lambda|^{-s}\,. \tag{3}$$

  The eta-invariant is an analytic invariant introduced by Atiyah, Patodi and Singer [1] defined for an elliptic and self-adjoint operator. We note that as in [1, Prop. 4.20], we may remove some spectral symmetry and the eta-invariant of L^{o} coincides with the eta-invariant of the operator $\star d$ restricted to $\Omega^1(X, \mathfrak{t}) \cap \mathrm{Im}(\star d)$.
- $\eta_{\mathrm{grav}}(\mathrm{G})$ denotes the eta-invariant for the operator $\star d$ acting on $\Omega^1(X, \mathbb{R})$, so that,

$$\eta(\star d) = N \cdot \eta_{\mathrm{grav}}(\mathrm{G})\,, \tag{4}$$

  where the eta-invariant on the left hand side of (4) is defined on $\Omega^1(X, \mathfrak{t})$ and $N = \dim \mathbb{T}$,
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$$\mathrm{CS}_s(A^{\mathrm{G}}) := \frac{1}{4\pi}\int_X s^*\,\mathrm{Tr}(A^{\mathrm{G}} \wedge dA^{\mathrm{G}} + \frac{2}{3}A^{\mathrm{G}} \wedge A^{\mathrm{G}} \wedge A^{\mathrm{G}})\,, \tag{5}$$

  is the gravitational Chern-Simons term, where A^{G} is the Levi-Civita connection and s a trivializing section of twice the tangent bundle of X. More explicitly, let $H = \mathrm{Spin}(6)$, $Q = TX \oplus TX$ viewed as a principal Spin(6)-bundle over X, $\mathrm{G} \in \Gamma(S^2(T^*X))$ a Riemannian metric on X, $\phi : Q \to \mathrm{SO}(X)$ a principal bundle morphism, and $A^{LC} \in \mathcal{A}_{SO(X)} := \{A \in (\Omega^1(\mathrm{SO}(X)) \otimes \mathfrak{so}(3))^{\mathrm{SO}(3)} \mid A(\xi^\sharp) = \xi,\ \forall \xi \in \mathfrak{so}(3)\}$ the Levi-Civita connection. Then $A^{\mathrm{G}} := \phi^* A^{LC} \in \mathcal{A}_Q := \{A \in (\Omega^1(Q) \otimes \mathfrak{h})^H \mid A(\xi^\sharp) = \xi,\ \forall \xi \in \mathfrak{h}\}$.

An Atiyah-Patodi-Singer theorem, [2, Prop. 4.19], says that the combination,

$$\eta_{\mathrm{grav}}(\mathrm{G}) + \frac{1}{3}\frac{\mathrm{CS}(A^{\mathrm{G}})}{2\pi}\,, \tag{6}$$

is a topological invariant depending only on a two-framing of X. Recall that a two-framing is a choice of a homotopy equivalence class Π of trivializations of $TX \oplus TX$, twice the tangent bundle of X. Note that Π is represented by the trivializing section $s\colon X \to Q$ above. The possible two-framings correspond to $\mathbb{Z}$. The identification with $\mathbb{Z}$ is given by the signature defect defined by,

$$\delta(X, \Pi) = \mathrm{sign}(W) - \frac{1}{6}p_1(2TW, \Pi)\,,$$

where W is a 4-manifold with boundary X and $p_1(2TW,\Pi)$ is the relative Pontrjagin number associated to the framing Π of the bundle $TX \oplus TX$. The canonical two-framing Π^c corresponds to $\delta(X,\Pi^c)=0$.

Remark 6. Before we present the main quantities of interest in Definitions 7, 8, we note that both definitions implicitly require a choice of base h^0 for $H^0(X,\mathbb{R})$ to be well defined. We elaborate on this point in §2.

Definition 7. [14] Let $k \in \mathbb{Z}$ and X a closed, oriented three-manifold. The abelian Chern-Simons partition function, $Z_{\mathbb{T}}(X,k)$, is the quantity,

$$Z_{\mathbb{T}}(X,k) = \sum_{P \in \operatorname{Tors} H^2(X,\Lambda)} Z_{\mathbb{T}}(X,P,k)\,, \tag{7}$$

where,

$$Z_{\mathbb{T}}(X,P,k) := k^{m_X} e^{ik\,\mathrm{CS}_{X,P}(A_P)} e^{\pi i N\left(\frac{\eta_{\mathrm{grav}}(\mathrm{G})}{4} + \frac{1}{12}\frac{\mathrm{CS}(A^{\mathrm{G}})}{2\pi}\right)} \int_{\mathcal{M}_P} \sqrt{T_X}\,. \tag{8}$$

Definition 8 ([14])**.** Let $k \in \mathbb{Z}$, and let $(X,\phi,\xi,\kappa,\mathrm{G})$ be a closed oriented Sasakian three-manifold. Define the *symplectic abelian Chern-Simons partition function*,

$$\overline{Z}_{\mathbb{T}}(X,k) = \sum_{[P] \in \operatorname{Tors} H^2(X,\Lambda)} \overline{Z}_{\mathbb{T}}(X,P,k)\,, \tag{9}$$

where

$$\overline{Z}_{\mathbb{T}}(X,P,k) = k^{m_X} e^{ik\,\mathrm{CS}_{X,P}(A_P)} e^{i\pi\left(\frac{N}{4} - \frac{1}{2}\eta_0\right)} \int_{\mathcal{M}_P} K_X \cdot \omega_P\,. \tag{10}$$

The main motivation for this work is the conjectural equivalence of the rigorous topological invariants $Z_{\mathbb{T}}(X,k)$ and $\overline{Z}_{\mathbb{T}}(X,k)$. Note that this conjecture arises simply due to the fact that the rigorous definitions of $Z_{\mathbb{T}}(X,k)$ and $\overline{Z}_{\mathbb{T}}(X,k)$ are derived from the same heuristic Chern-Simons partition function in physics. We note that part of this conjectural equivalence is motivated by [11] which argues that $\sqrt{T_X}$ is proportional to a specific scalar multiple of the natural unit symplectic volume form $\omega \in \Omega^{2gN}(\mathcal{M}_X,\mathbb{R})$ by using the group structure on the moduli space $\mathcal{M}_X$,

$$\sqrt{T_X} = C \cdot \Big(\frac{1}{\sqrt{|\operatorname{Tors} H^2(X,\Lambda)|}} \cdot \omega\Big)\,, \tag{11}$$

where $0 \neq C \in \mathbb{R}$. Note that [11] works with the case where X is endowed with a *regular* Sasakian structure, which corresponds to a principle $\mathbb{U}(1)$ bundle over a surface *without* orbifold points. This article studies the more general case of a three-manifold X that admits a Sasakian structure. We are able to calculate the square-root of T_X explicitly as a specific scalar multiple of a natural symplectic volume form on the moduli space $\mathcal{M}_X$ using a theorem of M. Rumin and N. Seshadri [24, Theorem 5.4]. We obtain the following

Theorem 9 (Main Theorem). *Let $(X, \phi, \xi, \kappa, \mathrm{G})$ be a closed Sasakian three-manifold. Then,*

$$\sqrt{T_X} = \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \cdot \omega\,. \tag{12}$$

We note that Theorem 9 combined with Theorem 5 leads to an explicit computation of the symplectic volume of the moduli space. Thus, we have the following,

Corollary 10. *Given a closed oriented Sasakian three-manifold $(X, \phi, \xi, \kappa, \mathrm{G})$, the symplectic volume of the moduli space $\mathcal{M}_X$ with respect to the symplectic volume form $\sqrt{T_X} \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R})$ is given by,*

$$\int_{\mathcal{M}_X} \sqrt{T_X} = \sqrt{|\operatorname{Tors} H^2(X, \Lambda)|} = \left| c_1(X) \cdot \prod_j \alpha_j \right|^{N/2}. \tag{13}$$

As a consequence of Theorem 9 we obtain the following verification of the above conjecture,

Corollary 11. *Let $k \in \mathbb{Z}$, and let $(X, \phi, \xi, \kappa, \mathrm{G})$ be a closed oriented Sasakian three-manifold. Then the magnitudes of $Z_{\mathbb{T}}(X, k)$ and $\overline{Z}_{\mathbb{T}}(X, k)$ agree identically,*

$$|Z_{\mathbb{T}}(X, k)| = \left|\overline{Z}_{\mathbb{T}}(X, k)\right|\,, \tag{14}$$

and

$$|Z_{\mathbb{T}}(X, k)| = k^{m_X} \cdot \frac{\left|\sum_{[P] \in \operatorname{Tors} H^2(X,\Lambda)} e^{ik\,\mathrm{CS}_{X,P}(A_P)}\right|}{\sqrt{|\operatorname{Tors} H^2(X, \Lambda)|}}. \tag{15}$$

2. Proof of the main theorem

In this section we prove Theorem 9 and compute the square root of the analytic torsion $\sqrt{T_X}$ as a symplectic volume form on the moduli space of flat abelian connections $\mathcal{M}_X$ in the case that X admits a Sasakian structure. For simplicity, we will assume $\mathbb{T} = \mathrm{U}(1)$ in this section and set $N = 1$.

Remark 12. The natural quantity that shows up in the symplectic abelian Chern-Simons path integral is ω multiplied by $1/|\operatorname{Vol}(I)|$, where

$$I := \{g \in \mathcal{G}_P | A_P \cdot g = A_P\} \simeq \mathrm{U}(1) < \mathcal{G}\,,$$

is the isotropy subgroup of the gauge group of a given abelian connection $A_P \in \mathcal{A}_P$. The volume of the isotropy group, $\operatorname{Vol}(I)$, requires a choice of measure on $I \simeq \mathrm{U}(1)$, which boils down to a choice of base h^0 for $H^0(X, \mathbb{R})$. We recall some of the details presently.

In our study of abelian Chern-Simons theory [14], the natural invariant metric $\mathrm{H}_{\mathcal{G}}$ on the group $\mathcal{G}$ is defined in terms of the Hodge star $\star$ for the given Sasakian metric G on X,

$$\mathrm{H}_{\mathcal{G}}(\theta_1, \theta_2) := \int_X \langle \theta_1 \wedge \star\theta_2 \rangle\,, \tag{16}$$

where $\theta_1,\ \theta_2 \in \text{Lie}\ \mathcal{G} \simeq \Omega^0(X,\mathbb{R})$. Observe that $\mathrm{H}_{\mathcal{G}}$ restricted to constant functions $\theta_1,\ \theta_2 \in \mathbb{R} \subset \text{Lie}\ \mathcal{G}$ is given as follows,

$$\begin{aligned}\mathrm{H}_{\mathcal{G}}(\theta_1,\theta_2) &= \int_X \langle \theta_1 \wedge \star\theta_2 \rangle \\ &= \Big(\int_X \star 1 \Big) \cdot \langle \theta_1, \theta_2 \rangle\,.\end{aligned}$$

We may therefore write $\sqrt{\mathrm{H}_{\mathcal{G}}} = \big(\int_X \star 1\big)^{1/2}$. Now we choose the measure $\sqrt{\mathrm{H}_{\mathcal{G}}}\, d\sigma$ on $I \simeq \mathrm{U}(1)$ such that $d\sigma = d\theta/2\pi$ setting $\int_{\mathrm{U}(1)} d\sigma = 1$. Let $\mathcal{H}^0(X,\mathbb{R})$ denote the harmonic 0-forms on X. Note that by definition of the de Rham map δ^0_{dR} : $\mathcal{H}^0(X,\mathbb{R}) \to H^0(X,\mathbb{R})$, this choice of measure may be viewed as a choice of base h^0 for $H^0(X,\mathbb{R}) \simeq \text{Lie}\ \mathrm{U}(1)$ such that $\delta^0_{\mathrm{dR}}(2\pi) = h^0$. We have,

$$\begin{aligned}\mathrm{Vol}(I) &:= \int_{\mathrm{U}(1)} \sqrt{\mathrm{H}_{\mathcal{G}}} d\sigma\,, \\ &= \sqrt{\mathrm{H}_{\mathcal{G}}},\ \text{since } \textstyle\int_{\mathrm{U}(1)} d\sigma = 1, \\ &= \Big[\int_X \star 1 \Big]^{1/2}. \qquad (17)\end{aligned}$$

Since the Hodge star $\star$ is defined in terms of the given Sasakian metric, we have,

$$\mathrm{Vol}(I) = \Big[\int_X \kappa \wedge d\kappa \Big]^{1/2} = [c_1(X)]^{1/2}\ .$$

A proof of Theorem 9 follows from [24, Theorem 5.4], where the analytic torsion is computed on a closed Sasakian three-manifold twisted by a unitary representation $\rho : \pi_1(X) \to \mathrm{U}(r)$. Combining this with a substitution of some known special values of the Riemann-Hurwitz zeta function completes the proof.

Let (M, G) be a closed oriented Riemannian manifold of dimension m and let $\rho\colon \pi_1(\mathrm{M}) \to \mathrm{U}(1)$ be a representation of the fundamental group of M. Recall that ρ corresponds to a flat principal U(1) bundle P over M equipped with a flat connection $A_\rho \in \mathcal{A}_P$. Given a representation $\chi\colon\ \mathrm{U}(1) \to \mathrm{Aut}\,\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we obtain an associated line bundle $\mathcal{E}_\chi := P \times_\chi \mathbb{F}$. Let,

$$d^\chi_{A_\rho} \colon \Omega^q(\mathrm{M}, \mathcal{E}_\chi) \to \Omega^{q+1}(\mathrm{M}, \mathcal{E}_\chi)\,,$$

denote the covariant derivative associated to A_ρ and let,

$$\Delta^\chi_q(\rho) := (d^\chi_{A_\rho})^* d^\chi_{A_\rho} + d^\chi_{A_\rho}(d^\chi_{A_\rho})^* \colon \Omega^q(\mathrm{M}, \mathcal{E}_\chi) \to \Omega^q(\mathrm{M}, \mathcal{E}_\chi)\,,$$

denote the corresponding Laplacian. Define the determinant line,

$$\det H^\bullet(\mathrm{M}, d^\chi_{A_\rho}) := \bigotimes_{j=0}^{3} \det H^j(\mathrm{M}, d^\chi_{A_\rho})^{(-1)^{j+1}}\,,$$

where a superscript -1 denotes the dual space. Let $|\cdot|_{L^2(\Omega^\bullet(X))}$ denote the L^2-metric on $\det H^\bullet(\mathrm{M}, d^\chi_{A_\rho})$ induced by the identification of $H^\bullet(\mathrm{M}, d^\chi_{A_\rho})$ with the harmonic forms $\mathcal{H}^\bullet(\mathrm{M}, d^\chi_{A_\rho})$ via the de Rham map $\delta^q_{\mathrm{dR}} \colon \mathcal{H}^q(\mathrm{M}, d^\chi_{A_\rho}) \to H^q(\mathrm{M}, d^\chi_{A_\rho})$.

Definition 13. [21] Let M be a closed oriented Riemannian manifold of dimension m and let $\rho\colon \pi_1(\mathrm{M}) \to \mathrm{U}(1)$ be a representation of the fundamental group of M and let $\chi\colon \mathrm{U}(1) \to \operatorname{Aut}\mathbb{F}$ be a representation of U(1) (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Let $\Delta_q^\chi(\rho)\colon \Omega^q(\mathrm{M}, \mathcal{E}_\chi) \to \Omega^q(\mathrm{M}, \mathcal{E}_\chi)$ denote the Laplacian in the representation χ. Let $\zeta_q(s)$ be the zeta-function for $\Delta_q^\chi(\rho)$ defined for $\mathrm{Re}(s) \gg 0$ by,

$$\zeta_q(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{tr}(e^{t\Delta_q} - \Pi_q)\, dt\,, \tag{18}$$

analytically continued to $\mathbb{C}$ and $\Pi_q\colon \Omega^q(\mathrm{M}, \rho) \to \mathcal{H}^q(\mathrm{M}, \rho)$ orthogonal projection. The *analytic torsion* is defined as,

$$T_{\mathrm{M}} = T_{\mathrm{M}}^\chi(\rho) := \exp\Big(\frac{1}{2}\sum_{q=0}^m (-1)^q q \zeta_q'(0)\Big)\,. \tag{19}$$

The *Ray-Singer metric* $\|\cdot\|_{RS}$ is defined as

$$\|\cdot\|_{RS} = T_{\mathrm{M}}\, |\cdot|_{L^2(\Omega^\bullet(X))}\,. \tag{20}$$

Note that [24] defines and studies a new type of analytic torsion on contact manifolds called the *contact analytic torsion*, denoted by T_X^C, and they also introduce a corresponding *contact Ray-Singer metric*, denoted $||\cdot||_C$. These quantities are defined in terms of the *contact complex* $(\mathcal{E}, D_H)$, originally introduced by M. Rumin [23], on a contact manifold (X, κ). Given the Reeb vector field $\xi \in \Gamma(X)$ for the contact form $\kappa \in \Omega^1(X, \mathbb{R})$, let $d_H\colon \Omega^j(X) \to \Omega^{j+1}(X)$ be defined as $d_H := d - \kappa \wedge \iota_\xi$, and $\mathcal{L}_\xi$ be the Lie derivative. Define $\Omega^1(H) := \{\alpha \in \Omega^1(X)\,|\, \iota_\xi \alpha = 0\}$ and $\Omega^2(V) := \{\beta \in \Omega^2(X)\,|\, \beta = \kappa \wedge \alpha, \text{ for } \alpha \in \Omega^1(X)\}$. Given a contact metric manifold $(X, \phi, \xi, \kappa, \mathrm{G})$, and $\star$ the usual Hodge star for the metric G, the *horizontal Hodge star* is defined as $\star_H := \star \circ (\kappa\wedge)$. The *contact complex* $(\mathcal{E}, D_H)$ is defined as

$$C^\infty(X) \xrightarrow{D_H\, =\, d_H} \Omega^1(H) \xrightarrow{D_H\, =\, D} \Omega^2(V) \xrightarrow{D_H\, =\, d} \Omega^3(X)\,, \tag{21}$$

with middle operator $D_H = D = \kappa \wedge (\mathcal{L}_\xi + d_H \star_H d_H)$. Note that this complex may be defined using only the choice of a contact 2-plane field [24], and we have introduced a contact metric structure in order to be more explicit. Also note that one can twist the contact complex with a flat bundle and define the twisted contact complex, contact analytic torsion and contact Ray-Singer metric as well [24]. Given a contact metric manifold $(X, \phi, \xi, \kappa, \mathrm{G})$, the contact analytic torsion and metric are defined using the *contact Laplacian* on $(\mathcal{E}, D_H)$,

$$\Delta_q^C = \begin{cases} (d_H^* d_H + d_H d_H^*)^2 & \text{if} \quad q = 0, 3\,, \\ D^* D + (d_H d_H^*)^2 & \text{if} \quad q = 1\,, \\ D D^* + (d_H^* d_H)^2 & \text{if} \quad q = 2\,. \end{cases} \tag{22}$$

This operator is *maximally hypoelliptic and invertible in the Heisenberg symbolic calculus* [24]; a key property that allows one to make sense of the zeta function for the contact Laplacian $\zeta(\Delta_q^C)(s)$. [24] introduce the *contact torsion function*

$$K(s) := \frac{1}{2}\sum_{q=0}^{3}(-1)^q w(q)\zeta(\Delta_q^C)(s)\,, \tag{23}$$

where for $q = 0, 1, 2, 3$

$$w(q) = \begin{cases} q\,, & q \leq 1\,, \\ q+1\,, & q > 1\,. \end{cases} \tag{24}$$

Note that our definition of $K(s)$ is the negative of the one that occurs in [24]. The *contact analytic torsion* is then defined to be

$$T_X^C := \exp\left(\frac{1}{2}K'(0)\right). \tag{25}$$

It is shown in [24] that the analytic torsion and Ray-Singer metric agree with their contact geometric counterparts on Sasakian manifolds. Note that our definition of T_X^C is the inverse of the definition used in [24].

Theorem 14 ([24, Theorem 4.2])**.** *Let $(X, \phi, \xi, \kappa, \mathrm{G})$ be a closed Sasakian (CR-Seifert) three-manifold, $\rho\colon \pi_1(X) \to U(N)$ a unitary representation, and $\chi_0\colon U(N) \to \mathrm{Aut}(\mathbb{C}^N)$ the standard representation. Let T_X and T_X^C denote the analytic torsion and the contact analytic torsion, respectively, in the standard representation; e.g. $T_X := T_X^{\chi_0}$. Then the analytic torsion T_X and the contact analytic torsion T_X^C agree,*

$$T_X(\rho) = T_X^C(\rho)\,. \tag{26}$$

Also, the Ray-Singer metric $\|\cdot\|_{RS}$ and the contact Ray-Singer metric $\|\cdot\|_C$ agree,

$$\|\cdot\|_{RS} = \|\cdot\|_C\,. \tag{27}$$

For $a \in (0, 1]$, let $\widetilde{\zeta}(s, a) = \sum_{n\in\mathbb{N}} \frac{1}{(n+a)^s}$ denote the Riemann-Hurwitz zeta function, and let $\widetilde{\zeta}(s) := \widetilde{\zeta}(s, 1)$ denote the Riemann zeta function. The main result that we need is given as follows.

Theorem 15 ([24, Theorem 5.4])**.** *Let $(X, \phi, \xi, \kappa, \mathrm{G})$ be a closed Sasakian three-manifold. Split $\mathcal{E}_\chi$ into irreducibles $\mathcal{E}_\chi^\theta$. Then the contact torsion function spectrally decomposes as,*

$$K(s) = \sum_{\mathcal{E}_\chi^\theta} K_\theta(s)\,, \tag{28}$$

such that,

- *On $\mathcal{E}_\chi^\theta$ with $\theta \in (0, 1)$, i.e. $\chi \circ \rho(h) = e^{2\pi i\theta} \neq 1$, we have,*

$$\begin{aligned} K_\theta(s) = &- \dim(\mathcal{E}_\chi^\theta)\chi(\Sigma^*)\big(\widetilde{\zeta}(2s, \theta) + \widetilde{\zeta}(2s, 1-\theta)\big) \\ &- \sum_{i,j}\frac{1}{\alpha_i^{2s}}\big(\widetilde{\zeta}(2s, \theta_{i,j}) + \widetilde{\zeta}(2s, 1-\theta_{i,j})\big)\,. \end{aligned} \tag{29}$$

- *Let* $\mathcal{E}_\chi^{0,i} = \ker(1 - \chi \circ \rho(c_i))$. *Then we have,*

$$K_0(s) = -\ K(X,\rho)(2\widetilde{\zeta}(2s)+1) - 2\widetilde{\zeta}(2s)\sum_i \dim(\mathcal{E}_\chi^{0,i})(\alpha_i^{-2s}-1)$$
$$- \sum_{\{(i,j):\theta_{i,j}\neq 0\}} \frac{1}{\alpha_i^{2s}}\left(\widetilde{\zeta}(2s,\theta_{i,j}) + \widetilde{\zeta}(2s,1-\theta_{i,j})\right).$$

where $K(X,\rho) := 2\dim H^0(X,\mathfrak{t}) - \dim H^1(X,\mathfrak{t})$.

Remark 16. We note that the proof of this theorem follows by application of the Riemann-Roch-Kawasaki formula [12], [10].

The case of interest for us is the trivial representation $\rho_0\colon \pi_1(X) \to \mathrm{U}(1)$. Since this is already scalar we have,

$$K(s) = K_0(s)\,, \tag{30}$$

where, by Theorem 15, we have

$$K_0(s) = -K(X,\rho)(2\zeta(2s)+1) - 2\zeta(2s)\sum_i(\alpha_i^{-2s}-1)\,. \tag{31}$$

Now we use the identification of the analytic torsion and the contact analytic torsion given in Theorem 14 to write $T_X^\chi(\rho_0) = \exp(K_0'(0)/2)$. We compute $K_0'(0)$ using Theorem 15. Using the special values of the Riemann-zeta function, $\zeta(0) = -1/2$ and $\zeta'(0) = -\ln(2\pi)/2$ [26], and $K(X,\rho) = 2\dim H^0(X,\mathfrak{t}) - \dim H^1(X,\mathfrak{t})$ [24, Eq. 42], we obtain,

$$K_0'(0)/2 = (2-2g)\ln(2\pi) - \sum_i \ln(\alpha_i)\,. \tag{32}$$

Thus,

$$T_X^\chi(\rho_0) = \frac{(2\pi)^{2-2g}}{\prod_i \alpha_i}\,. \tag{33}$$

It is easy to see that $T_X^{\mathrm{Ad}}(\rho) = T_X^\chi(\rho_0)$ when $\rho_0 \equiv 1$ is the trivial representation, χ is the standard representation, and $\rho : \pi_1(X) \to \mathrm{U}(1)$ is arbitrary. This follows because the spectra of the corresponding Laplacians are identical. That is, for the standard representation χ, the Laplacian at the trivial representation ρ_0 is given by,

$$\Delta_j^\chi(\rho_0) := d^*d + dd^*\colon \Omega^j(X,\mathbb{C}) \to \Omega^j(X,\mathbb{C})\,,$$

where $d^\chi_{A_{\rho_0}} = d$ is just the ordinary de Rham derivative. Also, for the adjoint representation,

$$\Delta_j^{\mathrm{Ad}}(\rho) := d^*d + dd^*\colon \Omega^j(X,\mathbb{R}) \to \Omega^j(X,\mathbb{R})\,,$$

since $d^{\mathrm{Ad}}_{A_\rho} = d$ for *any* representation ρ. Clearly, these operators have identical spectra. By Poincaré duality $H^3(X,d)^{-1}$ is canonically isomorphic to $H^0(X,d)$, and $H^1(X,d)^{-1}$ is canonically isomorphic to $H^2(X,d)$. Thus,

$$\|\cdot\|_{RS} \in |\det H^0(X,d_{A_\rho})|^{\otimes 2} \bigotimes |\det H^1(X,d_{A_\rho})^{-1}|^{\otimes 2}\,,$$

and we may define the square-root of $\|\cdot\|_{RS}$,

$$\sqrt{\|\cdot\|_{RS}} \in |\det H^0(X, d_{A_\rho})| \bigotimes |\det H^1(X, d_{A_\rho})^{-1}|\,.$$

Note that since the adjoint representation is trivial on $\mathbb{R}$, we have

$$\sqrt{\|\cdot\|_{RS}} \in |\det H^0(X, \mathbb{R})| \bigotimes |\det H^1(X, \mathbb{R})^{-1}|\,.$$

Remark 17. Observe that if ν^0 is an orthonormal base for $\mathcal{H}^0(X, \mathbb{R}) = \mathbb{R}$, then it may be identified as a scalar $\nu^0 \in \mathbb{R}$ such that,

$$\begin{aligned} 1 &= ||\nu^0||^2, \\ &= \int_X \nu^0 \wedge \star\nu^0, \\ &= |\nu^0|^2 \int_X \kappa \wedge d\kappa, \\ &= |\nu^0|^2 \cdot c_1(X)\,. \end{aligned}$$

Thus, $|\nu^0| = 1/|c_1(X)|^{1/2}$. In order to view the analytic torsion as a volume form on $\mathcal{M}_X$, we must choose a base h^0 for $H^0(X, \mathbb{R})$ and evaluate $\sqrt{T_X}$ at h_0. If we identify $H^0(X, \mathbb{R}) \simeq \mathbb{R}$ via the de Rham map δ^0_{dR}, then we make the same choice as in Remark 12 and choose $\delta^0_{\mathrm{dR}}(2\pi) = h^0$.

Choosing $h^0 \in H^0(X, \mathbb{R})$ as in Remark 17 and denoting the Ray-Singer metric evaluated at h^0 by $\|\cdot\|_{RS}|_{h^0}$, we define,

$$\sqrt{T_X} := \sqrt{\|\cdot\|_{RS}|_{h^0}} \in |\det H^1(X, \mathbb{R})^{-1}|\,. \tag{34}$$

We therefore have

$$\sqrt{T_X} = \frac{(2\pi)^{-Ng}}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \left|\bigwedge \delta^1_{\mathrm{dR}}(\nu^1)\right|^* , \tag{35}$$

where $\left|\bigwedge \delta^1_{\mathrm{dR}}(\nu^1)\right|^* : \bigwedge^{\max} H^1(X, \mathfrak{t}) \to \mathbb{R}^+$ is the volume form associated to the basis given by $\delta^1_{\mathrm{dR}}(\nu^1)$. Writing the above results concisely, if $(X, \phi, \xi, \kappa, \mathrm{G})$ is a closed Sasakian three-manifold, then,

$$\sqrt{T_X} = \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \cdot \omega\,, \tag{36}$$

where

$$\omega := \frac{\Omega^{gN}}{(gN)!(2\pi)^{2gN}}\,, \tag{37}$$

and

$$\Omega := \sum_{1 \le i \le gN} d\theta_i \wedge d\bar{\theta}_i\,. \tag{38}$$

Note that the generalization to the case of an arbitrary torus $\mathbb{T}$ is straightforward. We also point out that the extra factor of $(2\pi)^{gN}$ that occurs in Eq. (37) is due to the corresponding factor of $\sqrt{2\pi}$ in the norm of each orthonormal basis element for the first cohomology.

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