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ON A DECOMPOSITION OF NON-NEGATIVE RADON MEASURES

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Abstract. We establish a decomposition of non-negative Radon measures on $\mathbb{R}^d$ which extends that obtained by Strichartz [6] in the setting of α-dimensional measures. As consequences, we deduce some well-known properties concerning the density of non-negative Radon measures. Furthermore, some properties of non-negative Radon measures having their Riesz potential in a Lebesgue space are obtained.

1. Introduction – main results

Let d be a positive integer. Let $0 < \theta \leq 1$. We denote by dx the Lebesgue measure on $\mathbb{R}^d$. For any Lebesgue measurable subset E of $\mathbb{R}^d$, $|E|$ stands for its Lebesgue measure. For $1 \leq p < \infty$, $\|\cdot\|_p$ denotes the usual norm on the classical Lebesgue space $L^p = L^p(\mathbb{R}^d)$. The $d\theta$-dimensional Hausdorff measure on $\mathbb{R}^d$ is denoted by $\mathcal{H}_{d\theta}$ (see Section 2 for the definition of this measure and some of its basic properties). If μ is a measure on $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$, we denote by $\mu\lfloor A$ the restriction of μ to A.

A Borel measure μ on $\mathbb{R}^d$ is locally uniformly $d\theta$-dimensional if there exists a constant $C > 0$ such that

$$\mu\big(B(x,r)\big) \leq Cr^{d\theta}\,,$$

for every open ball $B(x,r)$ centered at x with radius $r \leq 1$.

This definition easily implies that μ is absolutely continuous with respect to $\mathcal{H}_{d\theta}$, but since $\mathcal{H}_{d\theta}$ is not σ-finite, the Radon-Nikodym theorem does not apply. Instead Strichartz proved in [6] the following substitute.

Proposition 1.1. *If μ is a locally uniformly $d\theta$-dimensional measure, then there exists a function φ and a measure ν such that $\mu = \varphi d\mathcal{H}_{d\theta} + \nu$, where ν has the property $\mathcal{H}_{d\theta}(A) < \infty$ implies $\nu(A) = 0$ for any Borel subset A of $\mathbb{R}^d$.*

Next, he gave the following definition motivated by Proposition 1.1.

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Definition 1.2. Let ν and μ be two Borel measures on $\mathbb{R}^d$. The measure ν is null with respect to μ on $\mathbb{R}^d$ and we will denote this with $\nu \lll \mu$, if for any Borel subset A of $\mathbb{R}^d$,

$$\mu(A) < \infty \Rightarrow \nu(A) = 0 \,.$$

In [6] the author established the following result concerning the density of non-negative Radon measures that are null with respect to $\mathcal{H}_{d\theta}$.

Proposition 1.3. *Suppose that ν is a non-negative Radon measure on $\mathbb{R}^d$. If $\nu \lll \mathcal{H}_{d\theta}$ then*

$$\limsup_{r\to 0}\ r^{-d\theta}\nu(B(x,r)) = 0$$

for $\mathcal{H}_{d\theta}$-almost every x.

A generalization of Proposition 1.1 and Proposition 1.3 was obtained in [4].

In the present note, we establish the following decomposition of non-negative Radon measures.

Proposition 1.4. *Suppose that μ is a non-negative Radon measure on $\mathbb{R}^d$. Let us consider the following subsets of $\mathbb{R}^d$:*

$$N_\theta = \left\{x \in \mathbb{R}^d : \limsup_{r\to 0} r^{-d\theta}\mu(B(x,r)) = 0\right\},$$

$$P_\theta = \left\{x \in \mathbb{R}^d : 0 < \limsup_{r\to 0} r^{-d\theta}\mu(B(x,\ r)) < \infty\right\},$$

$$E_\theta^\infty = \left\{x \in \mathbb{R}^d : \limsup_{r\to 0}\ r^{-d\theta}\mu(B(x,r)) = \infty\right\}.$$

Then $\mu = \mu\lfloor N_\theta + \mu\lfloor P_\theta + \mu\lfloor E_\theta^\infty$ and

(i) *for any Borel set $F \subset N_\theta$, $\mathcal{H}_{d\theta}(F) < \infty \Rightarrow \mu(F) = 0$,*

(ii) *P_θ is $\mathcal{H}_{d\theta}$ σ-finite and for any Borel set $F \subset P_\theta$, $\mathcal{H}_{d\theta}(F) = 0 \Rightarrow \mu(F) = 0$,*

(iii) *$\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$.*

The remark below shows that in the setting of non-negative Radon measures, Proposition 1.1 derives from Proposition 1.4.

Remark 1.5. If μ is a non-negative locally uniformly $d\theta$-dimensional measure, then

$$0 \le \limsup_{r\to 0} r^{-d\theta}\mu\big(B(x,r)\big) < \infty \,.$$

Therefore, by applying Proposition 1.4, μ has the following decomposition: $\mu = \mu\lfloor N_\theta + \mu\lfloor P_\theta$. In addition, on P_θ, $\mathcal{H}_{d\theta}$ and μ are σ-finite and μ is absolutely continuous with respect to $\mathcal{H}_{d\theta}$. Therefore, according to the Radon-Nikodym theorem there exists a function $\varphi \ge 0$ such that for all Borel sets $E \subset P_\theta$, we have

$$\mu(E) = \int_E \varphi(x)\ d\mathcal{H}_{d\theta}(x) \,.$$

As immediate consequences of Proposition 1.4, we have Proposition 1.3 and the following result.

Corollary 1.6. *Suppose that $0 < \theta < 1$ and μ is a non-negative Radon measure on $\mathbb{R}^d$ that is absolutely continuous with respect to the Lebesgue measure. Let*

$$A = \left\{ x \in \mathbb{R}^d : \limsup_{r \to 0} \ r^{-d\theta} \mu\big(B(x,r)\big) > 0 \right\}.$$

Then $\mathcal{H}_{d\theta}(A) = 0$.
In particular, for $u \in L^p$, $1 \leq p < \infty$, if E is defined by

$$E = \left\{ x \in \mathbb{R}^d : \limsup_{r \to 0} \ r^{-d\theta} \int_{B(x,r)} |u(x)|^p dx > 0 \right\},$$

then $\mathcal{H}_{d\theta}(E) = 0$.

Let us stress that Corollary 1.6 was already established in [8] in order to investigate the Lebesgue points for Sobolev functions.

For $0 < \gamma < 1$, we define the Riesz potential operator I_γ by

$$I_\gamma \mu(x) = \int_{\mathbb{R}^d} |x - y|^{d(\gamma - 1)} d\mu(y), \quad x \in \mathbb{R}^d,$$

for any suitable Radon measure μ on $\mathbb{R}^d$.
The next results that give some properties of non-negative Radon measures having their Riesz potential in a classical Lebesgue space also arise from Proposition 1.4.

Proposition 1.7. *Suppose that $0 < \gamma < 1$ and $1 < p < \infty$. Then for any non-negative Radon measure μ on $\mathbb{R}^d$ satisfying $I_\gamma \mu \in L^p$, we have*

$$\lim_{r \to 0} \int_{\mathbb{R}^d} \Big(r^{d(\gamma-1)} \mu\big(B(x,r)\big) \Big)^p dx = 0.$$

Proposition 1.8. *Suppose that $\frac{d}{d-1} < p < \infty$ and μ is a non-negative Radon measure. Then we have*

$$I_{\frac{1}{d}} \mu \in L^p \Rightarrow \lim_{r \to 0} r^{p'-d} \mu(B(x,r)) = 0 \quad \mu\text{-almost everywhere},$$

where $p' = \frac{p}{p-1}$.

Notice that Proposition 1.7 and Proposition 1.8 are related to the solvability in $L^p(\mathbb{R}^d, \mathbb{R}^d)$ of the equation

$$\operatorname{div} F = \mu \tag{1}$$

with measure data μ.
Indeed, Phuc and Torrès have obtained the following criterion.

Proposition 1.9 ([5])**.** *Suppose that μ is a non-negative Radon measure on $\mathbb{R}^d$ and $\frac{d}{d-1} < p < \infty$. Then the following conditions are equivalent:*
(i) *Equation* (1) *has a solution in $L^p(\mathbb{R}^d, \mathbb{R}^d)$.*
(ii) *$I_{\frac{1}{d}} \mu$ belongs to $L^p(\mathbb{R}^d, \mathbb{R})$.*

The remainder of this paper is organized as follows. In Section 2 we prove Proposition 1.4, Proposition 1.3 and Corollary 1.6. Section 3 is devoted to the proof of Proposition 1.7. In Section 4 we establish the proof of Proposition 1.8.

2. Proofs of Proposition 1.4, Proposition 1.3 and Corollary 1.6

In the sequel, we shall use the following notation.

Notation 2.1. For any non-empty subset B of $\mathbb{R}^d$, we denote by diam B its diameter.

Let us recall the definition of the ξ-dimensional Hausdorff measure $\mathcal{H}_\xi$ in $\mathbb{R}^d$, where $0 < \xi \leq d$ (see [3] for a detailed exposition on this measure). Let A be a subset of $\mathbb{R}^d$. For any $\delta > 0$,

$$\mathcal{H}_\xi^\delta(A) = \inf \left\{ \sum_{i\in I} (\mathrm{diam}\ U_i)^\xi \ : \ A \subset \underset{i\in I}{\cup} U_i,\ I \text{ countable and } \mathrm{diam}\ U_i < \delta \text{ for } i \in I \right\}$$

and

$$\mathcal{H}_\xi(A) = \lim_{\delta\to 0} \mathcal{H}_\xi^\delta(A)\,.$$

Remark 2.2. (i) If $0 < t < r$, then for any subset E of $\mathbb{R}^d$ we have

$$\mathcal{H}_t(E) < \infty \Rightarrow \mathcal{H}_r(E) = 0\,.$$

(ii) There exists a positive constant $C(d)$ such that for any Lebesgue measurable subset E of $\mathbb{R}^d$,

$$\mathcal{H}_d(E) = C(d)|E|\,.$$

The following result (see [8]) will be useful in the proof of Proposition 1.4.

Lemma 2.3. *Let μ be a non-negative Radon measure on $\mathbb{R}^d$. Let $0 < \lambda < \infty$. Suppose that F is a Borel subset of $\mathbb{R}^d$ such that*

$$\limsup_{r\to 0}\ r^{-d\theta}\mu(B(x,r)) > \lambda\,,$$

for each $x \in F$. Then there exists a constant $C = C(d,\ \theta)$ such that

$$\mathcal{H}_{d\theta}(F) \leq \frac{C}{\lambda}\mu(F)\,.$$

Proof of Proposition 1.4. a) Let F be a Borel subset of $\mathbb{R}^d$. Let $0 < \lambda < \infty$ and $0 < \delta < \infty$. Let us set $F_\delta^\lambda = \{x \in F : \sup\limits_{0<r\leq\delta} r^{-d\theta}\mu(B(x,r)) < \lambda\}$ and $F^\lambda = \{x \in F : \limsup\limits_{r\to 0}\ r^{-d\theta}\mu(B(x,r)) < \lambda\}$.

For any countable covering $\{U_i : i \in I\}$ of F such that diam $U_i < \frac{\delta}{2}$ for all $i \in I$, we have

$$U_i \cap F_\delta^\lambda \neq \emptyset \Rightarrow \exists\, x \in U_i \cap F_\delta^\lambda \Rightarrow \exists\, x \in F_\delta^\lambda \ : \ U_i \subset B(x,\ 2\ \mathrm{diam}\, U_i)$$

$$U_i \cap F_\delta^\lambda \neq \emptyset \Rightarrow \exists\, x \in F_\delta^\lambda \ : \ \mu(U_i) \leq \mu\big(B(x,\ 2\ \mathrm{diam}\ U_i)\big) < \lambda\ (2\ \mathrm{diam}\ U_i)^{d\theta}\,.$$

It follows that

$$\mu(F_\delta^\lambda) \leq \sum_{\substack{i\in I \\ U_i\cap F_\delta^\lambda\neq\emptyset}} \mu(U_i) \leq \lambda\, 2^{d\theta} \sum_{i\in I} (\mathrm{diam}\ U_i)^{d\theta}\,.$$

Hence,

$$\mu(F_\delta^\lambda) \leq \lambda\, 2^{d\theta}\mathcal{H}_{d\theta}^\delta(F) \leq \lambda\, 2^{d\theta}\mathcal{H}_{d\theta}(F)\,, \quad \lambda > 0,\ \delta > 0\,.$$

Notice that for any $\lambda > 0$, $\big(F_{\frac{1}{k}}^\lambda\big)_{k\geq 1}$ is an increasing sequence which converges to F^λ. So we have

$$\mu(F^\lambda) \leq \lambda\, 2^{d\theta}\, \mathcal{H}_{d\theta}(F)\,, \quad \lambda > 0\,. \tag{2}$$

b) Suppose that F is a Borel set such that $F \subset N_\theta$ and $\mathcal{H}_{d\theta}(F) < \infty$. Then, for any $\lambda > 0$, we have $F = F^\lambda = \big\{x \in F : \limsup_{r\to 0}\ r^{-d\theta}\mu(B(x,r)) < \lambda\big\}$. So by (2), we have $\mu(F) = 0$.
c) Suppose that F is a Borel set such that $F \subset P_\theta$ and $\mathcal{H}_{d\theta}(F) = 0$. It follows from (2) that for any $\lambda > 0$, $\mu(F^\lambda) = 0$. Since the increasing sequence $\big(F^k\big)_{k\geq 1}$ converges to F, we obtain $\mu(F) = 0$.

d) Let us set

$$A_\theta^{m,k} = \Big\{x \in B(0,m) : \frac{1}{k} < \limsup_{r\to 0}\ r^{-d\theta}\mu(B(x,r)) < \infty\Big\}\,, \quad k \in \mathbb{N}^*\,,\ m \in \mathbb{N}^*$$

and

$$B_\theta^{m,k} = \Big\{x \in B(0,m) : k < \limsup_{r\to 0}\ r^{-d\theta}\mu(B(x,r)) < \infty\Big\}\,, \quad k \in \mathbb{N}^*\,,\ m \in \mathbb{N}^*\,.$$

By Lemma 2.3, there exists a real constant $C = C(d,\theta)$ such that for any positive integers k and m

$$\mathcal{H}_{d\theta}(A_\theta^{m,k}) \leq Ck\mu(A_\theta^{m,k}) \leq Ck\mu\big(B(0,m)\big) < \infty \tag{3}$$

and

$$\mathcal{H}_{d\theta}(B_\theta^{m,k}) \leq \frac{C}{k}\mu\big(B(0,m)\big) < \infty\,. \tag{4}$$

Since

$$P_\theta = \bigcup_{(m,k)\in\mathbb{N}^*\times\mathbb{N}^*} A_\theta^{m,k}\,,$$

we deduce from (3) that P_θ is $\mathcal{H}_{d\theta}$ σ-finite.
From (4) we have

$$\lim_{k\to\infty} \mathcal{H}_{d\theta}(B_\theta^{k,m}) = 0\,, \quad m \in \mathbb{N}^*\,.$$

In addition, for any positive integer m, $(B_\theta^{k,m})_{k\geq 1}$ is a decreasing sequence which converges to $E_\theta^\infty \cap B(0,m)$. So, by (4),

$$\mathcal{H}_{d\theta}(E_\theta^\infty \cap B(0,m)) = 0\,, \quad m \in \mathbb{N}^*$$

and therefore $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$. $\square$

Proof of Proposition 1.3. Since ν is a non-negative Radon measure on $\mathbb{R}^d$ then Proposition 1.4 holds with μ replaced by ν. Let P_θ and E_θ^∞ be as in Proposition 1.4. Since by Proposition 1.4 $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$ then it is enough to prove that $\mathcal{H}_{d\theta}(P_\theta) = 0$. Let (m, k) be an element of $\mathbb{N}^* \times \mathbb{N}^*$. Let $A_\theta^{m,k}$ be as in the proof of Proposition 1.4. By (3), $\mathcal{H}_{d\theta}(A_\theta^{m,k}) < \infty$. Therefore $\nu(A_\theta^{m,k}) = 0$ by the hypothesis on ν. Using again (3), we obtain $\mathcal{H}_{d\theta}(A_\theta^{m,k}) = 0$. But

$$P_\theta = \bigcup_{(m,k)\in\mathbb{N}^*\times\mathbb{N}^*} A_\theta^{m,k}\,,$$

so $\mathcal{H}_{d\theta}(P_\theta) = 0$. □

Proof of Corollary 1.6. a) Let P_θ and E_θ^∞ be as in Proposition 1.4. Let us notice that $A = P_\theta \cup E_\theta^\infty$. Since Proposition 1.4 ensures that $\mathcal{H}_{d\theta}(E_\theta^\infty) = 0$ then it is enough to prove that $\mathcal{H}_{d\theta}(P_\theta) = 0$ to establish the general case.
Let (m, k) be an element of $\mathbb{N}^* \times \mathbb{N}^*$. Let $A_\theta^{m,k}$ be as in the proof of Proposition 1.4. By (3), $\mathcal{H}_{d\theta}(A_\theta^{m,k}) < \infty$. Since $0 < d\theta < d$, it follows from Remark 2.2 that $\mathcal{H}_d(A_\theta^{m,k}) = 0$ and $|A_\theta^{m,k}| = 0$. The absolute continuity of μ with respect to the Lebesgue measure implies $\mu(A_\theta^{m,k}) = 0$ and consequently $\mathcal{H}_{d\theta}(A_\theta^{m,k}) = 0$ thanks to (3). But

$$P_\theta = \bigcup_{(m,k)\in\mathbb{N}^*\times\mathbb{N}^*} A_\theta^{m,k}\,,$$

so $\mathcal{H}_{d\theta}(P_\theta) = 0$.
b) The particular case follows from the general case by defining a measure μ as $d\mu(x) = |u(x)|^p dx$. □

3. Proof of Proposition 1.7

Let us introduce the fractional maximal operator m_β, $1 < \beta < \infty$, defined by

$$m_\beta\mu(x) = \sup_{r>0} |B(x,r)|^{\frac{1}{\beta}-1}\mu(B(x,r))\,, \quad x \in \mathbb{R}^d\,,$$

for any non-negative Radon measure μ on $\mathbb{R}^d$.
We have the following result.

Proposition 3.1. *Suppose that $1 < \beta < \infty$ and $1 \le p < \infty$. Let μ be a non-negative Radon measure on $\mathbb{R}^d$ such that $m_\beta\mu \in L^p$. Then, we have*

$$\lim_{r\to 0}\int_{\mathbb{R}^d}\Big(r^{d(\frac{1}{\beta}-1)}\mu\big(B(x,\,r)\big)\Big)^p dx = 0\,.$$

Proof. By hypothesis, $0 < 1 - \frac{1}{\beta} < 1$.
Define $N_{1-\frac{1}{\beta}}$, $P_{1-\frac{1}{\beta}}$ and $E^\infty_{1-\frac{1}{\beta}}$ as in Proposition 1.4. Then $\mathcal{H}_{d(1-\frac{1}{\beta})}\big(E^\infty_{1-\frac{1}{\beta}}\big) = 0$ and there exists a countable family $\{A_i : i \in I\}$ of subsets of $\mathbb{R}^d$ satisfying $P_{1-\frac{1}{\beta}} = \bigcup\limits_{i\in I} A_i$ and $\mathcal{H}_{d(1-\frac{1}{\beta})}(A_i) < \infty$ for all $i \in I$.

So, according to Remark 2.2 we have $\left|P_{1-\frac{1}{\beta}} \cup E^{\infty}_{1-\frac{1}{\beta}}\right| = 0$.
Recall that

$$\lim_{r\to 0} r^{d(\frac{1}{\beta}-1)}\mu\big(B(x,r)\big) = 0\,, \quad x \in N_{1-\frac{1}{\beta}} = \mathbb{R}^d \setminus \big(P_{1-\frac{1}{\beta}} \cup E^{\infty}_{1-\frac{1}{\beta}}\big)\,.$$

In addition,

$$\omega_d^{\frac{1}{\beta}-1} r^{d(\frac{1}{\beta}-1)}\mu\left(B(x,\, r)\right) \le m_\beta\mu(x)\,, \quad x \in \mathbb{R}^d\,,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ and $m_\beta\mu \in L^p$.
An application of the dominated convergence theorem ends the proof. □

For the proof of Proposition 1.7 we need the following well-known connexions between the fractional maximal operator m_β and the Riesz potential operator $I_{\frac{1}{\beta}}$.

Proposition 3.2 ([1])**.** *Suppose that $1 < \beta < \infty$. Let μ be a non-negative Radon measure on $\mathbb{R}^d$. Then,*
(i) *$m_\beta\mu \le I_{\frac{1}{\beta}}\mu$,*
(ii) *if $1 - \frac{1}{\beta} > \frac{1}{p} > 0$, there is a real constant $C > 0$ not depending on μ such that*

$$C^{-1}\|m_\beta\mu\|_p \le \|I_{\frac{1}{\beta}}\mu\|_p \le C\|m_\beta\mu\|_p\,.$$

It follows that Proposition 1.7 is a consequence of Proposition 3.1.

4. Proof of Proposition 1.8

In the sequel, for $1 < p < \infty$, we shall denote by p' the conjugate of p: $p' = \frac{p}{p-1}$. For the proof of Proposition 1.8, we need some basic properties of the Bessel capacity of order $(t,\, p)$ $(t > 0,\ p > 1)$ denoted by $C_{t,p}$. So we refer the reader to [1], [2] or [7] for a detailed exposition on this capacity.
To prove the sufficiency part of Proposition 1.9, Phuc and Torrès remarked that if $I_{\frac{1}{d}}\mu \in L^p$ then the non-negative Radon measure μ belongs to the dual space of the Sobolev space $W^{1,\,p'}(\mathbb{R}^d)$. Therefore such a measure is absolutely continuous with respect to the Bessel capacity $C_{1,\,p'}$ (see Section 2 in [2]). Thus we may state the following result.

Proposition 4.1. *Suppose that $\frac{d}{d-1} < p < \infty$ and μ is a non-negative Radon measure such that $I_{\frac{1}{d}}\mu \in L^p$. Then for any Borel subset E of $\mathbb{R}^d$ we have*

$$C_{1,p'}(E) = 0 \Rightarrow \mu(E) = 0\,.$$

Another useful result is the following well-known relation between the Hausdorff measure and the Bessel capacity (see [1] for a proof).

Proposition 4.2. *Suppose that $1 < p \le d$. Then for any subset E of $\mathbb{R}^d$ we have*

$$\mathcal{H}_{d-p}(E) < \infty \Rightarrow C_{1,p}(E) = 0\,.$$

We may now prove Proposition 1.8.

Proof of Proposition 1.8. Let μ be a non-negative Radon measure on $\mathbb{R}^d$ such that $I_{\frac{1}{d}}\mu \in L^p$ and let $\frac{d}{d-1} < p < \infty$. Applying Proposition 1.4 to μ with $\theta = 1 - \frac{p'}{d}$, we get that $P_{1-\frac{p'}{d}}$ is $\mathcal{H}_{d-p'}$ σ-finite and $\mathcal{H}_{d-p'}\Big(E^{\infty}_{1-\frac{p'}{d}}\Big) = 0$. We then deduce from Proposition 4.2 that $C_{1,p'}\Big(P_{1-\frac{p'}{d}} \cup E^{\infty}_{1-\frac{p'}{d}}\Big) = 0$ and so $\mu\Big(P_{1-\frac{p'}{d}} \cup E^{\infty}_{1-\frac{p'}{d}}\Big) = 0$ by Proposition 4.1. We conclude that $\lim\limits_{r\to 0} r^{p'-d}\mu(B(x,r)) = 0$ μ-almost everywhere. $\square$

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