

PROLONGATION OF POISSON 2-FORM ON WEIL BUNDLES

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ABSTRACT. In this paper, M denotes a smooth manifold of dimension n , A a Weil algebra and M^A the associated Weil bundle. When (M, ω_M) is a Poisson manifold with 2-form ω_M , we construct the 2-Poisson form $\omega_{M^A}^A$, prolongation on M^A of the 2-Poisson form ω_M . We give a necessary and sufficient condition for that M^A be an A -Poisson manifold.

1. INTRODUCTION

1.1. Weil algebra and Weil bundle. In what follows, all structures are assumed to be of class C^∞ . We denote by M a smooth differential manifold, $C^\infty(M)$ the algebra of differentiable functions on M and by $\mathfrak{X}(M)$, the $C^\infty(M)$ -module of vectors field on M .

A Weil algebra is a real, unitary, commutative algebra of finite dimension with a unique maximal ideal of codimension 1 on \mathbb{R} , [15].

Let A be a Weil algebra and \mathfrak{m} be its maximal ideal. We have $A = \mathbb{R} \oplus \mathfrak{m}$ and the first projection

$$A = \mathbb{R} \oplus \mathfrak{m} \rightarrow \mathbb{R}$$

is a homomorphism of algebras which is surjective, called augmentation and the unique non zero integer $h \in \mathbb{N}$ such that $\mathfrak{m}^h \neq (0)$ and $\mathfrak{m}^{h+1} = (0)$ is the height of A , [15].

If M is a smooth manifold, and A a Weil algebra of maximal ideal \mathfrak{m} , an infinitely near point to $x \in M$ of kind A is a homomorphism of algebras

$$\xi: C^\infty(M) \rightarrow A$$

such that $[\xi(f) - f(x)] \in \mathfrak{m}$ for any $f \in C^\infty(M)$. i.e. the real part of $\xi(f)$ is exactly $f(x)$, [15].

We denote by M_x^A the set of all infinitely near points to $x \in M$ of kind A and $M^A = \bigcup_{x \in M} M_x^A$ the manifold of infinitely near points of kind A . We have $\dim M^A = \dim M \times \dim A$, [6].

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When both M and N are smooth manifolds and when $h: M \rightarrow N$ is a differentiable application, then the map

$$h^A: M^A \rightarrow N^A, \quad \xi \mapsto h^A(\xi),$$

such that, for any $g \in C^\infty(N)$,

$$[h^A(\xi)](g) = \xi(g \circ h)$$

is also differentiable. When h is a diffeomorphism, it is the same for h^A , [1].

Moreover, if $\varphi: A \rightarrow B$ is a homomorphism of Weil algebras, for any smooth manifold M , the map

$$\varphi_M: M^A \rightarrow M^B, \quad \xi \mapsto \varphi \circ \xi$$

is differentiable. In particular, the augmentation

$$A \rightarrow \mathbb{R}$$

defines for any smooth manifold M , the projection

$$\pi_M: M^A \rightarrow M,$$

which assigns every infinitely near point to $x \in M$ to its origin x . Thus (M^A, π_M, M) defines the bundle of infinitely near points or simply Weil bundle (see [3], [6], [15], [9]).

If (U, φ) is a local chart of M with coordinate functions (x_1, x_2, \dots, x_n) , the application

$$U^A \rightarrow A^n, \quad \xi \mapsto (\xi(x_1), \xi(x_2), \dots, \xi(x_n)),$$

is a bijection from U^A into an open of A^n . The manifold M^A is a smooth manifold modeled over A^n , that is to say an A -manifold of dimension n , [2], [13].

The set, $C^\infty(M^A, A)$ of differentiable functions on M^A with values in A is a commutative, unitary algebra over A . When one identifies \mathbb{R}^A with A , for $f \in C^\infty(M)$, the map

$$f^A: M^A \rightarrow A, \quad \xi \mapsto \xi(f)$$

is differentiable and the map

$$C^\infty(M) \rightarrow C^\infty(M^A, A), \quad f \mapsto f^A,$$

is an injective homomorphism of algebras and we have:

$$(f + g)^A = f^A + g^A; \quad (\lambda \cdot f)^A = \lambda \cdot f^A; \quad (f \cdot g)^A = f^A \cdot g^A$$

for $\lambda \in \mathbb{R}$, $f, g \in C^\infty(M)$.

We denote $\mathfrak{X}(M^A)$, the set of all vector fields on M^A . According to [2], [8] we have the following equivalent assertions:

Theorem 1. *The following assertions are equivalent:*

- (1) *A vector field on M^A is a differentiable section of the tangent bundle (TM^A, π_{M^A}, M^A) .*
- (2) *A vector field on M^A is a derivation of $C^\infty(M^A)$.*
- (3) *A vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A -linear.*

(4) A vector field on M^A is a linear map $X: C^\infty(M) \rightarrow C^\infty(M^A, A)$ such that

$$X(f \cdot g) = X(f) \cdot g^A + f^A \cdot X(g), \quad \text{for any } f, g \in C^\infty(M).$$

Consequently [8],

Theorem 2. *The map*

$$\mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X, Y) \longmapsto [X, Y] = X \circ Y - Y \circ X$$

is skew-symmetric A -bilinear and defines a structure of A -Lie algebra over $\mathfrak{X}(M^A)$.

Thus, if $\text{Der}_A[C^\infty(M^A, A)]$ denotes the $C^\infty(M^A, A)$ -module of derivations of $C^\infty(M^A, A)$ which are A -linear, a vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A -linear i.e. a A -linear map

$$X: C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \quad \text{for any } \varphi, \psi \in C^\infty(M^A, A).$$

Thus, we have

$$\mathfrak{X}(M^A) = \text{Der}_A [C^\infty(M^A, A)].$$

Proposition 1 ([2], [8]). *If $\theta: C^\infty(M) \rightarrow C^\infty(M)$ is a vector field on M , then there exists one and only one A -linear derivation*

$$\theta^A: C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

such that

$$\theta^A(f^A) = [\theta(f)]^A$$

for any $f \in C^\infty(M)$.

Proposition 2 ([2], [8]). *If θ, θ_1 and θ_2 are vector fields on M and if $f \in C^\infty(M)$, then we have:*

- (1) $(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A$;
- (2) $(f \cdot \theta)^A = f^A \cdot \theta^A$;
- (3) $[\theta_1, \theta_2]^A = [\theta_1^A, \theta_2^A]$.

Corollary 1. *The map*

$$\mathfrak{X}(M) \rightarrow \text{Der}_A [C^\infty(M^A, A)], \theta \mapsto \theta^A$$

is an injective homomorphism of \mathbb{R} -Lie algebras. If $\mu: A \rightarrow A$, is a \mathbb{R} -endomorphism, and $\theta: C^\infty(M) \rightarrow C^\infty(M)$ a vector field on M , then

$$\theta^A(\mu \circ f^A) = \mu \circ [\theta(f)]^A \quad \text{for any } f \in C^\infty(M).$$

1.2. Poisson manifold. We recall that a Poisson structure on a smooth manifold M is due to the existence of a bracket $\{, \}_M$ on $C^\infty(M)$ such that the pair $(C^\infty(M), \{, \}_M)$ is a real Lie algebra such that, for any $f \in C^\infty(M)$ the map

$$\text{ad}(f): C^\infty(M) \rightarrow C^\infty(M), \quad g \mapsto \{f, g\}_M$$

is a derivation of commutative algebra i.e.

$$\{f, g \cdot h\}_M = \{f, g\}_M \cdot h + g \cdot \{f, h\}_M$$

for $f, g, h \in C^\infty(M)$. In this case we say that $C^\infty(M)$ is a Poisson algebra and M is a Poisson manifold ([5], [14], [11]).

Let $\Omega_{\mathbb{R}}[C^\infty(M)]$ be the $C^\infty(M)$ -module of Kähler differentials of $C^\infty(M)$ and

$$\delta_M: C^\infty(M) \rightarrow \Omega_{\mathbb{R}}[C^\infty(M)], \quad f \mapsto \overline{f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f}$$

the canonical derivation which the image of δ_M generates the $C^\infty(M)$ -module $\Omega_{\mathbb{R}}[C^\infty(M)]$ i.e. for $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$x = \sum_{i \in I: \text{finite}} f_i \cdot \delta_M(g_i),$$

with $f_i, g_i \in C^\infty(M)$ for any $i \in I$ ([4], [11], [10]).

The manifold M is a Poisson manifold if and only if there exists a skew-symmetric 2-form

$$\omega_M: \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \rightarrow C^\infty(M)$$

such that for any f and g in $C^\infty(M)$,

$$\{f, g\}_M = -\omega_M[\delta_M(f), \delta_M(g)]$$

defines a structure of Lie algebra over $C^\infty(M)$, [11], [10]. In this case, we say that ω_M is the Poisson 2-form of the Poisson manifold M and we denote (M, ω_M) the Poisson manifold of Poisson 2-form ω_M .

The main goal of this paper is to study the prolongation of the Poisson 2-form ω_M of Poisson manifold on Weil bundles.

2. THE ALGEBRA OF KÄHLER FORMS ON $C^\infty(M^A, A)$

Definition 1. The $C^\infty(M^A)$ -module of Kähler differentials of $C^\infty(M^A)$ is the set

$$\Omega_{\mathbb{R}}[C^\infty(M^A)] = \frac{J}{J^2}$$

where J is the $C^\infty(M^A)$ -submodule of $C^\infty(M^A) \otimes_{\mathbb{R}} C^\infty(M^A)$ generated by the elements of the form $F \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes F$ with $F \in C^\infty(M^A)$. Thus, the map

$$d_{M^A}: C^\infty(M^A) \rightarrow \Omega_{\mathbb{R}}[C^\infty(M^A)], F \mapsto \overline{F \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes F}$$

is a derivation and the image of d_{M^A} generates $\Omega_{\mathbb{R}}[C^\infty(M^A)]$.

The A -algebra $C^\infty(M^A, A) \otimes_A C^\infty(M^A, A)$ admits a structure of $C^\infty(M^A, A)$ -module defined by the homomorphism of A -algebras

$$C^\infty(M^A, A) \rightarrow C^\infty(M^A, A) \otimes_A C^\infty(M^A, A), \varphi \mapsto \varphi \otimes 1_{C^\infty(M^A, A)}.$$

In this case, we say that $C^\infty(M^A, A) \otimes_A C^\infty(M^A, A)$ admits a structure of $C^\infty(M^A, A)$ -module defined by the first factor. The second factor is defined by

$$C^\infty(M^A, A) \rightarrow C^\infty(M^A, A) \otimes_A C^\infty(M^A, A), \varphi \mapsto 1_{C^\infty(M^A, A)} \otimes \varphi.$$

The map

$$C^\infty(M^A, A) \times C^\infty(M^A, A) \rightarrow C^\infty(M^A, A), (\varphi, \psi) \mapsto \varphi \cdot \psi$$

being A -bilinear, then there exists a unique A -linear map

$$m: C^\infty(M^A, A) \otimes_A C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

such that

$$m(\varphi \otimes \psi) = \varphi \cdot \psi.$$

The kernel of m is the $C^\infty(M^A, A)$ -submodule of $C^\infty(M^A, A) \otimes C^\infty(M^A, A)$ generated by the elements of the form $\varphi \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \varphi$ with $\varphi \in C^\infty(M^A, A)$.

We denote $\Omega_A[C^\infty(M^A, A)]$, the $C^\infty(M^A, A)$ -module of Kähler differentials of $C^\infty(M^A, A)$ which are A -linears. In this case, for $\varphi \in C^\infty(M^A, A)$, we denote $\overline{\varphi \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \varphi}$, the class of $\varphi \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \varphi$ in $C^\infty(M^A, A)$.

The map

$$C^\infty(M) \rightarrow \Omega_{\mathbb{A}}[C^\infty(M^A, A)], f \mapsto \overline{f^A \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes f^A}$$

is a derivation.

Thus,

Proposition 3. *There exists a unique A -linear derivation*

$$\delta_{M^A}^A: C^\infty(M^A, A) \rightarrow \Omega_A[C^\infty(M^A, A)]$$

such that

$$\delta_{M^A}^A(f^A) = [\delta_M(f)]^A$$

for any $f \in C^\infty(M)$.

Proof. Let

$$\begin{aligned} \delta_{M^A}^A: C^\infty(M^A, A) &\xrightarrow{\sigma^{-1}} A \otimes C^\infty(M^A) \xrightarrow{\text{id}_A \otimes d_{M^A}} A \\ &\otimes \Omega_{\mathbb{R}}[C^\infty(M^A)] \xrightarrow{\varpi} \Omega_A[C^\infty(M^A, A)] \end{aligned}$$

be that map, where

$$\sigma^{-1}: \varphi = \sum_{\alpha=1}^{\dim A} (a_\alpha^* \circ \varphi) \cdot a_\alpha \mapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a_\alpha^* \circ \varphi)$$

with $(a_\alpha)_{\alpha=1,\dots,\dim A}$ a basis of A and $(a_\alpha^*)_{\alpha=1,\dots,\dim A}$ the dual basis of the basis $(a_\alpha)_{\alpha=1,\dots,\dim A}$,

$$\begin{aligned} \text{id}_A \otimes d_{M^A} : \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a_\alpha^* \circ \varphi) &\mapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes d_{M^A}(a_\alpha^* \circ \varphi) \\ &= \sum_{\alpha=1}^{\dim A} a_\alpha \otimes \left[\overline{(a_\alpha^* \circ \varphi) \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes (a_\alpha^* \circ \varphi)} \right], \end{aligned}$$

$$\begin{aligned} \varpi : \sum_{\alpha=1}^{\dim A} a_\alpha \otimes d_{M^A}(a_\alpha^* \circ \varphi) &\mapsto \sum_{\alpha=1}^{\dim A} \left[\overline{(a_\alpha^* \circ \varphi)a_\alpha \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes (a_\alpha^* \circ \varphi)a_\alpha} \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \delta_{M^A}^A(\varphi) &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi) \\ &= \sum_{\alpha=1}^{\dim A} \left[\overline{(a_\alpha^* \circ \varphi)a_\alpha \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes (a_\alpha^* \circ \varphi)a_\alpha} \right]. \end{aligned}$$

For any $\varphi, \psi \in C^\infty(M^A, A)$, we have

$$\begin{aligned} \delta_{M^A}^A(\varphi + \psi) &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi + \psi) \\ &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\varphi) + \sigma^{-1}(\psi)) \\ &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\varphi)) + [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\psi)) \\ &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi) + [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\psi) \\ &= \delta_{M^A}^A(\varphi) + \delta_{M^A}^A(\psi). \end{aligned}$$

For any $\varphi \in C^\infty(M^A, A)$ and $a \in A$, we have

$$\begin{aligned} \delta_{M^A}^A(a \cdot \varphi) &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](a \cdot \varphi) \\ &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(a \cdot \varphi)) \\ &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](a \cdot \sigma^{-1}(\varphi)) \\ &= a \cdot [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\varphi)) \\ &= a \cdot \delta_{M^A}^A(\varphi). \end{aligned}$$

For any $\varphi, \psi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}
 \delta_{M^A}^A(\varphi \cdot \psi) &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi \cdot \psi) \\
 &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\varphi \cdot \psi)) \\
 &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\varphi) \cdot \sigma^{-1}(\psi)) \\
 &= [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\varphi)) \cdot \psi + \varphi \cdot [\varpi \circ (\text{id}_A \otimes d_{M^A})](\sigma^{-1}(\psi)) \\
 &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi) \cdot \psi + \varphi \cdot [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\psi) \\
 &= [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi) + [\varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}](\psi) \\
 &= \delta_{M^A}^A(\varphi) \cdot \psi + \varphi \cdot \delta_{M^A}^A(\psi).
 \end{aligned}$$

As

$$\delta_M: C^\infty(M) \rightarrow \Omega_{\mathbb{R}}[C^\infty(M)]$$

is a derivation, then the map

$$C^\infty(M) \rightarrow \Omega_A[C^\infty(M^A, A)], \quad f \mapsto [\delta_M(f)]^A$$

is a derivation. Thus, for any $f \in C^\infty(M)$

$$\begin{aligned}
 \delta_{M^A}^A(f^A) &= \varpi \circ (\text{id}_A \otimes d_{M^A}) \circ \sigma^{-1}(f^A) \\
 &= \overline{\sum_{\alpha=1}^{\dim A} [(a_\alpha^* \circ f^A)a_\alpha \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes (a_\alpha^* \circ f^A)a_\alpha]} \\
 &= \overline{\sum_{\alpha=1}^{\dim A} (a_\alpha^* \circ f^A)a_\alpha \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \sum_{\alpha=1}^{\dim A} (a_\alpha^* \circ f^A)a_\alpha} \\
 &= \overline{f^A \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes f^A} \\
 &= \overline{[f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f]^A}
 \end{aligned}$$

i.e.

$$\delta_{M^A}^A(f^A) = [\delta_M(f)]^A.$$

□

Proposition 4. *The map*

$$\Omega_{\mathbb{R}}[C^\infty(M)] \rightarrow \Omega_A[C^\infty(M^A, A)], \quad x \mapsto x^A$$

is an injective homomorphism of \mathbb{R} -modules.

Proof. Let

$$\Psi: \Omega_{\mathbb{R}}[C^\infty(M)] \rightarrow \Omega_A[C^\infty(M^A, A)], \quad x \mapsto x^A$$

be that map.

□

For any $x, y \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$\begin{aligned} \Psi(x + y) &= (x + y)^A \\ &= \left(\sum_{i \in I: \text{finite}} f_i \cdot \delta_M(f'_i) + \sum_{j \in I: \text{finite}} g_j \cdot \delta_M(g'_j) \right)^A \\ &= \left(\sum_{i \in I: \text{finite}} f_i \cdot \delta_M(f'_i) \right)^A + \left(\sum_{j \in I: \text{finite}} g_j \cdot \delta_M(g'_j) \right)^A \\ &= x^A + y^A. \end{aligned}$$

For any $x \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$ and for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \Psi(\lambda \cdot x) &= (\lambda \cdot x)^A \\ &= \left(\lambda \cdot \sum_{i \in I: \text{finite}} f_i \cdot \delta_M(f'_i) \right)^A \\ &= \lambda \cdot \left(\sum_{i \in I: \text{finite}} f_i \cdot \delta_M(f'_i) \right)^A \\ &= \lambda \cdot x^A. \end{aligned}$$

The pair $(\Omega_A[C^{\infty}(M^A, A)], \delta_{M^A}^A)$ satisfies the following universal property: for every $C^{\infty}(M^A, A)$ -module E and every A -derivation

$$\Phi: C^{\infty}(M^A, A) \rightarrow E,$$

there exists a unique $C^{\infty}(M^A, A)$ -linear map

$$\tilde{\Phi}: \Omega_A[C^{\infty}(M^A, A)] \rightarrow E$$

such that

$$\tilde{\Phi} \circ \delta_{M^A}^A = \Phi.$$

In other words, there exists a unique $\tilde{\Phi}$ which makes the following diagram commutative

$$\begin{array}{ccc} \Omega_A[C^{\infty}(M^A, A)] & & \\ \delta_{M^A}^A \uparrow & \searrow \tilde{\Phi} & \\ C^{\infty}(M^A, A) & \xrightarrow{\Phi} & E. \end{array}$$

This fact implies the existence of a natural isomorphism of $C^{\infty}(M^A, A)$ -modules

$$\text{Hom}_{C^{\infty}(M^A, A)}(\Omega_A[C^{\infty}(M^A, A)], E) \rightarrow \text{Der}_A[C^{\infty}(M^A, A), E], \quad \psi \mapsto \psi \circ \delta_{M^A}^A.$$

In particular, if $E = C^{\infty}(M^A, A)$, we have

$$\begin{aligned} \Omega_A[C^{\infty}(M^A, A)]^* &\simeq \text{Der}_A[C^{\infty}(M^A, A)] \\ &= \mathfrak{X}(M^A). \end{aligned}$$

For any $p \in \mathbb{N}$, $\Lambda^p(\Omega_A[C^\infty(M^A, A)]) = \mathfrak{L}_{sks}^p(\Omega_A[C^\infty(M^A, A)], C^\infty(M^A, A))$ denotes the $C^\infty(M^A, A)$ -module of skew-symmetric multilinear forms of degree p from $\Omega_A[C^\infty(M^A, A)]$ into $C^\infty(M^A, A)$ and

$$\Lambda(\Omega_A[C^\infty(M^A, A)]) = \bigoplus_{p \in \mathbb{N}} \Lambda^p(\Omega_A[C^\infty(M^A, A)])$$

the exterior $C^\infty(M^A, A)$ -algebra of $\Omega_A[C^\infty(M^A, A)]$.

$$\Lambda^0(\Omega_A[C^\infty(M^A, A)]) = C^\infty(M^A, A),$$

$$\Lambda^1(\Omega_A[C^\infty(M^A, A)]) = \Omega_A[C^\infty(M^A, A)]^*.$$

We denote,

$$\delta_{M^A}^A = \delta_{M^A} : \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)])$$

a unique derivation, of degree $+1$, which extends the canonical derivation

$$\delta_{M^A}^0 : C^\infty(M^A, A) \rightarrow \Omega_A[C^\infty(M^A, A)].$$

For any $\varphi, \psi, \psi_1, \psi_2, \dots, \psi_p \in C^\infty(M^A, A)$ and $\omega \in \Omega_A[C^\infty(M^A, A)]^*$, we get

1. $\delta_{M^A}(\varphi \cdot \delta_{M^A}(\psi_1) \wedge \dots \wedge \delta_{M^A}(\psi_p)) = \delta_{M^A}(\varphi) \wedge \delta_{M^A}(\psi_1) \wedge \dots \wedge \delta_{M^A}(\psi_p)$.
2. $\delta_{M^A}^1[\psi \cdot \delta_{M^A}^0(\varphi)] = \delta_{M^A}^0(\psi) \wedge \delta_{M^A}^0(\varphi)$.
3. $\delta_{M^A}^1(\varphi \cdot \omega) = \delta_{M^A}^0(\varphi) \wedge \omega + \varphi \cdot \delta_{M^A}^1(\omega)$.

Proposition 5. *If $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, then $\eta^A \in \Lambda^p(\Omega_A[C^\infty(M^A, A)])$.*

Proof. Indeed, for any $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, η is of the form $\delta_M(f_1) \wedge \dots \wedge \delta_M(f_p)$ with $f_1, f_2, \dots, f_p \in C^\infty(M)$.

$$\begin{aligned} \eta^A &= [\delta_M(f_1) \wedge \dots \wedge \delta_M(f_p)]^A \\ &= [\delta_M(f_1)]^A \wedge \dots \wedge [\delta_M(f_p)]^A \\ &= \delta_{M^A}^0(f_1^A) \wedge \dots \wedge \delta_{M^A}^0(f_p^A). \end{aligned}$$

Thus, the $C^\infty(M^A, A)$ -module $\Lambda^p(\Omega_A[C^\infty(M^A, A)])$ is generated by elements of the form

$$\eta^A = \delta_{M^A}^0(\varphi_1) \wedge \dots \wedge \delta_{M^A}^0(\varphi_p)$$

with $\varphi_1 = f_1^A, \dots, \varphi_p = f_p^A \in C^\infty(M^A, A)$. □

The algebra

$$\Lambda(\Omega_A[C^\infty(M^A, A)]) = \bigoplus_{p \in \mathbb{N}} \Lambda^p(\Omega_A[C^\infty(M^A, A)])$$

is the algebra of Kähler forms on $C^\infty(M^A, A)$.

The pair $(\Lambda(\Omega_A[C^\infty(M^A, A)]), \delta_{M^A}^A)$ is a differential complex and the map

$$A \times \Omega_{\mathbb{R}}[C^\infty(M)] \mapsto \Omega_A[C^\infty(M^A, A)], (a, x) \mapsto a \cdot x^A$$

induces the morphism of the differential complex $(A \otimes \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)]), \text{id}_A \otimes \delta_M)$ into the differential complex $(\Lambda(\Omega_A[C^\infty(M^A, A)]), \delta_{M^A}^A)$.

3. LIE DERIVATIVE WITH RESPECT TO A DERIVATION ON M^A

Let

$$\theta: C^\infty(M) \rightarrow C^\infty(M)$$

be a derivation and

$$\sigma_\theta: [\Omega_{\mathbb{R}}[C^\infty(M)]]^p \longrightarrow \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)]),$$

be the $C^\infty(M)$ -skew-symmetric multilinear map such that for any $x_1, x_2, \dots, x_p \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$\sigma_\theta(x_1, x_2, \dots, x_p) = \sum_{i=1}^p (-1)^{i-1} \tilde{\theta}(x_i) \cdot x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p,$$

where

$$\tilde{\theta}: \Omega_{\mathbb{R}}[C^\infty(M)] \rightarrow C^\infty(M)$$

is a unique $C^\infty(M)$ -linear map such that $\tilde{\theta} \circ \delta_M = \theta$. Then,

$$\sigma_{\theta^A}^A: [\Omega_A[C^\infty(M^A, A)]]^p \rightarrow \Lambda^p(\Omega_A[C^\infty(M^A, A)])$$

is a unique $C^\infty(M^A, A)$ -skew-symmetric multilinear map such that

$$\sigma_{\theta^A}^A(x_1^A, x_2^A, \dots, x_p^A) = [\sigma_\theta(x_1, x_2, \dots, x_p)]^A.$$

We denote

$$\widetilde{\sigma_{\theta^A}^A}: \Lambda^p(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda^{p-1}(\Omega_A[C^\infty(M^A, A)]),$$

the unique $C^\infty(M^A, A)$ -skew-symmetric multilinear map such that

$$\widetilde{\sigma_{\theta^A}^A}(x_1^A \wedge x_2^A \wedge \cdots \wedge x_p^A) = \sigma_{\theta^A}^A(x_1^A, x_2^A, \dots, x_p^A)$$

i.e. $\sigma_{\theta^A}^A$ induces a derivation

$$i_{\theta^A} = \widetilde{\sigma_{\theta^A}^A}: \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)])$$

of degree -1 .

Proposition 6. *For any $\theta \in \text{Der}_{\mathbb{R}}[C^\infty(M)]$ and for any $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, we have*

$$i_{\theta^A}(\eta^A) = [i_\theta(\eta)]^A.$$

Proof. If $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)])$, then there exists $f_1, f_2, \dots, f_p \in C^\infty(M)$, such that $\eta = \delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)$. Thus,

$$\begin{aligned} i_{\theta^A}(\eta^A) &= i_{\theta^A}([\delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)]^A) \\ &= i_{\theta^A}([\delta_M(f_1)]^A \wedge \cdots \wedge [\delta_M(f_p)]^A) \\ &= \sigma_{\theta^A}^A([\delta_M(f_1)]^A, \dots, [\delta_M(f_p)]^A) \end{aligned}$$

$$\begin{aligned}
 &= [\sigma_\theta(\delta_M(f_1), \dots, \delta_M(f_p))]^A \\
 &= [i_\theta(\delta_M(f_1) \wedge \dots \wedge \delta_M(f_p))]^A \\
 &= [i_\theta(\eta)]^A.
 \end{aligned}$$

For $p = 1$, we have

$$\begin{aligned}
 i_{\theta^A} &= \widetilde{\sigma_{\theta^A}} : \Lambda^1(\Omega_A[C^\infty(M^A, A)]) \\
 &= \Omega_A[C^\infty(M^A, A)]^* \rightarrow \Lambda^0(\Omega_A[C^\infty(M^A, A)]) = C^\infty(M^A, A),
 \end{aligned}$$

and for any $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$i_{\theta^A}(y^A) = \widetilde{\theta^A}(y^A).$$

For $p = 2$, we have

$$\sigma_{\theta^A}^A : \Omega_A[C^\infty(M^A, A)] \times \Omega_A[C^\infty(M^A, A)] \rightarrow C^\infty(M^A, A)$$

and for any $x, y \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$\sigma_{\theta^A}^A(x^A, y^A) = \widetilde{\theta^A}(x^A) \cdot y^A - \widetilde{\theta^A}(y^A) \cdot x^A.$$

Thus, the map

$$i_{\theta^A} : \Lambda^2(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Omega_A[C^\infty(M^A, A)]^*$$

is the unique $C^\infty(M^A, A)$ -linear map such that

$$i_{\theta^A}(x^A \wedge y^A) = \sigma_{\theta^A}^A(x^A, y^A) = \widetilde{\theta^A}(x^A) \cdot y^A - \widetilde{\theta^A}(y^A) \cdot x^A.$$

□

Definition 2. The Lie derivative with respect to $D \in \text{Der}_A[C^\infty(M^A, A)]$ is the derivation of degree 0

$$\mathfrak{L}_D = i_D \circ \delta_{M^A}^A + \delta_{M^A}^A \circ i_D : \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)]).$$

Proposition 7. For any $\theta \in \mathfrak{X}(M)$, the map

$$\mathfrak{L}_{\theta^A} : \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)])$$

is a unique A -linear derivation such that

$$\mathfrak{L}_{\theta^A}(\eta^A) = [\mathfrak{L}_\theta(\eta)]^A,$$

for any $\eta \in \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$.

Proof. For any $\eta \in \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$, we have

$$\begin{aligned}
\mathfrak{L}_{\theta^A}(\eta^A) &= i_{\theta^A} [\delta_{M^A}^A(\eta^A)] + \delta_{M^A}^A [i_{\theta^A}(\eta^A)] \\
&= i_{\theta^A} ([\delta_M(\eta)]^A) + \delta_{M^A}^A ([i_\theta(\eta)]^A) \\
&= (i_\theta[\delta_M(\eta)])^A + (\delta_M[i_\theta(\eta)])^A \\
&= (i_\theta[\delta_M(\eta)] + \delta_M[i_\theta(\eta)])^A \\
&= [\mathfrak{L}_\theta(\eta)]^A.
\end{aligned}$$

□

Proposition 8. For any $\theta \in \mathfrak{X}(M)$, for any $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f \in C^\infty(M)$, we have

- (1) $\mathfrak{L}_{f^A \cdot \theta^A}(x^A) = [\mathfrak{L}_{f \cdot \theta}(x)]^A.$
- (2) $\mathfrak{L}_{\theta^A}(f^A \cdot x^A) = [\mathfrak{L}_\theta(f \cdot x)]^A.$
- (3) $\mathfrak{L}_{\theta^A}[\delta_{M^A}^A(f^A)] = [\mathfrak{L}_\theta(\delta_M(f))]^A.$

Proof. For any $\theta \in \mathfrak{X}(M)$, for any $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f \in C^\infty(M)$, we have

1.
$$\begin{aligned}
\mathfrak{L}_{f^A \cdot \theta^A}(x^A) &= i_{f^A \cdot \theta^A} [\delta_{M^A}^A(x^A)] + \delta_{M^A}^A [i_{f^A \cdot \theta^A}(x^A)] \\
&= f^A \cdot i_{\theta^A} [\delta_{M^A}^A(x^A)] + \delta_{M^A}^A [f^A \cdot i_{\theta^A}(x^A)] \\
&= f^A \cdot i_{\theta^A} ([\delta_M(x)]^A) + \delta_{M^A}^A (f^A \cdot [i_\theta(x)]^A) \\
&= f^A \cdot i_{\theta^A} ([\delta_M(x)]^A) + i_{\theta^A}(x^A) \cdot \delta_{M^A}^A(f^A) + f^A \cdot \delta_{M^A}^A [i_{\theta^A}(x^A)] \\
&= f^A \cdot (i_\theta[\delta_M(x)]^A) + [i_\theta(x)]^A \cdot [\delta_M(f)]^A + f^A \cdot (\delta_M[i_\theta(x)]^A) \\
&= (f \cdot i_\theta[\delta_M(x)] + i_\theta(x) \cdot \delta_M(f) + f \cdot \delta_M[i_\theta(x)])^A \\
&= (f \cdot i_\theta[\delta_M(x)] + \delta_M(f \cdot [i_\theta(x)]))^A \\
&= (i_{f \cdot \theta}[\delta_M(x)] + \delta_M[i_{f \cdot \theta}(x)])^A \\
&= [\mathfrak{L}_{f \cdot \theta}(x)]^A.
\end{aligned}$$

Thus,

$$\mathfrak{L}_{f^A \cdot \theta^A}(x^A) = [\mathfrak{L}_{f \cdot \theta}(x)]^A.$$

2.
$$\begin{aligned}
\mathfrak{L}_{\theta^A}(f^A \cdot x^A) &= i_{\theta^A} [\delta_{M^A}^A(f^A \cdot x^A)] + \delta_{M^A}^A [i_{\theta^A}(f^A \cdot x^A)] \\
&= i_{\theta^A} [\delta_{M^A}^A(f^A) \Lambda x^A + f^A \cdot \delta_{M^A}^A(x^A)] + \delta_{M^A}^A [f^A \cdot i_{\theta^A}(x^A)] \\
&= \theta^A(f^A) \cdot x^A - \delta_{M^A}^A(f^A) \cdot \widetilde{\theta^A}(x^A) + f^A \cdot \theta^A(x^A) \\
&\quad + \delta_{M^A}^A(f^A) \cdot \widetilde{\theta^A}(x^A) + f^A \cdot \delta_{M^A}^A [i_{\theta^A}(x^A)]
\end{aligned}$$

$$\begin{aligned}
 &= (\theta(f) \cdot x - \delta_M(f) \cdot \tilde{\theta}(x) + f \cdot \theta(x) \\
 &\quad + \delta_M(f) \cdot \tilde{\theta}(x) + f \cdot \delta_M[i_\theta(x)])^A \\
 &= (i_\theta[\delta_M(f)\Lambda x + f \cdot \delta_M(x)] + \delta_M[f \cdot i_\theta(x)])^A \\
 &= (i_\theta[\delta_M(f \cdot x)] + \delta_M[i_\theta(f \cdot x)])^A \\
 &= [\mathfrak{L}_\theta(f \cdot x)]^A.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \mathfrak{L}_{\theta^A} [\delta_{M^A}^A(f^A)] &= i_{\theta^A} [\delta_{M^A}^A(\delta_{M^A}^A(f^A))] + \delta_{M^A}^A [i_{\theta^A}(\delta_{M^A}^A(f^A))] \\
 &= 0 + \delta_{M^A}^A [\theta^A(f^A)] \\
 &= \delta_{M^A}^A([\theta(f)]^A) \\
 &= (\delta_M[\theta(f)])^A \\
 &= (0 + \delta_M[\theta(f)])^A \\
 &= (i_\theta[\delta_M(\delta_M(f))] + \delta_M[i_\theta(\delta_M(f))])^A \\
 &= [\mathfrak{L}_\theta(\delta_M(f))]^A.
 \end{aligned}$$

□

Proposition 9. For any $D \in \text{Der}_A[C^\infty(M^A, A)]$, $X \in \Omega_A[C^\infty(M^A, A)]$, and $\varphi \in C^\infty(M^A, A)$, we have

- (1) $\mathfrak{L}_{\varphi \cdot D}(X) = \varphi \cdot \mathfrak{L}_D(X) + \tilde{D}(X) \cdot \delta_{M^A}^A(\varphi)$;
- (2) $\mathfrak{L}_D(\varphi \cdot X) = D(\varphi) \cdot X + \varphi \cdot \mathfrak{L}_D(X)$;
- (3) $\mathfrak{L}_D[\delta_{M^A}^A(\varphi)] = \delta_{M^A}^A[D(\varphi)]$.

Proof. For any $D \in \text{Der}_A[C^\infty(M^A, A)]$, $X \in \Omega_A[C^\infty(M^A, A)]$, and $\varphi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}
 1. \quad \mathfrak{L}_{\varphi \cdot D}(X) &= i_{\varphi \cdot D}[\delta_{M^A}^A(X)] + \delta_{M^A}^A[i_{\varphi \cdot D}(X)] \\
 &= \varphi \cdot i_D[\delta_{M^A}^A(X)] + \delta_{M^A}^A[\varphi \cdot i_D(X)] \\
 &= \varphi \cdot i_D[\delta_{M^A}^A(X)] + i_D(X) \cdot \delta_{M^A}^A(\varphi) + \varphi \cdot \delta_{M^A}^A[i_D(X)] \\
 &= \varphi \cdot \mathfrak{L}_D(X) + \tilde{D}(X) \cdot \delta_{M^A}^A(\varphi).
 \end{aligned}$$

2.
$$\begin{aligned} \mathfrak{L}_D(\varphi \cdot X) &= i_D[\delta_{M^A}^A(\varphi \cdot X)] + \delta_{M^A}^A[i_D(\varphi \cdot X)] \\ &= i_D[\delta_{M^A}^A(\varphi)\Lambda X + \varphi \cdot \delta_{M^A}^A(X)] + \delta_{M^A}^A[\varphi \cdot i_D(X)] \\ &= \tilde{D}[\delta_{M^A}^A(\varphi)] \cdot X - \delta_{M^A}^A(\varphi) \cdot \tilde{D}(X) + \varphi \cdot i_D[\delta_{M^A}^A(X)] \\ &\quad + \delta_{M^A}^A(\varphi) \cdot \tilde{D}(X) + \varphi \cdot \delta_{M^A}^A[i_D(X)] \\ &= D(\varphi) \cdot X + \varphi \cdot i_D[\delta_{M^A}^A(X)] + \varphi \cdot \delta_{M^A}^A[i_D(X)] \\ &= D(\varphi) \cdot X + \varphi \cdot (i_D[\delta_{M^A}^A(X)] + \delta_{M^A}^A[i_D(X)]) \\ &= D(\varphi) \cdot X + \varphi \cdot \mathfrak{L}_D(X). \end{aligned}$$
3.
$$\begin{aligned} \mathfrak{L}_D[\delta_{M^A}^A(\varphi)] &= i_D[\delta_{M^A}^A(\delta_{M^A}^A(\varphi))] + \delta_{M^A}^A[i_D(\delta_{M^A}^A(\varphi))] \\ &= 0 + \delta_{M^A}^A[\tilde{D} \circ \delta_{M^A}^A(\varphi)] \\ &= \delta_{M^A}^A[D(\varphi)]. \end{aligned}$$

□

4. THE POISSON 2-FORM ON WEIL BUNDLES

We recall that, when M is a smooth manifold, A a Weil algebra and M^A the associated Weil bundle, the A -algebra $C^\infty(M^A, A)$ is a Poisson algebra over A if there exists a bracket $\{\cdot, \cdot\}$ on $C^\infty(M^A, A)$ such that the pair $(C^\infty(M^A, A), \{\cdot, \cdot\})$ is a Lie algebra over A satisfying

$$\{\varphi, \psi_1 \cdot \psi_2\} = \{\varphi, \psi_1\} \cdot \psi_2 + \psi_1 \cdot \{\varphi, \psi_2\}$$

for any $\varphi, \psi_1, \psi_2 \in C^\infty(M^A, A)$. When $C^\infty(M^A, A)$ is a Poisson A -algebra, we will say that the manifold M^A is a A -Poisson manifold [1],[7].

When $(M, \{\cdot, \cdot\})$ is a Poisson manifold, the map

$$\text{ad}: C^\infty(M) \rightarrow \text{Der}_{\mathbb{R}}[C^\infty(M)], \quad f \rightarrow \text{ad}(f)$$

such that $[\text{ad}(f)](g) = \{f, g\}$ for any $g \in C^\infty(M)$, is a derivation. Thus:

Proposition 10. *There exists a derivation*

$$\text{ad}^A: C^\infty(M^A, A) \rightarrow \text{Der}_A[C^\infty(M^A, A)]$$

such that

$$\text{ad}^A(f^A) = [\text{ad}(f)]^A.$$

Let's consider the following diagram commutative:

$$\begin{array}{ccc} C^\infty(M^A, A) & \xrightarrow{\tilde{\tau}} & \text{Der}_A[C^\infty(M^A, A)] \\ \gamma_M \uparrow & & \uparrow \Phi \\ C^\infty(M) & \xrightarrow{\text{ad}} & \text{Der}_{\mathbb{R}}[C^\infty(M)] \end{array}$$

i.e.

$$\tilde{\tau} \circ \gamma_M = \Phi \circ \text{ad},$$

where

$$\gamma_M : C^\infty(M) \rightarrow C^\infty(M^A, A), \quad f \mapsto f^A$$

and

$$\Phi : \text{Der}_{\mathbb{R}}[C^\infty(M)] \rightarrow \text{Der}_A[C^\infty(M^A, A)], \quad \theta \rightarrow \theta^A.$$

For any $f \in C^\infty(M)$, we have

$$\tilde{\tau} \circ \gamma_M(f) = \tilde{\tau}(f^A)$$

and

$$\Phi \circ \text{ad}(f) = \Phi[\text{ad}(f)] = [\text{ad}(f)]^A.$$

Thus, there exists $\text{ad}^A = \tilde{\tau}$ such that

$$\text{ad}^A(f^A) = [\text{ad}(f)]^A.$$

As

$$\text{ad}^A : C^\infty(M^A, A) \rightarrow \text{Der}_A[C^\infty(M^A, A)]$$

is a derivation, then there exists a unique $C^\infty(M^A, A)$ -linear map

$$\widetilde{\text{ad}}^A : \Omega_A[C^\infty(M^A, A)] \rightarrow \text{Der}_A[C^\infty(M^A, A)]$$

such that

$$\widetilde{\text{ad}}^A \circ \delta_{M^A}^A = \text{ad}^A.$$

Let's consider the canonical isomorphism

$$\sigma_{M^A} : \Omega_A[C^\infty(M^A, A)]^* \rightarrow \text{Der}_A[C^\infty(M^A, A)], \quad \Psi \mapsto \Psi \circ \delta_{M^A}^A$$

and let

$$\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A : \Omega_A[C^\infty(M^A, A)] \xrightarrow{\widetilde{\text{ad}}^A} \text{Der}_A[C^\infty(M^A, A)] \xrightarrow{\sigma_{M^A}^{-1}} \Omega_A[C^\infty(M^A, A)]^*$$

be the map.

Proposition 11. *If (M, ω_M) is a Poisson manifold, then the map,*

$$\omega_{M^A}^A : \Omega_A[C^\infty(M^A, A)] \times \Omega_A[C^\infty(M^A, A)] \rightarrow C^\infty(M^A, A)$$

such that for any $X, Y \in \Omega_A[C^\infty(M^A, A)]$

$$\omega_{M^A}^A(X, Y) = -[\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(X)](Y)$$

is a skew-symmetric 2-form on $\Omega_A[C^\infty(M^A, A)]$ such that

$$\omega_{M^A}^A(x^A, y^A) = [\omega_M(x, y)]^A,$$

for any x and y in $\Omega_{\mathbb{R}}[C^\infty(M)]$.

Proof. For any $X \in \Omega_A[C^\infty(M^A, A)]$, we have $X = \sum_{i \in I: \text{fni}} \varphi_i \cdot \delta_{M^A}^A(\psi_i)$, with $\varphi_i \in C^\infty(M^A, A)$, $\psi_i \in C^\infty(M^A, A)$.

$$\begin{aligned}
\omega_{M^A}(X, X) &= -[\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(X)](X) \\
&= - \sum_{j \in I: \text{finite}} \varphi_j \cdot [\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(X)] \delta_{M^A}^A(\psi_j) \\
&= - \sum_{j \in I: \text{finite}} \varphi_j \cdot [\widetilde{\text{ad}}^A(X)](\psi_j) \\
&= - \sum_{j, k \in I: \text{finite}} \varphi_j \cdot \varphi_k \cdot [\widetilde{\text{ad}}^A(\delta_{M^A}^A(\psi_k))](\psi_j) \\
&= - \sum_{j, k \in I: \text{finite}} \varphi_j \cdot \varphi_k \cdot [\text{ad}^A(\psi_k)](\psi_j) \\
&= - \sum_{j, k \in I: \text{finite}} \varphi_j \cdot \varphi_k \cdot \{\psi_k, \psi_j\} \\
&= 0.
\end{aligned}$$

For any X_1, X_2 and $Y \in \Omega_A[C^\infty(M^A, A)]$ and for any $\varphi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}
\omega_{M^A}[(\varphi \cdot X_1 + X_2), Y] &= -[\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(\varphi \cdot X_1 + X_2)](Y) \\
&= -(\sigma_{M^A}^{-1}[\varphi \cdot \widetilde{\text{ad}}^A(X_1) + \widetilde{\text{ad}}^A(X_2)])(Y) \\
&= -\varphi \cdot (\sigma_{M^A}^{-1}[\widetilde{\text{ad}}^A(X_1)](Y) + (\sigma_{M^A}^{-1}[\widetilde{\text{ad}}^A(X_2)])(Y)) \\
&= \varphi \cdot \omega_{M^A}(X_1, Y) + \omega_{M^A}(X_2, Y).
\end{aligned}$$

For any x and y in $\Omega_{\mathbb{R}}[C^\infty(M)]$,

$$x^A = \sum_{i \in I: \text{fni}} f_i^A \cdot \delta_{M^A}^A(f_i^A) \quad \text{and} \quad y^A = \sum_{j \in I: \text{fni}} g_j^A \cdot \delta_{M^A}^A(g_j^A).$$

Thus,

$$\begin{aligned}
\omega_{M^A}^A(x^A, y^A) &= -[\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(x^A)](y^A) \\
&= -\left[\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A\left(\sum_{i \in I: \text{finite}} f_i^A \cdot \delta_{M^A}^A(f_i^A)\right)\right]\left(\sum_{j \in I: \text{finite}} g_j^A \cdot \delta_{M^A}^A(g_j^A)\right) \\
&= - \sum_{i, j \in I: \text{finite}} f_i^A \cdot g_j^A \cdot [\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(\delta_{M^A}^A(f_i^A))](\delta_{M^A}^A(g_j^A))
\end{aligned}$$

$$\begin{aligned}
 &= - \sum_{i,j \in I: \text{ finite}} f_i^A \cdot g_j^A \cdot [\sigma_{M^A}^{-1} \circ \widetilde{\text{ad}}^A(\delta_{M^A}^A(f_i^A))](\delta_{M^A}^A(g_j^A)) \\
 &= - \sum_{i,j \in I: \text{ finite}} f_i^A \cdot g_j^A \cdot [\sigma_{M^A}^{-1}(\widetilde{\text{ad}}^A \circ \delta_{M^A}^A(f_i^A))](\delta_{M^A}^A(g_j^A)) \\
 &= - \sum_{i,j \in I: \text{ finite}} f_i^A \cdot g_j^A \cdot [\sigma_{M^A}^{-1}(\text{ad}^A(f_i^A))](\delta_{M^A}^A(g_j^A)) \\
 &= - \sum_{i,j \in I: \text{ finite}} f_i^A \cdot g_j^A \cdot [\sigma_{M^A}^{-1}(\text{ad}(f_i^A)^A)](\delta_M(g_j^A))^A \\
 &= - \left[\sum_{i,j \in I: \text{ finite}} f_i \cdot g_j \cdot [\sigma_M^{-1}(\text{ad}(f_i))](\delta_M(g_j)) \right]^A \\
 &= - \left[\sum_{i,j \in I: \text{ finite}} f_i \cdot g_j \cdot [\sigma_M^{-1}(\widetilde{\text{ad}} \circ \delta_M(f_i))](\delta_M(g_j)) \right]^A \\
 &= - \left[\sigma_M^{-1} \circ \widetilde{\text{ad}} \left(\sum_{i \in I: \text{ finite}} f_i \cdot \delta_M(f_i) \right) \left(\sum_{j \in I: \text{ finite}} g_j \cdot \delta_M(g_j) \right) \right]^A \\
 &= [\omega_M(x, y)]^A.
 \end{aligned}$$

□

Proposition 12. *When (M, ω_M) is a Poisson manifold of Poisson 2-form ω_M , then $(M^A, \omega_{M^A}^A)$ is a Poisson manifold.*

Proof. For any f and g in $C^\infty(M)$,

$$\begin{aligned}
 \omega_{M^A}^A(\delta_{M^A}^A(f^A), \delta_{M^A}^A(g^A)) &= \omega_{M^A}^A([\delta_M(f)]^A, [\delta_M(g)]^A) \\
 &= [\omega_M(\delta_{M^A}(f), \delta_M(g))]^A \\
 &= -\{f, g\}_M^A,
 \end{aligned}$$

and

$$\omega_{M^A}^A(x^A, y^A) = [\omega_M(x, y)]^A,$$

for any $x, y \in \Omega_{\mathbb{R}}[C^\infty(M)]$. We deduce that $(M^A, \omega_{M^A}^A)$ is a Poisson manifold. □

Theorem 3. *The manifold M^A is a Poisson manifold if and only if there exists a skew-symmetric 2-form*

$$\omega_{M^A}^A: \Omega_A[C^\infty(M^A, A)] \times \Omega_A[C^\infty(M^A, A)] \rightarrow C^\infty(M^A, A)$$

such that for any φ and ψ in $C^\infty(M^A, A)$,

$$\{\varphi, \psi\}_{M^A} = -\omega_{M^A}^A(\delta_{M^A}^A(\varphi), \delta_{M^A}^A(\psi))$$

defines a structure of A -Lie algebra over $C^\infty(M^A, A)$. Moreover, for any f and g in $C^\infty(M)$,

$$\{f^A, g^A\}_{M^A} = \{f, g\}_M^A.$$

Proof. Indeed, according to the previous proposition, the bracket

$$\{\varphi, \psi\}_{M^A} = -\omega_{M^A}^A(\delta_{M^A}^A(\varphi), \delta_{M^A}^A(\psi))$$

defines a structure of A -Lie algebra over $C^\infty(M^A, A)$. For any f and g in $C^\infty(M)$,

$$\{f^A, g^A\}_{M^A} = -\omega_{M^A}^A(\delta_{M^A}^A(f^A), \delta_{M^A}^A(g^A)) = \{f, g\}_M^A.$$

In this case, we will say that $\omega_{M^A}^A$ is the Poisson 2-form of the A -Poisson manifold M^A and we denote $(M^A, \omega_{M^A}^A)$ the A -Poisson manifold of Poisson 2-form $\omega_{M^A}^A$. \square

Proposition 13. *When (M, ω_M) is a Poisson manifold of Poisson 2-form ω_M , then for any $x, y \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and for any $f, g \in C^\infty(M)$, we get*

- (1) $[\widetilde{\text{ad}}^A(x^A)](f^A) = ([\widetilde{\text{ad}}(x)](f))^A.$
- (2) $[\widetilde{\widetilde{\text{ad}}^A}(x^A)](y^A) = ([\widetilde{\widetilde{\text{ad}}}(x)](y))^A.$
- (3) $\mathfrak{L}_{\widetilde{\text{ad}}^A[\delta_{M^A}^A(f^A)]}^{\sim}(g^A) = (\mathfrak{L}_{\widetilde{\text{ad}}[\delta_M(f)]}^{\sim}(g))^A.$

Proof. 1. For any $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$, $x^A = g^A \cdot \delta_{M^A}^A(h^A)$ with g and h in $C^\infty(M)$, and for any $f, g \in C^\infty(M)$, we have

$$\begin{aligned} [\widetilde{\text{ad}}^A(x^A)](f^A) &= [\widetilde{\text{ad}}^A(g^A \cdot \delta_{M^A}^A(h^A))](f^A) \\ &= [g^A \cdot \widetilde{\text{ad}}^A \circ \delta_{M^A}^A(h^A)](f^A) \\ &= [g^A \cdot \text{ad}^A(h^A)](f^A) \\ &= g^A \cdot [\text{ad}(h)]^A(f^A) \\ &= (g \cdot [\text{ad}(h)](f))^A \\ &= (g \cdot \widetilde{\text{ad}}[\delta_M(h)](f))^A \\ &= (\widetilde{\text{ad}}[g \cdot \delta_M(h)](f))^A \\ &= (\widetilde{\text{ad}}[g \cdot \delta_M(h)](f))^A \\ &= ([\widetilde{\text{ad}}(x)](f))^A. \end{aligned}$$

2. When $y \in \Omega_{\mathbb{R}}[C^\infty(M)]$, $y^A = g^A \cdot \delta_{M^A}^A(h^A)$ with g and h in $C^\infty(M)$

$$\begin{aligned}
 \widetilde{[\text{ad}^A(x^A)]}(y^A) &= \widetilde{[\text{ad}^A(x^A)]}(g^A \cdot \delta_{M^A}^A(h^A)) \\
 &= g^A \cdot (\widetilde{[\text{ad}^A(x^A)]} \circ \delta_{M^A}^A)(h^A) \\
 &= g^A \cdot \widetilde{[\text{ad}^A(x^A)]}(h^A) \\
 &= g^A \cdot \widetilde{[\text{ad}^A(x^A)]}(h^A) \\
 &= (g \cdot \widetilde{[\text{ad}^A(x^A)]})(h^A) \\
 &= (g \cdot \widetilde{[\text{ad}^A(x^A)]} \circ \delta_M)(h)^A \\
 &= (\widetilde{[\text{ad}^A(x^A)]}(g \cdot \delta_M(h)))^A \\
 &= (\widetilde{[\text{ad}^A(x^A)]}(y))^A.
 \end{aligned}$$

□

Proposition 14. *When (M, ω_M) is a Poisson manifold of Poisson 2-form ω_M , then for any $X, Y \in \Omega_A[C^\infty(M^A, A)]$ and for any $\varphi, \psi \in C^\infty(M^A, A)$, we get*

- (1) $\widetilde{[\text{ad}^A(X)]}(\varphi) = -\omega_{M^A}^A(X, \delta_{M^A}^A(\varphi));$
- (2) $\widetilde{[\text{ad}^A(X)]}(Y) = -\omega_{M^A}^A(X, Y);$
- (3) $\mathfrak{L}_{\widetilde{\text{ad}^A[\delta_{M^A}^A(\varphi)]}} \delta_{M^A}^A(\psi) = \delta_{M^A}^A(\{\varphi, \psi\}_{M^A}).$

Proof. When X and $Y \in \Omega_A[C^\infty(M^A, A)]$, $X = \sum_{i \in I: \text{fni}} \varphi_i \cdot \delta_{M^A}^A(\varphi'_i)$, $Y =$

$$\begin{aligned}
 &\sum_{j \in J: \text{fni}} \psi_j \cdot \delta_{M^A}^A(\psi'_j) \text{ with } \varphi_i, \varphi'_i, \psi_j, \psi'_j \in C^\infty(M^A, A) \\
 1. \quad \widetilde{[\text{ad}^A(X)]}(\varphi) &= \left[\widetilde{\text{ad}^A} \left(\sum_{i \in I: \text{fni}} \varphi_i \cdot \delta_{M^A}^A(\varphi'_i) \right) \right](\varphi) \\
 &= \sum_{i \in I: \text{fni}} \varphi_i \cdot (\widetilde{\text{ad}^A}[\delta_{M^A}^A(\varphi'_i)])(\varphi) \\
 &= \sum_{i \in I: \text{fni}} \varphi_i \cdot [\text{ad}^A(\varphi'_i)](\varphi) \\
 &= \sum_{i \in I: \text{fni}} \varphi_i \cdot \{\varphi'_i, \varphi\}_{M^A} \\
 &= - \sum_{i \in I: \text{fni}} \varphi_i \cdot \omega_{M^A}^A(\delta_{M^A}^A(\varphi'_i), \delta_{M^A}^A(\varphi)) \\
 &= -\omega_{M^A}^A \left(\sum_{i \in I: \text{fni}} \varphi_i \cdot \delta_{M^A}^A(\varphi'_i), \delta_{M^A}^A(\varphi) \right) \\
 &= -\omega_{M^A}^A(X, \delta_{M^A}^A(\varphi)).
 \end{aligned}$$

$$\begin{aligned}
2. \quad \widetilde{[\text{ad}^A(X)]}(Y) &= \widetilde{[\text{ad}^A(X)]}\left(\sum_{j \in J: \text{ fini}} \psi_j \cdot \delta_{MA}^A(\psi'_j)\right) \\
&= \sum_{j \in J: \text{ fini}} \psi_j \cdot \widetilde{[\text{ad}^A(X)]}(\delta_{MA}^A(\psi'_j)) \\
&= \sum_{j \in J: \text{ fini}} \psi_j \cdot \left(\widetilde{[\text{ad}^A(X)]} \circ \delta_{MA}^A\right)(\psi'_j) \\
&= \sum_{j \in J: \text{ fini}} \psi_j \cdot \widetilde{[\text{ad}^A(X)]}(\psi'_j) \\
&= - \sum_{j \in J: \text{ fini}} \psi_j \cdot \omega_{MA}^A(X, \delta_{MA}^A(\psi'_j)) \\
&= -\omega_{MA}^A\left(X, \sum_{j \in J: \text{ fini}} \psi_j \cdot \delta_{MA}^A(\psi'_j)\right) \\
&= -\omega_{MA}^A(X, Y);
\end{aligned}$$

$$\begin{aligned}
3. \quad \mathfrak{L}_{\widetilde{\text{ad}^A[\delta_{MA}^A(\varphi)]}} \delta_{MA}^A(\psi) &= \mathfrak{L}_{\text{ad}^A(\varphi)} \delta_{MA}^A(\psi) \\
&= \delta_{MA}^A[\text{ad}^A(\varphi)(\psi)] \\
&= \delta_{MA}^A(\{\varphi, \psi\}_{MA}).
\end{aligned}$$

□

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