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A COMPLEMENT OF POSITIVE WEAK ALMOST DUNFORD-PETTIS OPERATORS ON BANACH LATTICES

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Abstract. In this paper, we give some necessary and sufficient conditions such that each positive operator between two Banach lattices is weak almost Dunford-Pettis, and we derive some interesting results about the weak Dunford-Pettis property in Banach lattices.

1. Introduction and notation

A norm bounded subset A of a Banach lattice E is said to be almost Dunford-Pettis set, if every disjoint weakly null sequence (f_n) in E' converges uniformly to zero on A, that is, $\lim_{n\to\infty} \sup_{x\in A} f_n(x) = 0$.

Recall from [3] an operator $T\colon X \to F$ from a Banach space X into a Banach lattice F is called weak almost Dunford-Pettis if T carries each relatively weakly compact set in X to an almost Dunford-Pettis set in F, equivalently, whenever $f_n(T(x_n)) \to 0$ for every weakly null sequence (x_n) in X and for every disjoint weakly null sequence (f_n) in F'.

A Banach space X has the weak Dunford-Pettis property (wDP property for short), if every relatively weakly compact set in E is almost Dunford-Pettis, equivalently, whenever $f_n(x_n) \to 0$ for every weakly null sequence (x_n) in E and for every disjoint weakly null sequence (f_n) in E' (see Corollary 2.6 of [3]). Note that the Banach lattice E has the wDP property if, and only if, the identity operator $I_E\colon E \to E$ is weak almost Dunford-Pettis.

The sequence (x_n) of a Banach lattice E is disjoint if $|x_n| \wedge |x_m| = 0$ whenever $n \neq m$ (we denote by $x_n \perp x_m$). A lattice seminorm ρ on a Banach lattice E is a seminorm such that for every x, $y \in E$, $|x| \leq |y|$ we have $\rho(x) \leq \rho(y)$.

An operator $T\colon E \to F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. If $T\colon E \to F$ is a positive operator between two Banach lattices, then its adjoint $T'\colon F' \to E'$, defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$, is also positive. For the theory of Banach lattices and positive operators, we refer the reader to monographs [1, 4].

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Recently, Bouras and Moussa [3] introduced and studied the class of positive weak almost Dunford-Pettis operators on Banach lattices, and give some characterizations of this class of operators in terms of sequences. In this paper, using the notion of lattice seminorm on Banach lattice, we give some necessary and sufficient conditions such that each positive operator is weak almost Dunford-Pettis (Theorem 2.3 and Theorem 2.5), and we establish a new characterization of positive weak almost Dunford-Pettis operator (Theorem 2.7). Next, we derive some consequences about the weak Dunford-Pettis property in Banach lattices (Corollaries 2.4, 2.6 and 2.8).

2. Main results

We start this article by a characterization of order bounded weak almost Dunford-Pettis operator between two Banach lattices.

Proposition 2.1. *Let $T\colon E \to F$ be an order bounded operator from a Banach lattice E into another Banach lattice F. Then the following statements are equivalent:*

(1) *T is weak almost Dunford-Pettis operator.*
(2) *T carries the solid hull of each relatively weakly compact subset of E to an almost Dunford-Pettis subset of F.*

Proof. (2) $\Rightarrow$ (1) Obvious.
(1) $\Rightarrow$ (2) Let W be a relatively weakly compact subset of E, and (f_n) be a disjoint weakly null sequence in F'. Let (x_n) be a disjoint sequence in $S(W)^+ = S(W) \cap E^+$ where $S(W)$ is the solid hull of W, then from Theorem 4.34 of [1] $x_n \xrightarrow{w} 0$ in E. Thus, by our hypothesis we see that $f_n(Tx_n) \to 0$ for every disjoint sequence (x_n) in $S(W)^+$ and every disjoint weakly null sequence (f_n) in F', and by Theorem 2.4 of [2] we get that $T(S(W))$ is an almost Dunford-Pettis subset of F, and this completes the proof. □

In order to prove the next theorem, we need the following lemma.

Lemma 2.2. *Let E be a Banach lattice, and let (x_n) be a norm bounded sequence in E^+. Then the sequence defined for $n \geq 2$ by*

$$y_n = \Big(x_n - 4^n \sum_{i=1}^{n-1} x_i - 2^{-n} \sum_{i=1}^{\infty} 2^{-i} x_i\Big)^+,$$

is a disjoint sequence of E^+.

Proof. Let $n > m \geq 2$, then

$$0 \leq y_n \leq (x_n - 4^n x_m)^+ ,$$

and

$$\begin{aligned} 0 \leq 4^n y_m &\leq 4^n (x_m - 4^{-n} x_n)^+ \\ &= (4^n x_m - x_n)^+ \\ &= (x_n - 4^n x_m)^- . \end{aligned}$$

Since $(x_n - 4^n x_m)^+ \perp (x_n - 4^n x_m)^-$, we conclude that $y_n \perp y_m$. □

Let $T\colon E \to F$ be a positive operator between two Banach lattices E and F, and let B be a norm bounded subset of F', note that $\rho_{T',B}(x) = \sup\{T'|f|(|x|)\colon f \in B\}$ defines a lattice seminorm on E. For $T = I_E$ we have the lattice seminorm on E defined by $\rho_B(x) = \sup\{|f|(|x|)\colon f \in B\}$.

The next result gives some necessary conditions such that the adjoint T' of a positive operator T is weak almost Dunford-Pettis.

Theorem 2.3. *Let E and F be two Banach lattices, and a positive operator T from E into F. If the adjoint operator T' from F' into E' is weak almost Dunford-Pettis, then for every relatively weakly compact set B in F', for each weakly null sequence (x_n) in E and for each $\epsilon > 0$ there exist y in E^+ and a natural number m such that*

$$\rho_{T',B}\left((|x_n| - y)^+\right) \leq \epsilon$$

holds for every $n > m$.

Proof. Assume by way of contradiction that there exist a relatively weakly compact subset B in F', a weakly null sequence (x_n) in E and some $\epsilon > 0$ such that for each y in E^+ and each natural number m we have

$$\rho_{T',B}\left((|x_n| - y)^+\right) > \epsilon$$

for at least one $n > m$, hence $(|x_n| - y)^+(T|f_n|) > \epsilon$ for at least one $f_n \in B$. Now, an easy inductive argument shows that there exist a subsequence of (x_n) (which we shall denote by (x_n) again) and a sequence $(f_n) \subset B$ such that

$$T'|f_n|\left(\left(|x_n| - 4^n \sum_{i=1}^{n-1} |x_i|\right)^+\right) > \epsilon$$

holds for all $n \geq 2$. Put $x = \sum_{i=1}^{\infty} 2^{-i}|x_i|$ and $y_n = (|x_n| - 4^n \sum_{i=1}^{n-1}|x_i| - 2^{-n}x)^+$. By Lemma 2.2 the sequence (y_n) is disjoint. As $0 \leq y_n \leq |x_n|$ and (x_n) is a weakly null sequence in E, so from Lemma 4.34 of [1], we see that, $y_n \xrightarrow{w} 0$. Note that the canonical injection $\tau : E \to E''$ is a lattice homomorphism, we obtain $\tau(y_n)$ is a disjoint weakly null sequence in E''. Since T' is positive weak almost Dunford-Pettis operator, then from Proposition 2.1 we obtain that $T'(S(B))$ is an almost Dunford-Pettis set in E' (where $S(B)$ is the solid hull of B), we get that $T'|f_n|(y_n) = \tau(y_n)(T'|f_n|) \to 0$. Now, we have

$$\begin{aligned} 0 < \epsilon < T'|x_n|\left(\left(|x_n| - 4^n \sum_{i=1}^{n-1}|x_i|\right)^+\right) \\ \leq T'|f_n|(y_n) + T'|f_n|(2^{-n}x) \to 0 \end{aligned}$$

which is impossible, and this completes the proof. □

Corollary 2.4. *Let E be a Banach lattice. If the topological dual Banach lattice E' has the wDP property, then for every relatively weakly compact set B in E', for each weakly null sequence (x_n) in E and for each $\epsilon > 0$ there exist y in E^+ and a natural number m such that*

$$\rho_B\left((|x_n| - y)^+\right) \leq \epsilon$$

holds for every $n > m$.

The following Theorem gives some sufficient conditions such that a positive operator between two Banach lattices is weak almost Dunford-Pettis.

Theorem 2.5. *Let E and F be two Banach lattices, and a positive operator T from E into F. Suppose that for every relatively weakly compact set B in F', each weakly null sequence (x_n) in E and each $\epsilon > 0$ there exist y in E^+ and a natural number m such that*

$$\rho_{T',B}\left((|x_n| - y)^+\right) \leq \epsilon$$

holds for every $n > m$. Then, T is weak almost Dunford-Pettis.

Proof. Let (x_n) be a disjoint weakly null sequence in E and (f_n) be a disjoint weakly null sequence in F'. Put $B = \{f_n : n \in N\}$ and let $\epsilon > 0$ by our hypothesis there exist y in E^+ and a natural number i such that

$$T'|f_n|\left((|x_n| - y)^+\right) \leq \rho_{T',B}\left((|x_n| - y)^+\right) \leq \frac{\epsilon}{2}$$

holds for all $n > i$.

On the other hand, in view of Lamma 4.34 of [1] we see that $|f_n| \xrightarrow{w} 0$ in F', and hence there exists some natural j such that $T'|f_n|(y) \leq \frac{\epsilon}{2}$ holds for every $n \geq j$.

Thus,

$$\begin{aligned} |f_n(Tx_n)| &\leq T'|f_n|(|x_n|) \\ &\leq T'|f_n|\left((|x_n| - y)^+\right) + T'|f_n|(y) \\ &\leq \epsilon \end{aligned}$$

for every $n \geq m = \sup\{i, j\}$, this prove that $f_n(Tx_n) \to 0$, and from Theorem 2.5 of [3], we conclude that T is positive weak almost Dunford-Pettis. □

Corollary 2.6. *Let E be a Banach lattice. Suppose that for every relatively weakly compact set B in E', each weakly null sequence (x_n) in E and each $\epsilon > 0$ there exist y in E^+ and a natural number m such that*

$$\rho_B\left((|x_n| - y)^+\right) \leq \epsilon$$

holds for every $n > m$. Then, E has the wDP property.

Let $T\colon E \to F$ be a positive operator between two Banach lattices E and F. Let A be a norm bounded subset of E, then it is clear that $\rho_{T,A}(f) = \sup\{|f|(T|x|)\colon x \in A\}$ defines a lattice seminorm on F'. For the identity operator $I_E\colon E \to E$, we have the lattice seminorm on E' defined by $\rho_A(f) = \sup\{|f|(|x|)\colon x \in A\}$. Using the concept of the lattice seminorm $\rho_{T,A}$ in Banach lattice F', we obtain a new characterization of positive weak almost Dunford-Pettis operator.

Theorem 2.7. *Let E and F be two Banach lattices. Then, a positive operator T from E into F is weak almost Dunford-Pettis if, and only if, for every relatively* *weakly compact set A in E, for each weakly null sequence (f_n) in F' and for each $\epsilon > 0$ there exist g in $(F')^+$ and a natural number m such that*

$$\rho_{T,A}\left((|f_n| - g)^+\right) \leq \epsilon$$

holds for every $n > m$.

Proof. For the "if" part, let (x_n) be a disjoint weakly null sequence in E and (f_n) be a disjoint weakly null sequence in F'. Put $A = \{x_n : n \in N\}$ and let $\epsilon > 0$ by our hypothesis there exist g in $(F')^+$ and a natural number m_1 such that

$$(|f_n| - g)^+(T\,|x_n|) \leq \rho_{T,A}\left((|f_n| - g)^+\right) \leq \frac{\epsilon}{2}$$

holds for all $n > m_1$.

On the other hand, in view of Lemma 4.34 of [1] we see that $|x_n| \xrightarrow{w} 0$, and hence there exists some natural m_2 such that $g(T(|x_n|)) \leq \frac{\epsilon}{2}$ holds for every $n > m_2$.

Thus,

$$\begin{aligned} |f_n(Tx_n)| &\leq |f_n|\,(T\,|x_n|) \\ &\leq (|f_n| - g)^+(T\,|x_n|) + g(T\,|x_n|) \\ &\leq \epsilon \end{aligned}$$

for every $n > m = \sup\{m_1, m_2\}$, this prove that $f_n(Tx_n) \to 0$, and from Theorem 2.5 of [3], we conclude that T is positive weak almost Dunford-Pettis.

For the "only if" part, assume by way of contradiction that there exist a relatively weakly compact subset A in E, a weakly null sequence (f_n) in F' and some $\epsilon > 0$ such that for each g in $(F')^+$ and each natural number m we have

$$\rho_{T,A}\left((|f_n| - g)^+\right) > \epsilon$$

for at least one $n > m$, hence $(|f_n| - g)^+(T\,|x_n|) > \epsilon$ for at least one $x_n \in A$. Now, an easy inductive argument shows that there exist a subsequence of (f_n) (which we shall denote by (f_n) again) and a sequence $(x_n) \subset A$ such that

$$\left(|f_n| - 4^n \sum_{i=1}^{n-1} |f_i|\right)^+ (T\,|x_n|) > \epsilon$$

holds for all $n \geq 2$. Put $f = \sum_{i=1}^{\infty} 2^{-i}\,|f_i|$ and $g_n = (|f_n| - 4^n \sum_{i=1}^{n-1} |f_i| - 2^{-n} f)^+$. By Lemma 2.2 the sequence (g_n) is disjoint of $(F')^+$. As $0 \leq g_n \leq |f_n|$ and (f_n) is a weakly null sequence in F', so from Lemma 4.34 of [1], we see that, $g_n \xrightarrow{w} 0$. Since T is positive weak almost Dunford-Pettis operator, then by Proposition 2.1 we have $T(S(A))$ is an almost Dunford-Pettis set in F (where $S(A)$ is the solid hull of A), we get that $g_n(T\,|x_n|) \to 0$. Now, we have

$$\begin{aligned} 0 < \epsilon &< \left(|f_n| - 4^n \sum_{i=1}^{n-1} |f_i|\right)^+ (T\,|x_n|) \\ &\leq g_n(T\,|x_n|) + 2^{-n} f(T\,|x_n|) \to 0 \end{aligned}$$

which is impossible, and this completes the proof. □

As a simple consequence of Theorem 2.7, we obtain the following result.

Corollary 2.8. *A Banach lattice E has the wDP property if, and only if, for every relatively weakly compact set A in E, for each weakly null sequence (f_n) in E' and for each $\epsilon > 0$ there exist g in $(E')^+$ and a natural number m such that*

$$\rho_A\left((|f_n| - g)^+\right) \leq \epsilon$$

holds for every $n > m$.

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