

ON THE EXISTENCE OF NON-LINEAR FRAMES

SHAH JAHAN, VARINDER KUMAR, AND S.K. KAUSHIK

ABSTRACT. A stronger version of the notion of frame in Banach space called Strong Retro Banach frame (SRBF) is defined and studied. It has been proved that if \mathcal{X} is a Banach space such that \mathcal{X}^* has a SRBF, then \mathcal{X} has a Bi-Banach frame with some geometric property. Also, it has been proved that if a Banach space \mathcal{X} has an approximative Schauder frame, then \mathcal{X}^* has a SRBF. Finally, the existence of a non-linear SRBF in the conjugate of a separable Banach space has been proved.

1. INTRODUCTION

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in the context of nonharmonic Fourier series. Frames now a days are widely used in various branches of mathematics and engineering. Feichtinger and Grochenig [6] generalized the notion of frame to Banach spaces and introduced the concept of atomic decomposition in a Banach space. Also, Grochenig [7] introduced a more general concept namely Banach frame for Banach spaces. For a nice and comprehensive survey of frames and related concepts one may refer to [1, 4].

Various other generalizations of frames for Banach spaces were defined and studied by many authors namely Schauder frames by Han and Larson [8] and also studied by Casazza et al. [2, 3], frames by Terekhin [17]. Banach frames in conjugate Banach spaces, called retro Banach frames, were introduced and studied by Jain et al. [9] and further studied in [13]. Approximative atomic decompositions in Banach spaces were studied in [10]. Schauder frames in conjugate Banach spaces were defined and studied in [12] while approximative Schauder frames were studied in [11]. The notion of Bi-Banach frame in a Banach space was defined and studied in [14] wherein they noted that a Schauder frame for a Banach space is a Bi-Banach frame but the converse is not true.

In the present paper, we shall consider a stronger notion of frame in a Banach space called strong Retro Banach frame (SRBF). It has been proved that if \mathcal{X} is a Banach space such that \mathcal{X}^* has a SRBF, then \mathcal{X} has a Bi-Banach frame with some geometric property. Also, it has been proved that if a Banach space \mathcal{X} has an approximative Schauder frame, then \mathcal{X}^* has a SRBF. Finally, a result related to

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the existence of a non-linear SRBF in the conjugate of a separable Banach space has been proved.

Throughout this paper \mathcal{X} will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R}, \mathbb{C})$, \mathcal{X}^* denotes the conjugate space of \mathcal{X} and $L(\mathcal{X}, \mathcal{X})$ denote the Banach space of all continuous linear mappings of \mathcal{X} into \mathcal{X} . For a sequence $\{x_n\} \in \mathcal{X}$ and $\{f_n\} \in \mathcal{X}^*$, $[x_n]$ denotes the closed linear span of $\{x_n\}$ in the norm topology of \mathcal{X} and $[\overline{f_n}]$ the closed linear span of $\{f_n\}$ in the weak star topology of \mathcal{X}^* . A sequence space S is called a *BK-space* if it is a Banach space and the co-ordinate functionals are continuous on S . That is the relations $x_n = \{\alpha_j^{(n)}\}$, $x = \{\alpha_j\} \in S$, $\lim_{n \rightarrow \infty} x_n = x$ imply $\lim_{n \rightarrow \infty} \alpha_j^{(n)} = \alpha_j$ ($j = 1, 2, 3, \dots$). Also, if $V \subseteq \mathcal{X}^*$, then we define $\gamma(v) = \inf_{\substack{x \in \mathcal{X} \\ x \neq 0}} \sup_{\substack{f \in v \\ \|f\| \leq 1}} \left| f\left(\frac{x}{\|x\|}\right) \right|$.

A sequence $\{x_n\} \subset \mathcal{X}$ is said to be a *Markusevic basis* (*M-basis*) for \mathcal{X} if $\{x_n\}$ is complete in \mathcal{X} and there exists a sequence $\{f_n\}$ in \mathcal{X}^* biorthogonal to $\{x_n\}$, called an associated sequence of coefficient functional (a.s.c.f.), which is total on \mathcal{X} .

Definition 1.1 ([9]). Let \mathcal{X} be a Banach space and \mathcal{X}_d^* be a BK-space. Let $\{x_n\} \subset \mathcal{X}$ and $J: \mathcal{X}_d^* \rightarrow \mathcal{X}^*$ be given. The pair $(\{x_n\}, J)$ is called a *retro Banach frame* for \mathcal{X}^* with respect to \mathcal{X}_d^* if

- (a) $\{f(x_n)\} \in \mathcal{X}_d^*$, for all $f \in \mathcal{X}^*$.
- (b) There exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$(1.1) \quad A\|f\|_{\mathcal{X}^*} \leq \|\{f(x_n)\}\|_{\mathcal{X}_d^*} \leq B\|f\|_{\mathcal{X}^*}, \quad \text{for all } f \in \mathcal{X}^*.$$

- (c) J is a bounded linear operator such that

$$J(\{f(x_n)\}) = f, \quad \text{for all } f \in \mathcal{X}^*.$$

The constants A and B are called lower and upper bounds of the retro Banach frame $(\{x_n\}, J)$. The inequality (1.1) is called the retro Banach frame inequality.

A retro Banach frame $(\{x_n\}, J)$ is said to be exact if there exists a sequence $\{f_n\} \subset \mathcal{X}^*$ such that $f_i(x_j) = \delta_{i,j}$, for all $i, j \in \mathbb{N}$.

Definition 1.2. Let \mathcal{X} be a Banach space with dual \mathcal{X}^* . A pair $(\{x_n\}, \{f_n\})$ (where $\{f_n\} \subset \mathcal{X}^*$ and $\{x_n\} \subset \mathcal{X}$) is called a *Bi-Banach frame* for \mathcal{X} if there exist associated Banach spaces \mathcal{X}_d and $(\mathcal{X}^*)_d$ and bounded linear operators $S: \mathcal{X}_d \rightarrow \mathcal{X}$, $T: (\mathcal{X}^*)_d \rightarrow \mathcal{X}^*$ such that $(\{f_n\}, S)$ is a Banach frame for \mathcal{X} and $(\{x_n\}, T)$ is retro Banach frame for \mathcal{X}^* .

A Bi-Banach frame $(\{x_n\}, \{f_n\})$ is called *tight* if both the retro Banach frame $(\{x_n\}, T)$ and the Banach frame $(\{f_n\}, S)$ are tight.

The following results are stated in the form of lemmas which will be used in the subsequent work.

Lemma 1.3 ([16]). *Let \mathcal{X} be a Banach space and $\{f_n\} \subset \mathcal{X}^*$ be a sequence such that $\{x \in \mathcal{X} : f_n(x) = 0, \forall n \in \mathbb{N}\} = \{0\}$. Then \mathcal{X} is linearly isometric to the Banach space $\mathcal{X}_d = \{\{f_n(x)\} : x \in \mathcal{X}\}$, where the norm is given by $\|\{f_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}$.*

Lemma 1.4 ([16]). *Let \mathcal{X} be a separable normed linear space and let $\{x_n^*\}$ be a sequence in \mathcal{X}^* such that $\frac{x_n^*}{\|x_n^*\|} \xrightarrow{w^*} 0$ and that for the linear subspace $[x_n^*]$ of \mathcal{X}^* , $\gamma([x_n^*]) \geq 0$. Then there exist a norm $|\cdot|$ on \mathcal{X} equivalent to the initial norm on \mathcal{X} such that $(\mathcal{X}, |\cdot|)$ is strictly convex and satisfies the following property*

$$(1.2) \quad \text{If } \lim_{n \rightarrow \infty} f_k(x_n) = f_k(x_0) (k = 1, 2, \dots), \quad \text{then } \lim_{n \rightarrow \infty} |x_n| \geq |x_0|.$$

2. MAIN RESULT

Approximative Schauder frames in Banach spaces were studied in [11] and the notion of Bi-Banach frame was studied in [14]. In the following definition, we gave a stronger notion called Strong Retro Banach frame (SRBF). The idea of defining this notion is to correlate this notion with the existing notions like approximative Schauder frames and Bi-Banach frames.

Definition 2.1. Let $\{x_n\} \subset \mathcal{X}$ be an exact RBF for \mathcal{X}^* with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Let $X_n = [x_1, x_2, \dots, x_n]$, $n \in \mathbb{N}$. If there exists a sequence $\{v_n\}$, where each $v_n: X_n \rightarrow X_n$ is a continuous linear mapping, such that $x = \lim_{n \rightarrow \infty} v_n \sum_{i=1}^n f_i(x)x_i$, $x \in \mathcal{X}$, then $(\{x_n\}, \{f_n\}, \{v_n\})$ is called a strong RBF (or SRBF) for \mathcal{X}^* .

Remark 2.2. If we define $u_n: \mathcal{X} \rightarrow \mathcal{X}$, $n \in \mathbb{N}$ by

$$u_n(x) = v_n \sum_{i=1}^n f_i(x)x_i, \quad n \in \mathbb{N}.$$

Then one may observe that if $(\{x_n\}, \{f_n\}, \{v_n\})$ is a SRBF, then $\lim_{n \rightarrow \infty} u_n(x) = x$ and $\dim u_n(\mathcal{X}) = \dim (v_n \sum_{i=1}^n f_i(x)x_i)(\mathcal{X}) \leq n < \infty$ and so $\{x_n\}$ is an approximative basis of \mathcal{X} .

In the following result, we prove that the existence of SRBF in the conjugate of a Banach space guarantees the existence of a Bi-Banach frame in the Banach space along with some geometric property.

Theorem 2.3. *Let X be a Banach space and $(\{x_n\}, \{f_n\}, \{v_n\})$ be a SRBF for \mathcal{X}^* with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Then $(\{x_n\}, \{f_n\})$ is a Bi-Banach frame for \mathcal{X} such that $\gamma([f_n]) > 0$.*

Proof. Clearly, by definition of SRBF, $\{x \in \mathcal{X} : f_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$. Therefore, by Lemma 1.3, there exists an associated Banach space $\mathcal{X}_d = \{\{f_n(x)\}; x \in \mathcal{X}\}$ with norm given by $\|\{f_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}}$, $x \in \mathcal{X}$. Define $J: \mathcal{X}_d \rightarrow \mathcal{X}$ by $J(\{f_n(x)\}) = x$, $x \in \mathcal{X}$. Then J is a bounded linear operator such that $(\{f_n\}, J)$ is a Banach frame for \mathcal{X} . Hence $(\{x_n\}, \{f_n\})$ is a Bi-Banach frame for \mathcal{X} . Let for each $n \in \mathbb{N}$, $v_n: X_n \rightarrow X_n$ be a continuous linear mapping given by $\lim_{n \rightarrow \infty} v_n \sum_{i=1}^n f_i(x)x_i = x$. Write $v_n(x_j) = \sum_{i=1}^n a_{ji}^{(n)} x_i$, $j = 1, 2, \dots, n$, $n \in \mathbb{N}$, where

$a_{ji}^{(n)} = f_i(v_n(x_j))$, for all $i, j = 1, 2, 3, \dots, n, n \in \mathbb{N}$. Thus

$$v_n \left(\sum_{i=1}^n f_i(x) x_i \right) = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}^{(n)} f_i(x) \right) x_j, \quad x \in \mathcal{X}, \quad n \in \mathbb{N}.$$

Define

$$h_{n,j} = \sum_{i=1}^n a_{ij}^{(n)} f_i, \quad j = 1, 2, 3, \dots, n, \quad n \in \mathbb{N}.$$

Then $h_{n,i} \in [f_i]_{i=1}^n$ ($j = 1, 2, \dots, n; n \in \mathbb{N}$). Hence, we conclude that $\gamma([f_n]) > 0$. \square

In the following result, we prove a weak duality type result.

Theorem 2.4. *Let $(\{x_n\}, \{f_n\}, \{v_n\})$ be a SRBF for \mathcal{X}^* with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Then there exists a sequence of continuous linear mappings $\{\tau_n\}$ ($\tau_n: V_n \rightarrow V_n$, where $V_n = [f_1, f_2, \dots, f_n], n \in \mathbb{N}$) such that*

$$f(x) = \lim_{n \rightarrow \infty} \left(\tau_n \sum_{i=1}^n f(x_i) f_i \right) (x)$$

Proof. For each $k = 1, 2, 3, \dots, n, n \in \mathbb{N}$, define $\tau_n: \text{span}\{f_1, f_2, \dots, f_n\} \rightarrow \text{span}\{f_1, f_2, \dots, f_n\}$ by

$$\tau_n(f_k) = \sum_{i=1}^n a_{ik}^{(n)} f_i = \sum_{i=1}^n f_k(v_n(x_i)) f_i.$$

Extend each τ_n to $[f_1, f_2, \dots, f_n]$. Then

$$\begin{aligned} \left(\tau_n \sum_{i=1}^n f(x_i) f_i \right) (x) &= \sum_{i=1}^n f(x_i) (\tau_n(f_i))(x) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ji}^{(n)} f_j(x) \right) f(x_i) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n f_j(v_n(x_j)) f_j(x) \right) f(x_i) \\ &= \sum_{i=1}^n v_n \left(\sum_{j=1}^n f_j(x_j) f_j(x) \right) f(x_i) \\ &= f \left(v_n \left(\sum_{i=1}^n f_i(x) x_i \right) \right) \\ &= \rightarrow f(x) \quad \text{as } n \rightarrow \infty \end{aligned}$$

\square

Approximative Schauder frames were defined and studied in [11]. In the following result, we prove that if a Banach space \mathcal{X} has an approximative Schauder frame, then its dual space has a SRBF.

Theorem 2.5. *If a Banach space \mathcal{X} has an approximative Schauder frame, then \mathcal{X}^* has a SRBF.*

Proof. Let $\{u_n\}$ be a sequence of finite rank continuous linear mapping from \mathcal{X} to \mathcal{X} such that $\lim_{n \rightarrow \infty} u_n(x) = x, x \in \mathcal{X}$. Let $\{x_n\}$ be a Markusevic basis for \mathcal{X} with a.s.c.f. $\{f_n\} \subset \mathcal{X}^*$ such that

$$(2.1) \quad \bigcup_n u_n^*(\mathcal{X}^*) \subset [f_n].$$

Since each u_n is finite dimensional, we may write

$$u_n(x) = \sum_{i=1}^{p_n} \psi_{ni}(x)\phi_{ni}, \quad x \in \mathcal{X}, n \in \mathbb{N},$$

where $\{\phi_{ni}\}_{i=1}^{p_n}$ is a basis for $\{u_n(\mathcal{X})\}$ with associated sequence $\{\psi_{ni}\}_{i=1}^{p_n} \subset \mathcal{X}^*$. Let $\{g_{ni}\}_{i=1}^{p_n}$ be a sequence in \mathcal{X}^* that is biorthogonal to $\{\phi_{ni}\}_{i=1}^{p_n}$. Then

$$\begin{aligned} u_n^*(g_{nj})(x) &= g_{nj}(u_n(x)) \\ &= g_{nj}\left(\sum_{i=1}^{p_n} \psi_{ni}(x)\phi_{ni}\right) \\ &= \sum_{i=1}^{p_n} \psi_{ni}(x)g_{nj}(\phi_{ni}) \\ &= \psi_{nj}(x), \quad j = 1, 2, \dots, p_n. \end{aligned}$$

Hence, $\psi_{nj} \in [f_n], j = 1, 2, \dots, p_n, n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be given. Then, for any $\epsilon > 0$, there exists an integer $m_n(\epsilon)$ such that for each $i = 1, 2, \dots, p_n$ one can find $\bar{\phi}_{ni} \in [x_1, x_2, \dots, x_{m_n}]$ and $\bar{\psi}_{ni} \in [f_1, f_2, \dots, f_{m_n}]$ such that

$$(2.2) \quad \|\phi_{ni} - \bar{\phi}_{ni}\| < \epsilon \quad \text{and} \quad \|\psi_{ni} - \bar{\psi}_{ni}\| < \epsilon, \quad i = 1, 2, 3, \dots, p_n.$$

Write

$$\bar{v}_{m_n}(x) = \sum_{i=1}^{p_n} \bar{\psi}_{ni}(x)\bar{\phi}_{ni}, \quad x \in \mathcal{X}.$$

Then

$$\begin{aligned} \|\bar{v}_{m_n}(x) - u_n(x)\| &= \left\| \sum_{i=1}^{p_n} (\bar{\psi}_{ni}(x) - \psi_{ni}(x))\bar{\phi}_{ni} + \sum_{i=1}^{p_n} \psi_{ni}(x)(\bar{\phi}_{ni} - \phi_{ni}) \right\| \\ &\leq \left(\sum_{i=1}^{p_n} \|\bar{\psi}_{ni} - \psi_{ni}\| \|\bar{\phi}_{ni}\| + \sum_{i=1}^{p_n} \|\psi_{ni}\| \|\bar{\phi}_{ni} - \phi_{ni}\| \right) \|x\|, \quad x \in \mathcal{X}. \end{aligned}$$

Therefore, by (2.2), taking $\epsilon = \frac{1}{n}$ and $\{m_n\}$ to be an increasing sequence, we obtain

$$(2.3) \quad \|\bar{v}_{m_n} - u_n\| < \frac{1}{n}.$$

Now, observe that

$$(2.4) \quad f_i \left(x - \sum_{j=1}^k f_j(x)x_j \right) = 0, \\ \text{for all } x \in \mathcal{X}, i = 1, 2, \dots, m_n; \quad k = m_n, m_n + 1, \dots$$

Also, $\bar{\psi}_{ni} \in [f_1, f_2, \dots, f_{m_n}]$. So, for $x \in \mathcal{X}$, we have

$$(2.5) \quad \begin{aligned} \bar{v}_{m_n} \left(\sum_{j=1}^k f_j(x)x_j \right) &= \sum_{i=1}^{p_n} \bar{\psi}_{ni} \left(\sum_{j=1}^k f_j(x)x_j \right) \bar{\phi}_{ni} \\ &= \sum_{i=1}^{p_n} \bar{\psi}_{ni}(x) \bar{\phi}_{ni} \\ &= \bar{v}_{m_n}(x), \quad \text{for all } x \in \mathcal{X} \quad \text{and} \quad k \geq m_n. \end{aligned}$$

Therefore

$$(2.6) \quad \begin{aligned} \lim_{n \rightarrow \infty} \bar{v}_{m_n} \left(\sum_{j=1}^{m_n} f_j(x)x_j \right) &= \lim_{n \rightarrow \infty} \bar{v}_{m_n}(x) \\ &= \lim_{n \rightarrow \infty} u_n(x) \\ &= x, \quad x \in \mathcal{X}. \end{aligned}$$

Define a sequence $\{T_n\}$ by $T_n(x) = \sum_{i=1}^n f_i(x)x_i$, $n \in \mathbb{N}$. Write

$$v_k = T_k|_{[x_1, x_2, \dots, x_k]}, \quad k = 1, 2, \dots, m_1 - 1$$

and

$$v_k = \bar{v}_{m_n}|_{[x_1, x_2, \dots, x_k]}, \quad k = m_n, m_n + 1, \dots, m_n - 1, \quad n = 1, 2, 3 \dots$$

Then, each v_k is a continuous linear mapping defined on $[x_1, x_2, \dots, x_k]$ with range given by

$$v_k([x_1, x_2, \dots, x_k]) = [x_1, x_2, \dots, x_k]; \quad k = 1, 2, 3, \dots, m_1 - 1$$

$$v_k([x_1, x_2, \dots, x_k]) \subset [x_1, x_2, \dots, x_k], \quad k = m_n, m_n + 1, \dots, m_{n+1} - 1, \quad n \in \mathbb{N}.$$

Using (2.5) and (2.6), we obtain $\lim_{n \rightarrow \infty} \bar{v}_{m_n}(x) = x$. Hence $\lim_{n \rightarrow \infty} v_n(\sum_{i=1}^n f_i(x)x_i) = x$. □

In view of the proof of Theorem 2.5, one may observe that existence of approximative Schauder frame in a Hilbert space is a sufficient condition for the space having Markusevic basis to have a SRBF. More precisely we have

Corollary 2.6. *Let \mathcal{H} be a Hilbert space with an approximative Schauder frame. Then every Markusevic basis of \mathcal{H} give rise to a SRBF for \mathcal{H} .*

In the following example, we show that in general, a SRBF do not have strong duality

Example 2.7. Let \mathcal{X} be a Banach space with a Schauder basis and such that \mathcal{X}^* is separable but fails to have approximative property. Let $\{x_n\}$ be a shrinking Markusevic basis of \mathcal{X} with associated sequence of coefficient functional $\{f_n\} \subset \mathcal{X}^*$. Define

$$u_n(x) = \sum_{i=1}^n f_i(x)x_i, \quad x \in \mathbb{N}.$$

Then $\{x_n\}$ is an approximative Schauder frame for \mathcal{X} satisfying $\bigcup_{n=1}^\infty u_n^*(\mathcal{X}^*) \subset [f_n]$. Therefore $(\{x_n\}, \{f_n\}, \{v_n\})$ is a SRBF for \mathcal{X}^* . However, \mathcal{X}^{**} has no SRBF.

One may observe that in Definition 2.1, each v_n is linear. Now, we would like to drop this condition of linearity and in the process define non-linear SRBF.

Definition 2.8. A SRBF $(\{x_n\}, \{f_n\}, \{v_n\})$ is called non-linear SRBF if each v_n is continuous but not necessarily linear.

Finally, we prove the following result related to the existence of a non-linear SRBF.

Theorem 2.9. *If \mathcal{X} is a separable Banach Space, then \mathcal{X}^* has a non-linear SRBF.*

Proof. Let $\{x_n\}$ be a Markusevic basis with a sequence of coefficient functional $\{f_n\} \subset \mathcal{X}^*$ such that $\gamma([f_n]) > 0$. Then, by Lemma 1.4, there is a norm $|\cdot|$ on \mathcal{X} that is equivalent to the original norm $\|\cdot\|$ on \mathcal{X} such that \mathcal{X} with this new norm $|\cdot|$ is strictly convex. Therefore, by [15, Corollary 3.3, page 110], for every finite dimensional subspace \mathcal{G} of \mathcal{X} and for every $x \in \mathcal{X} \setminus \mathcal{G}$, there is a unique $\pi_{\mathcal{G}}(x) \in \mathcal{G}$ such that $|x - \pi_{\mathcal{G}}(x)| = \text{dist}(x, \mathcal{G}) = \min_{g \in \mathcal{G}} |x - g|$ and such that the mapping $\pi_{\mathcal{G}}: \mathcal{X} \rightarrow \mathcal{G}$ is continuous (here note that, in general, $\pi_{\mathcal{G}}$ is non-linear). Let $\mathfrak{N}(a, b)$ denote a positive integer depending on a and b . For each n , choose an increasing sequence of positive integers $\{m_n\}$ with $m_1 = \mathfrak{N}(1, 1)$, $m_2 = \mathfrak{N}(m_1, \frac{1}{2})$, $m_3 = \mathfrak{N}(m_2, \frac{1}{3})$, \dots , $m_n = \mathfrak{N}(m_{n-1}, \frac{1}{n})$, for all $n \geq 2$ and satisfying

$$\text{dist} \left(a, [x_i]_{i=m_{n-1}+1}^{m_n} \right) \leq \left(1 + \frac{1}{n} \right) \text{dist} \left(a, [x_i]_{i=m_{n-1}+1}^\infty \right),$$

where $a \in [x_i]_{i=1}^{m_{n-1}}$. Define $\{v_n\}$ by $v_k = T_k|_{[x_1, \dots, x_k]}$, $k = 1, 2, \dots, m_1 - 1$, where $T_k(x) = \sum_{i=1}^k f_i(x)x_i$ and for any $b = \sum_{i=1}^k a_i x_i \in [x_i]_{i=1}^k$, ($k = m_n, m_n + 1, \dots, m_{n+1} - 1$; $n \in \mathbb{N}$)

$$v_k(b) = \sum_{i=1}^{m_{n-1}} a_i x_i - \pi_{\mathcal{G}} \left(\sum_{i=1}^{m_{n-1}} a_i x_i \right),$$

where $\mathcal{G} = [x_i]_{i=m_{n-1}+1}^{m_n}$. Then each v_n is continuous (in general, non-linear) with range given by

$$\begin{aligned} v_k([x_1, \dots, x_k]) &= [x_1, \dots, x_k], \quad k = 1, 2, 3, \dots, m_1 - 1 \\ v_k([x_1, \dots, x_k]) &\subset [x_1, \dots, x_k], \quad (k = m_n, m_n + 1, \dots, m_{n+1} - 1; n \in \mathbb{N}). \end{aligned}$$

Let $x \in \mathcal{X}$ be any element. Then

$$f_i \left(v_k \left(\sum_{j=1}^k f_j(x)x_j \right) \right) = f_i(x), \quad i = 1, 2, \dots, m_n; \quad k = m_n, m_n+1, \dots, m_{n+1}-1; \quad n \in \mathbb{N}.$$

This gives

$$(2.7) \quad \lim_{k \rightarrow \infty} f_i \left(v_k \left(\sum_{j=1}^k f_j(x)x_j \right) \right) = f_i(x), \quad i = 1, 2, \dots$$

In view of Lemma 1.4, we have

$$(2.8) \quad \lim_{k \rightarrow \infty} \left| v_k \left(\sum_{i=1}^k f_i(x)x_i \right) \right| \geq |x|.$$

Also, we have

$$\begin{aligned} v_k \left(\sum_{i=1}^k f_i(x)x_i \right) &= \left| \sum_{i=1}^{m_{n-1}} f_i(x)x_i - \pi_{\mathcal{G}} \sum_{i=1}^{m_{n-1}} f_i(x)x_i \right| \\ &= \text{dist} \left(\sum_{i=1}^{m_{n-1}} f_i(x)x_i, \mathcal{G} \right) \\ &\leq \left(1 + \frac{1}{n} \right) \text{dist} \left(\sum_{i=1}^{m_{n-1}} f_i(x)x_i, [x_i]_{i=m_{n-1}+1}^{\infty} \right) \\ &\leq \left(1 + \frac{1}{n} \right) \left| \sum_{i=1}^{m_{n-1}} f_i(x)x_i + \left(x - \sum_{i=1}^{m_{n-1}} f_i(x)x_i \right) \right| \\ &= \left(1 + \frac{1}{n} \right) |x|, \quad k = m_n, m_n + 1, \quad m_{n+1} - 1; \quad n \in \mathbb{N}. \end{aligned}$$

Thus, by (2.8), we have

$$(2.9) \quad \lim_{k \rightarrow \infty} \left| v_k \left(\sum_{i=1}^k f_i(x)x_i \right) \right| = |x|.$$

Hence, we conclude that

$$\lim_{k \rightarrow \infty} \left| v_k \left(\sum_{i=1}^k f_i(x)x_i \right) - x \right| = 0.$$

Since $|\cdot|$ is equivalent to the initial norm of \mathcal{X} , we obtain

$$\lim_{n \rightarrow \infty} v_n \left(\sum_{i=1}^n f_i(x)x_i \right) = x, \quad x \in \mathcal{X}.$$

□

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S. JAHAN, DEPARTMENT OF MATHEMATICS,
 UNIVERSITY OF DELHI, DELHI-110 007, INDIA
E-mail: chowdharyshahjahan@gmail.com

V. KUMAR, DEPARTMENT OF MATHEMATICS,
 SHAHEED BHAGAT SINGH COLLEGE,
 UNIVERSITY OF DELHI, DELHI-110 017, INDIA
E-mail: varinder1729@gmail.com

S.K. KAUSHIK, DEPARTMENT OF MATHEMATICS,
 KIRORI MAL COLLEGE,
 UNIVERSITY OF DELHI, DELHI-110 007, INDIA
E-mail: shikk2003@yahoo.co.in