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ON MINIMAL IDEALS IN THE RING OF REAL-VALUED CONTINUOUS FUNCTIONS ON A FRAME

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Abstract. Let $\mathcal{R}L$ be the ring of real-valued continuous functions on a frame L. The aim of this paper is to study the relation between minimality of ideals I of $\mathcal{R}L$ and the set of all zero sets in L determined by elements of I. To do this, the concepts of coz-disjointness, coz-spatiality and coz-density are introduced. In the case of a coz-dense frame L, it is proved that the f-ring $\mathcal{R}L$ is isomorphic to the f-ring $C(\Sigma L)$ of all real continuous functions on the topological space ΣL. Finally, a one-one correspondence is presented between the set of isolated points of ΣL and the set of atoms of L.

1. Introduction

In studying the ring $C(X)$ of all real continuous functions on a topological space X, zero sets are a powerful tool, defined by $Z(f) = \{x \in X : f(x) = 0\}$, for $f \in C(X)$. For a frame L, the ring $\mathcal{R}L$ is defined as a pointfree version of $C(X)$. In the pointfree topology, cozero elements are considered as a dual concept of zero sets. Cozero elements are defined by $coz(\alpha) = \bigvee\{\alpha(p,0) \vee \alpha(0,q) : p,q \in \mathbb{Q}\}$, for $\alpha \in \mathcal{R}L$.

Recall that in [12], considering the prime elements of a given frame L as pointfree points of L, the trace of an element α of $\mathcal{R}L$ on any point p of L is defined as a real number denoted by $\alpha[p]$. Then the zero set of α is defined by $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$. The real number $\alpha[p]$ is defined by the Dedekind cut $(L(p,\alpha), U(p,\alpha))$, where $L(p,\alpha) = \{r \in \mathbb{Q} : \alpha(-,r) \leq p\}$ and $U(p,\alpha) = \{s \in \mathbb{Q} : \alpha(s,-) \leq p\}$). Also, the map $\widetilde{p} : \mathcal{R}L \to \mathbb{R}$ given by $\widetilde{p}(\alpha) = \alpha[p]$ is an f-ring homomorphism (Propositions 2.2, 2.4).

The main results of this paper are based on a theorem about the ring $C(X)$ which gives some equivalent conditions regarding the minimality of an ideal I of $C(X)$ as follows:

Theorem 1.1 ([16]). *Let X be a completely regular space. The following are equivalent.*

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(1) *I is a minimal ideal.*

(2) $|Z[I]| = 2$.

(3) *There is an isolated point* $x_0 \in X$ *such that* $Z[I] = \{X, X - \{x_0\}\}$.

In [9], Themba Dube obtained a pointfree version of Theorem 1.1, by using $coz[I]$ of $\mathcal{R}L$ instead of $Z[I]$ of $C(X)$ as follows:

Theorem 1.2 ([9]). *An ideal I of $\mathcal{R}L$ is minimal if and only if $coz[I]$ consists only of two elements.*

From the cited theorem, it will follow that achieving equivalence between minimality of ideal I and the condition $|Z[I]| = 2$ is useful. For finding the equivalence, we require the equivalence of conditions $|Z[I]| = 2$ and $|\,coz(I)| = 2$. So, we need the equivalence $coz(\alpha) = coz(\beta)$ if and only if $Z(\alpha) = Z(\beta)$, for every $\alpha, \beta \in \mathcal{R}L$. In the Theorem 3.11, it is proved for a frame L, the following are evidently equivalent:

(1) L is coz-spatial.

(2) For every $\alpha, \beta \in \mathcal{R}L$, $Z(\alpha) = Z(\beta)$ if and only if $coz(\alpha) = coz(\beta)$.

Coz-disjoint frames are introduced in Section 3; also every completely regular frame is coz-disjoint (Proposition 3.5). In the coz-disjoint frame L, if P is a prime ideal of $\mathcal{R}L$, then $|\bigcap Z[P]| \leq 1$ (Theorem 3.8). For every $\alpha \in \mathcal{R}L$, M_α is defined by $\{\beta \in \mathcal{R}L : Z(\alpha) \subseteq Z(\beta)\}$. If L is a coz-disjoint frame, it is shown that $M_\alpha = M_p$ if and only if M_α is a prime ideal in Proposition 3.15.

In the last section, we study and analyze the three following conditions, without coz-spatiality, and with some other concepts like coz-disjointness, coz-density, and coz-spatiality.

(1) I is a minimal ideal.

(2) $|Z[I]| = 2$.

(3) There exists $p \in \Sigma L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$.

(1)⇒(2)⇒(3) are proved in Propositions 4.3 and 4.4 with the assumptions of coz-disjointness and coz-density. For (3)⇒(1), we suppose the concept of coz-density, which is weaker than weakly spatiality (Corollary 4.18).

In Proposition 4.17, it is proved that if L is a coz-dense frame then $\mathcal{R}L$ is isomorphic to a ring $C(X)$ for some topological space X. In fact, $\mathcal{R}L \simeq C(\Sigma L)$ as two f-rings. In Corollary 4.22, we construct a one-one correspondence between the set of isolated points of ΣL and the set of atoms of L. The relations among coz-dense, coz-spatial, weakly spatial, spatial, coz-disjoint, and completely regular conditions are explained in Remark 4.23.

2. Preliminaries

We recall some basic notions and facts about frames and spaces. For further information see [4, 17] on frames and [14] on spaces.

A *frame* is a complete lattice L in which the distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$ holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$. A *frame homomorphism* (*frame map*) between frames is a map which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An element x of a frame L is said to be:

(1) *prime* (*point*) if $x < \top$ and, for a, $b \in L$, $a \wedge b \leq x$ implies $a \leq x$ or $b \leq x$,

(2) an *atom* if $\bot < x$ and, for any $a \in L$, $\bot \leq a \leq x$ imply $a = \bot$ or $a = x$.

The *pseudocomplement* of an element a of a frame L is the element

$$a^{\star} = \bigvee \{x \in L : x \wedge a = \bot\}.$$

An element a of a frame L is said to be *rather below* an element $b \in L$, written $a \prec b$, provided that $a^{\star} \vee b = \top$. On the other hand, a is *completely below* b, written $a \prec\!\!\prec b$, if there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$.

We recall the contravariant functor Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ $(a \in L)$ as its open sets. Also, for a frame map $h\colon L \to M$, $\Sigma h\colon \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_*\colon M \to L$ is the *right adjoint* of h characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$.

Recall from [4] that the frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\top = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

Note that the pairs (p, q) in $\mathcal{L}(\mathbb{R})$ and the open intervals $\langle p, q\rangle = \{x \in \mathbb{R} : p < x < q\}$ in the frame $\mathfrak{O}\mathbb{R}$ have the same role; in fact there is a frame isomorphism $\lambda\colon \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ such that $\lambda(p, q) = \langle p, q\rangle$. In other word, $\mathcal{L}(\mathbb{R})$ is the frame generated by $\mathbb{Q} \times \mathbb{Q}$ with equations {R1, R2, R3, R4}, so we have the following lemma.

Lemma 2.1. *Let $f\colon \mathbb{Q} \times \mathbb{Q} \to L$ be a function satisfying the following relations:*

(R1′) $f(p, q) \wedge f(r, s) = f(p \vee r, q \wedge s)$,

(R2′) $f(p, q) \vee f(r, s) = f(p, s)$ *whenever* $p \leq r < q \leq s$,

(R3′) $f(p, q) = \bigvee\{f(r, s) | p < r < s < q\}$,

(R4′) $\top = \bigvee\{f(p, q) \mid p, q \in \mathbb{Q}\}$.

Then there exists a unique frame map $g\colon \mathcal{L}(\mathbb{R}) \to L$ such that $g(p, q) = f(p, q)$ for every p, $q \in \mathbb{Q}$.

The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [4].

Corresponding to every continuous operation $\diamond\colon \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+$, $\cdot$, $\wedge$, $\vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(u,w) : (r,s) \diamond (u,w) \leq (p,q)\},$$

where $(r,s) \diamond (u,w) \leq (p,q)$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p,q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p,q) = \bot$.

The *cozero map* is the map $coz\colon \mathcal{R}L \to L$, defined by

$$coz(\alpha) = \bigvee\{\alpha(p,0) \vee \alpha(0,q) : p,q \in \mathbb{Q}\} = \alpha((-,0) \vee (0,-))$$

where

$$(0,-) = \bigvee\{(0,q)) : q \in \mathbb{Q}, q > 0\} \quad \text{and} \quad (-,0) = \bigvee\{(p,0)) : p \in \mathbb{Q}, p < 0\}.$$

For $A \subseteq \mathcal{R}L$, we write $\mathrm{Coz}[A]$ to denote the family of cozero-elements $\{coz(\alpha) : \alpha \in A\}$. On the other hand, the family $\mathrm{Coz}[\mathcal{R}L]$ of all cozero-elements in L will also be denoted, for simplicity, by $\mathrm{Coz}\, L$. It is known that L is completely regular if and only if $\mathrm{Coz}(\mathcal{R}L)$ generates L. For more details about the *cozero map* and its properties, which are used in this paper, see [4].

Here we see the necessary notations, definitions, and results of [10].

Proposition 2.2 ([10])**.** *Let L be a frame. If $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, then $(L(p,\alpha), U(p,\alpha))$ is a Dedekind cut for a real number, denoted by $\widetilde{p}(\alpha)$.*

Proposition 2.3 ([10])**.** *If p is a prime element of a frame L, then there exists a unique map $\widetilde{p}\colon \mathcal{R}L \longrightarrow \mathbb{R}$ such that for each $\alpha \in \mathcal{R}L$, $r \in L(p,\alpha)$, and $s \in U(p,\alpha)$, we have $r \leq \widetilde{p}(\alpha) \leq s$.*

By the following proposition, $\tilde{p}$ is an f-ring homomorphism.

Proposition 2.4 ([10])**.** *If p is a prime element of a frame L, then $\widetilde{p}\colon \mathcal{R}L \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism. Also, $\widetilde{p}$ is a linear map with $\widetilde{p}(\mathbf{1}) = 1$.*

Let L be a frame and p be a prime element of L. Throughout this paper for every $f \in \mathcal{R}L$, we define $f[p] = \widetilde{p}(f)$.

Recall [12] for $\alpha \in \mathcal{R}L$, $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$. For $A \subseteq \mathcal{R}L$, we write $Z[A]$ to denote the family of zero-sets $\{Z(\alpha) : \alpha \in A\}$. On the other hand, the family $Z[\mathcal{R}L]$ of all zero-sets in L will also be denoted, for simplicity, by $Z[L]$. The following lemma and proposition play important roles in this paper.

Lemma 2.5. *Let p be a prime element of frame L. For $\alpha \in \mathcal{R}L$, $\alpha[p] = 0$ if and only if $coz(\alpha) \leq p$. Hence $Z(\alpha) = \Sigma L - \Sigma_{coz(\alpha)}$.*

Proposition 2.6. *For every α, $\beta \in \mathcal{R}L$, we have*

(1) *For every $n \in \mathbb{N}$, $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.*

(2) *$Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$.*

(3) *$Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.*

(4) *If* α *is a unit of* $\mathcal{R}L$*, then* $Z(\alpha) = \emptyset$.

(5) $Z(L)$ *is closed under the countable intersection.*

In [15], using the technique of sublocales, the authors present zero sublocales. A sublocale S of a frame L is a zero sublocale if it is of the form

$$\big(f^*(0,-)\big)^* \wedge \big(f^*(-,0)\big)^*$$

for some localic map $f\colon L \to \mathcal{L}(\mathbb{R})$ as the right Galois adjoint of a frame homomorphism $h = f^*\colon \mathcal{L}(\mathbb{R}) \to L$. The zero sets used in this paper are different from the zero sublocales.

3. COZ-DISJOINTNESS AND COZ-SPATIALITY

We introduce the concept of coz-disjoint for frames as follows:

Definition 3.1. A frame L is called *coz-disjoint* if for every two distinct prime elements p, $q \in L$, $coz(L) \cap \downarrow p \neq coz(L) \cap \downarrow q$.

Proposition 3.2 ([11])**.** *If* p *is a prime element of a frame* L *and* $M_p = \{\alpha \in \mathcal{R}L : \alpha[p] = 0\} = \ker \widetilde{p}$*, then* M_p *is a maximal ideal.*

Lemma 3.3. *Let* L *be a frame and* p, $q \in L$ *be prime elements. The following statements are equivalent:*

(1) $coz(L) \cap \downarrow p = coz(L) \cap \downarrow q$.

(2) $\tilde{p} = \tilde{q}$.

(3) $M_p = M_q$.

Proof. (1)⇔(2) By [10, Corollary 3.10(1)].

(2)⇒(3) Obvious.

(3)⇒(2) Let $\alpha \in \mathcal{R}L$ and $r = \tilde{p}(\alpha)$. So, $0 = \tilde{p}(\alpha) - r = \tilde{p}(\alpha) - \tilde{p}(\mathbf{r}) = \tilde{p}(\alpha - \mathbf{r})$. That is to say $\tilde{q}(\alpha - \mathbf{r}) = 0$, and hence $\tilde{q}(\alpha) = r$. Therefore $\tilde{p} = \tilde{q}$. □

The foregoing lemma gives directly the following corollary.

Corollary 3.4. *Let* L *be a frame. The following statements are equivalent:*

(1) L *is a coz-disjoint frame.*

(2) *If* $M_p = M_q$*, then* $p = q$.

We regard the Stone-Čech compactification of L, denoted βL, as the frame of completely regular ideals of L (for more details, see [5]). We denote the right adjoint of the join map $j_L\colon \beta L \to L$ by r_L and recall that $r_L(a) = \{x \in L : x \prec\!\!\prec a\}$. We define $M^I = \{\alpha \in \mathcal{R}L : r_L(coz(\alpha)) \subseteq I\}$, for all $1_{\beta L} \neq I \in \beta L$. If $M^I = M^J$ then $I = J$ (see [7]).

Proposition 3.5. *Every completely regular frame is coz-disjoint.*

Proof. Let $p, q \in \Sigma L$ and $M_p = M_q$. So, $M^{r_L(p)} = M_p = M_q = M^{r_L(q)}$. That is to say, $r_L(p) = r_L(q)$, and hence $p = q$. □

Remark 3.6. The converse of Proposition 3.5 is not true. To see this, let L be a frame such that $\Sigma L = \emptyset$. Define a frame $M = L \cup \{*\}$ where $* \notin L$ and $x < *$ for all $x \in L$. We have $*$ is the top element of M, and $\top_L$ is the only prime element of M, so M is a coz-disjoint frame, but M is not regular, because for every $x, y \in L$, $x^* \vee y \neq \top_M$. This means that $x \not\prec y$.

Proposition 3.7 ([12]). *Every prime ideal in $\mathcal{R}L$ is contained in a unique maximal ideal.*

Theorem 3.8. *Let L be a coz-disjoint frame. If P is a prime ideal of $\mathcal{R}L$, then $|\bigcap Z[P]| \leq 1$.*

Proof. If $p, q \in \bigcap Z[P]$, then $P \subseteq M_p$ and $P \subseteq M_q$. By Proposition 3.7, $M_p = M_q$. Therefore, by Corollary 3.4, $p = q$ and the proof is complete. □

By Proposition 3.5, Theorem 3.8 is a more general result than [12, Proposition 6.14].

Strongly z-ideals in $\mathcal{R}L$ are introduced in [12], and it is proved there that every strongly z-ideal of $\mathcal{R}L$ is a z-ideal. We define the concept of a coz-spatial frame and study the relation between coz-spatial frames and strongly z-ideals.

Definition 3.9. Let L be a frame and I be an ideal of $\mathcal{R}L$.

(1) I is called a z-ideal if for any $\alpha \in \mathcal{R}L$ and $\beta \in I$, $coz(\alpha) = coz(\beta)$ (or $coz(\alpha) \leq coz(\beta)$) implies $\alpha \in I$ [6].
(2) I is called a *strongly z-ideal* if $Z(\alpha) \subseteq Z(\beta)$ and $\alpha \in I$ imply $\beta \in I$.
(3) L is called *coz-spatial* if $coz(\alpha) \not\leq coz(\beta)$ implies that there exists a prime element $p \in L$ such that $coz(\alpha) \not\leq p$ and $coz(\beta) \leq p$.

Note that in studying the ring $C(X)$, z-ideals play important role (for more details see [1, 2, 3, 14]).

Remark 3.10. Every spatial frame is coz-spatial, but the converse is not necessarily true. To see this, consider the frame M discussed in Remark 3.6. We have $\mathrm{Coz}(M) = \{\bot, *\}$ and it is directly checked that M is coz-spatial.

Theorem 3.11. *Let L be a frame. The following statements are equivalent:*

(1) *L is coz-spatial.*
(2) *Every z-ideal of $\mathcal{R}L$ is a strongly z-ideal.*
(3) *For every $\alpha, \beta \in \mathcal{R}L$, $Z(\alpha) = Z(\beta)$ if and only if $coz(\alpha) = coz(\beta)$.*

Proof. (1)⇒(2) Let I be a z-ideal. Suppose $Z(\alpha) \subseteq Z(\beta)$ and $\alpha \in I$. Assume $coz(\beta) \not\leq coz(\alpha)$. Since L is coz-spatial, there exists a prime element $p \in L$ such that $coz(\beta) \not\leq p$ and $coz(\alpha) \leq p$. So $p \in Z(\alpha)$ and $p \notin Z(\beta)$. This contradiction shows that $coz(\beta) \leq coz(\alpha)$. Since I is a z-ideal and $\alpha \in I$, $\beta \in I$. Therefore I is a strongly z-ideal.

(2)⇒(1) Suppose $coz(\alpha) \not\leq coz(\beta)$. Let $I = \{\gamma \in \mathcal{R}L : coz(\gamma) \leq coz(\beta)\}$. Then, I is a z-ideal such that $\beta \in I$ and $\alpha \notin I$. By hypothesis I is a strongly z-ideal. Hence $Z(\beta) \not\subseteq Z(\alpha)$. So there is $p \in \Sigma L$ such that $p \in Z(\beta)$ and $p \notin Z(\alpha)$, and so $coz(\beta) \leq p$ and $coz(\alpha) \not\leq p$. Therefore L is coz-spatial.

(1)⇒(3) Suppose $Z(\alpha) = Z(\beta)$. If $coz(\alpha) \not\leq coz(\beta)$, since L is coz-spatial, there exists $p \in \Sigma L$ such that $coz(\alpha) \not\leq p$ and $coz(\beta) \leq p$, hence $Z(\alpha) \neq Z(\beta)$ and obtain a contradiction. Therefore $coz(\alpha) = coz(\beta)$.

(3)⇒(1) Suppose $coz(\alpha) \not\leq coz(\beta)$. If $Z(\beta) \subseteq Z(\alpha)$, we have $Z(\alpha\beta) = Z(\alpha)$, and hence $coz(\alpha\beta) = coz(\beta)$. Thus $coz(\alpha) \leq coz(\beta)$, which is a contradiction. So $Z(\beta) \not\subseteq Z(\alpha)$, that is to say, there exist $p \in \Sigma L$ such that $coz(\alpha) \not\leq p$ and $coz(\beta) \leq p$. Therefore L is coz-spatial. □

The next corollary can easily be deduced from Theorems 1.2 and 3.11.

Corollary 3.12. *Let L be a coz-spatial frame. An ideal I of $\mathcal{R}L$ is minimal if and only if $Z[I]$ consists only of two elements.*

For every $\alpha \in \mathcal{R}L$, we put $M_\alpha = \{\beta \in \mathcal{R}L : Z(\alpha) \subseteq Z(\beta)\}$. One can easily conclude the following proposition.

Proposition 3.13. *For every $\alpha \in \mathcal{R}L$, M_α is a strongly z-ideal of $\mathcal{R}L$.*

Proposition 3.14. *Let L be a coz-disjoint. For $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, $M_\alpha = M_p$ if and only if $Z(\alpha) = \{p\}$.*

Proof. (⇒) It is clear that $p \in Z(\alpha)$. Let $q \in Z(\alpha)$. If $\beta \in M_\alpha$, then $Z(\alpha) \subseteq Z(\beta)$. So $\beta[q] = 0$ that is to say $\beta \in M_q$. So $M_p = M_q$. Therefore, by Corollary 3.4, $p = q$. The converse is obvious. □

Proposition 3.15. *Let L be coz-disjoint. For $\alpha \in \mathcal{R}L$, M_α is a prime ideal if and only if there exists $p \in \Sigma L$ such that $M_\alpha = M_p$.*

Proof. (⇒) By Proposition 3.14, it is enough to show that $|Z(\alpha)| = 1$. If $Z(\alpha) = \emptyset$, then

$$M_\alpha = \{\beta \in \mathcal{R}L : \emptyset = Z(\alpha) \subseteq Z(\beta)\} = \mathcal{R}L\,,$$

which is a contradiction. Now, suppose p, $q \in Z(\alpha)$. Hence $M_\alpha \subseteq M_p \cap M_q$ and, by [7, Lemma 4.3] and Proposition 3.13, $M_p \cap M_q$ is a prime ideal, hence $M_p = M_q$. Therefore, by Corollary 3.4, $p = q$. By Proposition 3.2, the converse is obvious. □

4. Zero sets of minimal ideals

In the case of a coz-spatial frame L, $coz(\alpha) = coz(\beta)$ if and only if $Z(\alpha) = Z(\beta)$. So, $|\,coz[I]| = 2$ if and only if $|Z[I]| = 2$. Thus, by Theorems 1.2 and 3.11, for a coz-spatial frame L, we have I is minimal if and only if $|Z[I]| = 2$. Because coz-spatiality is rather strong, in this section the following three conditions are studied, using coz-disjoint and coz-density.

(1) I is a minimal ideal.
(2) $|Z[I]| = 2$.
(3) There exists $p \in \Sigma L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$.

We recall that a ring A is reduced if it has no nonzero nilpotent element. It is easy to check that for every frame L, $\mathcal{R}L$ is reduced. In a reduced ring A, every minimal ideal is generated by an idempotent element.

Definition 4.1. A frame L is called *coz-dense* if $\Sigma_{coz(\alpha)} = \emptyset$ implies $\alpha = \mathbf{0}$.

Recall [11] that L is *weakly spatial* if $x = \top$, whenever $\Sigma_x = \Sigma L$.

Remark 4.2.

(1) Every coz-spatial frame is coz-dense. To see this, let $\alpha \neq \mathbf{0}$, hence $coz(\alpha) \nleq coz(\mathbf{0})$. Since L is coz-spatial, there is a prime element $p \in L$ such that $coz(\alpha) \nleq p$. Therefore $\Sigma_{coz(\alpha)} \neq \emptyset$.

(2) Every weakly spatial frame is coz-dense (see Lemma 3.5 in [12]).

Proposition 4.3. *Let L be a coz-disjoint and coz-dense frame. If I is a nonzero minimal ideal of $\mathcal{R}L$, then $|Z[I]| = 2$.*

Proof. Let $\mathbf{0} \neq \delta$ be an idempotent in $\mathcal{R}L$ such that $I = \delta\mathcal{R}L$. Then $M = (\mathbf{1} - \delta)\mathcal{R}L$ is a maximal ideal hence $\delta \neq \mathbf{0}$ and is not unit. Since L is coz-dense, $\Sigma L \neq Z(\delta) \neq \emptyset$. Suppose that $p \in \Sigma L - Z(\delta)$. By Proposition 2.4, $\delta[p](\mathbf{1} - \delta)[p] = \tilde{p}(\delta(\mathbf{1} - \delta)) = 0$, which follows that $(\mathbf{1} - \delta)[p] = 0$, and hence $M_{\mathbf{1}-\delta} \subseteq M_p$. Since $M \subseteq M_{\mathbf{1}-\delta}$, we conclude that $M = M_{\mathbf{1}-\delta} = M_p$, and so, by Proposition 3.14, $Z(\mathbf{1} - \delta) = \{p\}$ and $Z(\delta) = \Sigma L - \{p\}$. Therefore $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$ and the proof is complete. □

Proposition 4.4. *Let L be a coz-disjoint frame. If I is a proper ideal of $\mathcal{R}L$ such that $|Z[I]| = 2$, then there exists $p \in \Sigma L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$.*

Proof. Let $\mathbf{0} \neq \alpha \in I$. We prove that $Z(\alpha) = \Sigma L - \{p\}$ for some $p \in \Sigma L$. Let $p, q \in \Sigma L - Z(\alpha)$ be distinct prime elements. Then, by Corollary 3.4, $M_p \neq M_q$. Hence there is $\beta \in \mathcal{R}L$ such that $\beta[p] = 0$ and $\beta[q] \neq 0$. Since, by Proposition 2.6, $Z(\alpha) \subset Z(\alpha) \cup \{p\} \subseteq Z(\alpha\beta) \in Z[I]$ and $|Z[I]| = 2$, we can conclude that $Z(\alpha\beta) = \Sigma L$. So $\alpha[q]\beta[q] = 0$, that is to say $\alpha[q] = 0$ or $\beta[q] = 0$, which is a contradiction. Therefore, there exists $p \in \Sigma L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$. □

Let $f\colon \Sigma L \to \mathbb{R}$ be a continuous function. For every p, $q \in \mathbb{Q}$, define $\widehat{f}(p, q) = \bigvee\{a \in L : f(\Sigma_a) \subseteq \langle p, q\rangle\}$, where $\langle p, q\rangle = \{x \in \mathbb{R} : p < x < q\}$. Then, we have the following lemma.

Lemma 4.5. *Let L be a frame and $f\colon \Sigma L \to \mathbb{R}$ be a continuous function. Then the following relations hold:*

(1) $\widehat{f}(p, q) \wedge \widehat{f}(r, s) = \widehat{f}(p \vee r, q \wedge s)$.

(2) $\widehat{f}(p, q) \vee \widehat{f}(r, s) = \widehat{f}(p, s)$ *whenever* $p \leq r < q \leq s$.

(3) $\widehat{f}(p, q) = \bigvee\{\widehat{f}(r, s) | p < r < s < q\}$.

(4) $\top = \bigvee\{\widehat{f}(p, q) | p, q \in \mathbb{Q}\}$.

Hence $\widehat{f}\colon \mathcal{L}(\mathbb{R}) \to L$ is a frame map.

Proof. By Lemma 2.1, it is obvious. □

Remark 4.6. Let L be a regular frame. If f, $g\colon L \to M$ are two frame maps such that for every $x \in L$, $f(x) \leq g(x)$, then $f = g$, by regularity of L. In particular, if α, $\beta \in \mathcal{R}L$ and for every r, $s \in \mathbb{Q}$, $\alpha(r, s) \leq \beta(r, s)$, then $\alpha = \beta$.

Proposition 4.7. *Let L be a frame. The map $\varphi\colon C(\Sigma L) \to \mathcal{R}L$ given by $\varphi(f) = \widehat{f}$ is an f-ring homomorphism.*

Proof. Let $f,\ g \in C(\Sigma L)$ and $\diamond \in \{+, \cdot, \vee, \wedge\}$ be an operation. We prove that $\widehat{f \diamond g} = \widehat{f} \diamond \widehat{g}$. Let $r,\ s \in \mathbb{Q}$ and $\langle t,u\rangle \diamond \langle v,w\rangle \subseteq \langle r,s\rangle$. Then

$$\begin{aligned}
\widehat{f}(t,u) \wedge \widehat{g}(v,w) &= \bigvee\{a : f(\Sigma_a) \subseteq \langle t,u\rangle\} \wedge \bigvee\{b : g(\Sigma_b) \subseteq \langle v,w\rangle\} \\
&= \bigvee\{a \wedge b : f(\Sigma_a) \subseteq \langle t,u\rangle, g(\Sigma_b) \subseteq \langle v,w\rangle\} \\
&\leq \bigvee\{a \wedge b : f \diamond g(\Sigma_a \cap \Sigma_b) \subseteq \langle t,u\rangle \diamond \langle v,w\rangle\} \\
&= \bigvee\{a \wedge b : f \diamond g(\Sigma_{a\wedge b}) \subseteq \langle t,u\rangle \diamond \langle v,w\rangle\} \\
&\leq \bigvee\{c : f \diamond g(\Sigma_c) \subseteq \langle r,s\rangle\} \\
&= \widehat{f \diamond g}(r,s)\,.
\end{aligned}$$

Hence $\widehat{f} \diamond \widehat{g}(r,s) = \bigvee\{\widehat{f}(t,u) \wedge \widehat{g}(v,w) : \langle t,u\rangle \diamond \langle v,w\rangle \subseteq \langle r,s\rangle\} \leq \widehat{f \diamond g}(r,s)$. By Remark 4.6, $\widehat{f \diamond g} = \widehat{f} \diamond \widehat{g}$. □

Remark 4.8. Let L be a frame and $t \in \mathbb{R}$. We can easily check $\widehat{\mathbf{t}} = \mathbf{t}$, in particular $\widehat{\mathbf{1}} = \mathbf{1}$. Thus for every $f,\ g \in C(\Sigma L)$, $\widehat{f + tg} = \widehat{f} + \widehat{\mathbf{t}}\widehat{g} = \widehat{f} + t\widehat{g}$. Therefore φ is a linear function.

Lemma 4.9. *Let L be a frame, $f\colon \Sigma L \to \mathbb{R}$ be a continuous function, $p \in \Sigma L$, and $r,\ s \in \mathbb{Q}$. Then $r < f(p) < s$ if and only if there exists $a \in L$ such that $a \not\leq p$ and $f(\Sigma_a) \subseteq \langle r,s\rangle$.*

Proof. Assume that $r < f(p) < s$. Since f is continuous, $f^{-1}(\langle r,s\rangle)$ is an open set in ΣL. Thus there is $a \in L$ such that $f^{-1}(\langle r,s\rangle) = \Sigma_a$. Therefore $f(\Sigma_a) \subseteq \langle r,s\rangle$ and $p \in \Sigma_a$, and so, $a \not\leq p$. The converse is obvious. □

Remark 4.10. There is a homeomorphism $\tau\colon \Sigma\mathcal{L}(\mathbb{R}) \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r,s) \not\leq p$ for all prime elements p of $\mathcal{L}(\mathbb{R})$ and all $r,\ s \in \mathbb{Q}$ (see Proposition 1 of [4, page 12]).

Proposition 4.11. *Let L be a frame and $f\colon \Sigma L \to \mathbb{R}$ be a continuous function. Then $\tau \circ \Sigma\widehat{f} = f$.*

Proof. Let $p \in \Sigma L$ and $r,\ s \in \mathbb{Q}$ such that $r < f(p) < s$. By Lemma 4.9, there exists $a \in L$ such that $a \not\leq p$ and $f(\Sigma_a) \subseteq \langle r,s\rangle$. Thus

$$\widehat{f}(r,s) = \bigvee\{a : f(\Sigma_a) \subseteq \langle r,s\rangle\} \not\leq p\,.$$

Since $\Sigma\widehat{f}(p) = \widehat{f}_*(p)$, where $\widehat{f}_*$ is the right adjoint of $\widehat{f}$, $(r,s) \not\leq \Sigma\widehat{f}(p)$. Thus, by Remark 4.10, $r < \tau(\Sigma\widehat{f}(p)) < s$, and so $f(p) = \tau(\Sigma\widehat{f}(p))$. Therefore $f = \tau \circ \Sigma\widehat{f}$. □

According to the foregoing proposition, we can immediately conclude the following corollary.

Corollary 4.12. *The f-ring homomorphism $\varphi\colon C(\Sigma L) \to \mathcal{R}L$ is a monomorphism.*

Proposition 4.13. *Let L be a frame and $f\colon \Sigma L \to \mathbb{R}$ be a continuous function, then $Z(\widehat{f}) = Z(f)$.*

Proof. Assume that $f(p) = 0$. Let $a \in L$ be such that $f(\Sigma_a) \subseteq \langle -\infty, 0\rangle \cup \langle 0, +\infty\rangle$. Hence, for every $q \in \Sigma_a$, $f(q) \neq 0$. Thus $p \notin \Sigma_a$, and so $a \leq p$. Hence $coz(\widehat{f}) = \bigvee\{a : f(\Sigma_a) \subseteq \langle -\infty, 0\rangle \cup \langle 0, +\infty\rangle\} \leq p$. Therefore, by Lemma 2.5, $p \in Z(\widehat{f})$. Conversely, suppose $f(p) \neq 0$. Since f is continuous, there exists $a \in L$ such that $p \in \Sigma_a$ and $f(\Sigma_a) \subseteq \langle -\infty, 0\rangle \cup \langle 0, +\infty\rangle$. So $a \not\leq p$ and $a \leq coz(\widehat{f})$, and hence $coz(\widehat{f}) \not\leq p$. Thus $p \notin Z(\widehat{f})$. Therefore $Z(\widehat{f}) = Z(f)$. □

Notation 4.14. *Let $p \in \Sigma L$ be an isolated point. We define*

$$\chi_p(x) = \begin{cases} 1, & x = p \\ 0, & x \in \Sigma L - \{p\}\,. \end{cases}$$

It is clear that $\chi_p\colon \Sigma L \to \mathbb{R}$ is a continuous map. Define $\epsilon_p = \widehat{\chi_p} \in \mathcal{R}L$.

Lemma 4.15. *Let L be a frame and $p \in \Sigma L$ be an isolated point. Then*

(1) *ϵ_p is an idempotent element of $\mathcal{R}L$.*

(2) *$Z(\epsilon_p) = \Sigma L - \{p\}$.*

(3) *Assume that L is a coz-dense frame. For every $\alpha \in \mathcal{R}L$, if $Z(\alpha) = \Sigma L - \{p\}$, then $\epsilon_p \alpha = \alpha$.*

Proof. (1) By Proposition 4.7, $\epsilon_p^2 = \widehat{\chi_p}^2 = \widehat{\chi_p^2} = \widehat{\chi_p} = \epsilon_p$.

(2) By Proposition 4.13, $Z(\epsilon_p) = Z(\chi_p) = \Sigma L - \{p\}$.

(3) By Remark 4.8, $\epsilon_p - \mathbf{1} = \widehat{\chi_p - \mathbf{1}}$, and hence $Z(\epsilon_p - \mathbf{1}) = Z(\chi_p - \mathbf{1}) = \{p\}$, by Proposition 4.13. Now, let $\beta = \alpha(\epsilon_p - \mathbf{1})$. Then, by Proposition 2.6, $Z(\beta) = Z(\alpha) \cup Z(\epsilon_p - \mathbf{1}) = \Sigma L$. Thus, for every prime element $q \in L$, $coz(\beta) \leq q$. Since L is coz-dense, $\beta = \mathbf{0}$. Therefore, $\epsilon_p \alpha = \alpha$. □

Remark 4.16. Let $\alpha \in \mathcal{R}L$. Define $\overline{\alpha}\colon \Sigma L \to \mathbb{R}$ by $\overline{\alpha}(p) = \widetilde{p}(\alpha)$. Then, there is an f-ring homomorphism $\psi\colon \mathcal{R}L \to C(\Sigma L)$ given by $\psi(\alpha) = \tau \circ \Sigma\alpha$, by [10, Proposition 3.9]. Moreover, if $p \leq q$, then $\psi(\alpha)(p) = \tilde{q}(\alpha)$ for every $\alpha \in \mathcal{R}L$. In particular, $(\tau \circ \Sigma\alpha)(p) = \psi(\alpha)(p) = \tilde{p}(\alpha)$ for every $p \in \Sigma L$. Therefore $\overline{\alpha} = \tau \circ \Sigma\alpha$, so $\overline{\alpha}$ is continuous.

Proposition 4.17. *If L is a coz-dense frame, then $\psi\colon \mathcal{R}L \to C(\Sigma L)$ is an isomorphism, and $\psi^{-1} = \varphi$.*

Proof. First we show that ψ is a monomorphism. Let $\overline{\alpha} = \mathbf{0}$. Then for every $p \in \Sigma L$, we have $coz(\alpha) \leq p$. Since L is coz-dense, $\alpha = \mathbf{0}$. Therefore ψ is a monomorphism and, by Proposition 4.11, ψ is onto. So ψ is an isomorphism. Also, for every $f \in C(\Sigma L)$, $\psi(\widehat{f}) = f$, by Proposition 4.11. Hence, $\psi \circ \varphi = \mathrm{id}$, that is to say $\psi^{-1} = \varphi$. □

Corollary 4.18. *Let L be a coz-dense frame and p be an isolated point of ΣL. If I is a nonzero z-ideal of $\mathcal{R}L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$, then I is a minimal ideal.*

Proof. Let $\mathbf{0} \neq \alpha \in I$. Then $Z(\alpha) = \Sigma L - \{p\}$. Therefore, by Lemma 4.15 (3), $\alpha = \epsilon_p \alpha \in \epsilon_p \mathcal{R}L$, and hence $I \subseteq \epsilon_p \mathcal{R}L$. It is enough to show that $\epsilon_p \mathcal{R}L$ is a minimal ideal. To see this, let $0 \neq J \subseteq \epsilon_p \mathcal{R}L$. If $\mathbf{0} \neq \beta \in J$, then $\beta \in \epsilon_p \mathcal{R}L$, hence $Z(\beta) \supseteq Z(\epsilon_p) = \Sigma L - \{p\}$, and so $Z(\beta) = \Sigma L - \{p\}$. Therefore $\beta = \epsilon_p \beta$. Let $k = \overline{\beta}(p) \neq 0$. By Proposition 4.17, $\overline{\epsilon}_p = \chi_p = \frac{\mathbf{1}}{\mathbf{k}}(\overline{\beta}) = \overline{\frac{\mathbf{1}}{\mathbf{k}}\beta}$, and also $\epsilon_p = \frac{\mathbf{1}}{\mathbf{k}}\beta \in J$. Thus, $J = \epsilon_p \mathcal{R}L$. Therefore $\epsilon_p \mathcal{R}L$ is a minimal ideal. Since $I \subseteq \epsilon_p \mathcal{R}L$, $I = \epsilon_p \mathcal{R}L$ is a minimal ideal. □

By Propositions 4.3, 4.4 and foregoing corollary, we can easily prove the following theorem.

Theorem 4.19. *Let L be a coz-disjoint and coz-dense frame and I be a nonzero ideal of $\mathcal{R}L$. Then the following statements are equivalent.*

(1) *I is a minimal ideal.*

(2) *$|Z[I]| = 2$.*

(3) *There exists $p \in \Sigma L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$.*

In Theorem 1.1 the minimal ideals of $C(X)$ are characterized by zero sets and isolated points of X. This is extended by Dube in Theorem 1.2 to arbitrary $\mathcal{R}L$, where the minimal ideals of $\mathcal{R}L$ are characterized by cozero elemens and atoms of L. The approaches in Theorem 1.1 and Theorem 1.2 are completely different. The approach used in the proof of Theorem 4.19 is similar to the proof of Theorem 1.1. This approach enables us to introduce the new concepts of coz-disjointness, coz-spatiality and coz-density, and find a relation between the rings $\mathcal{R}L$ and $C(\Sigma L)$. These concepts can be useful in further research; for example, by coz-density, we present a description of the socle of the ring $\mathcal{R}L$ based on minimal ideals of $\mathcal{R}L$ and zero sets in pointfree topology (for more details, see [13]).

For any $\top \neq a \in L$, the set $\mathcal{R}(a) = \{\alpha \in \mathcal{R}L : coz(\alpha) \leq a\}$ is an ideal of $\mathcal{R}L$. Let L be a completely regular frame. An ideal of $\mathcal{R}L$ is minimal if and only if it is of the form $\mathcal{R}(a)$, for some atom a of L ([9, Lemma 3.4]).

Proposition 4.20. *Let L be a coz-dense frame. If p is an isolated point of ΣL, then $\bigwedge \Sigma L - \{p\}$ is an atom of L.*

Proof. Let $a = \bigwedge \Sigma L - \{p\}$. If $\alpha \in \mathcal{R}(a)$, then for every $q \in \Sigma L - \{p\}$, $coz(\alpha) \leq q$, and so, by Lemma 2.5, $Z(\alpha) = \Sigma L$ or $Z(\alpha) = \Sigma L - \{p\}$. Thus $Z[\mathcal{R}(a)] = \{\Sigma L, \Sigma L - \{p\}\}$, and hence, by Corollary 4.18, $\mathcal{R}(a)$ is a minimal ideal of $\mathcal{R}L$. Therefore a is an atom of L. □

Proposition 4.21. *Let L be a coz-disjoint and coz-dense frame. If a is an atom of L, then there exists an isolated point p of ΣL such that $\{q \in \Sigma L : a \leq q\} = \Sigma L - \{p\}$.*

Proof. Let a be an atom of L. Then $\mathcal{R}(a) = \{\alpha \in \mathcal{R}L : coz(\alpha) \leq a\}$ is a minimal ideal of $\mathcal{R}L$, and so, by Theorem 4.19, there exists $p \in \Sigma L$ such that

$Z[\mathcal{R}(a)] = \{\Sigma L, \Sigma L - \{p\}\}$. If $\mathbf{0} \neq \alpha \in \mathcal{R}(a)$ and $a \leq q \in \Sigma L$, then, by Lemma 2.5, $q \in Z(\alpha)$. Hence $\{q \in \Sigma L : a \leq q\} \subseteq \Sigma L - \{p\}$. Let $q \in \Sigma L - \{p\}$ and $\mathbf{0} \neq \alpha \in \mathcal{R}(a)$. Then $\alpha[q] = 0$, and hence, by Lemma 2.5, $\bot \neq coz(\alpha) \leq a \wedge q$. Since a is an atom of L, $a \leq q$. □

Corollary 4.22. *Assume that L is a coz-disjoint and coz-dense frame. Then ΣL has an isolated point if and only if L has an atom. Moreover, the assignment $p \mapsto \bigwedge \Sigma L - \{p\}$ gives a bijection between the set of isolated points of ΣL and the set of atoms of L.*

Proof. Suppose that $\bigwedge \Sigma L - \{p\} = \bigwedge \Sigma L - \{q\} = a$. If $p \neq q$, then $p \in \Sigma L - \{q\}$. So $a \leq p$. Thus $\bigwedge \Sigma L = p \wedge \bigwedge \Sigma L - \{p\} = a$, and hence $a \leq r$ for all $r \in \Sigma L$, Since L is coz-dense, $a = 0$, which is a contradiction. This proves that the correspondence is one-one. It suffices to prove ontoness. Let a be an atom. Then, by Proposition 4.21, there is an isolated point $p \in \Sigma L$ such that $\{q \in \Sigma L : a \leq q\} = \Sigma L - \{p\}$, and so $a \leq \bigwedge \Sigma L - \{p\} = b$. By Proposition 4.20, b is an atom. Therefore $a = b = \Sigma L - \{p\}$. □

Remark 4.23. Here, we explain the relations among coz-spatial, coz-dense, coz-disjoint, weakly spatial, spatial and completely regular conditions. By Remark 3.10, being spatial implies being coz-spatial (weakly spatial) and, by Remark 4.2, being coz-spatial (or weakly spatial) implies being coz-dense. None of the conditions (coz-spatial, weakly spatial, and coz-disjoint) can imply either being spatial or completely regular. Also, by Proposition 3.5, completely regular implies coz-disjoint. Finally, coz-spatial does not imply weakly spatial. To see this, let L and M be two frames such that $L \cap M = \emptyset$ and $\Sigma L = \Sigma M = \emptyset$. Let $K = L \cup M$. For every $x \in L$ and $y \in M$ define $x < y$. So K is a frame such that $\Sigma K = \{\top_L\}$ and $coz(K) = \{\bot = \bot_L, \top = \top_M\}$. Then, K is not weakly spatial, but it is coz-spatial.

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