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Δ-WEAK CHARACTER AMENABILITY OF CERTAIN BANACH ALGEBRAS

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Abstract. In this paper we investigate Δ-weak character amenability of certain Banach algebras such as projective tensor product $A\widehat{\otimes}B$ and Lau product $A\times_\theta B$, where A and B are two arbitrary Banach algebras and $\theta\in\Delta(B)$, the character space of B. We also investigate Δ-weak character amenability of abstract Segal algebras and module extension Banach algebras.

1. Introduction

Let A be a Banach algebra and let $\varphi\in\Delta(A)$, consisting of all nonzero homomorphisms from A into $\mathbb{C}$. The concept of φ-amenability was first introduced by Kaniuth et al. in [6]. Specifically, A is called φ-amenable if there exist a $m\in A^{**}$ such that

(i) $m(\varphi)=1$;

(ii) $m(f\cdot a)=\varphi(a)m(f)\ \ (a\in A, f\in A^*)$.

Monfared in [10], introduced and studied the notion of character amenable Banach algebra. A was called character amenable if it has a bounded right approximate identity and it is φ-amenable for all $\varphi\in\Delta(A)$. Many aspects of φ-amenability have been investigated in [3, 6, 9].

Let A be a Banach algebra and $\varphi\in\Delta(A)\cup\{0\}$. Following [7], A is called Δ-weak φ-amenable if, there exists a $m\in A^{**}$ such that

(i) $m(\varphi)=0$;

(ii) $m(\psi\cdot a)=\psi(a)\ \ (a\in\ker(\varphi),\ \psi\in\Delta(A))$.

In this paper we use above definition with a slight difference. In fact we say that A is Δ-weak φ-amenable if, there exists a $m\in A^{**}$ such that

(i) $m(\varphi)=0$;

(ii) $m(\psi\cdot a)=\psi(a)\ \ (a\in A,\psi\in\Delta(A)\setminus\{\varphi\})$.

The aim of the present work is to study Δ-weak character amenability of certain Banach algebras such as projective tensor product $A\widehat{\otimes}B$, Lau product $A\times_\theta B$, where $\theta\in\Delta(B)$, abstract Segal algebras and module extension Banach algebras. Indeed, we show that $A\widehat{\otimes}B$ (resp. $A\times_\theta B$) is Δ-weak character amenable if and

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only if both A and B are Δ-weak character amenable. For abstract Segal algebra B with respect to A, we investigate relations between Δ-weak character amenability of A and B. Finally, for a Banach algebra A and A-bimodule X we show that $A \oplus_1 X$ is Δ-weak character amenable if and only if A is Δ-weak character amenable.

2. Δ-weak character amenability of $A\widehat{\otimes}B$

We commence this section with the following definition:

Definition 2.1. Let A be a Banach algebra. The net $(a_\alpha)_\alpha$ in A is called a Δ-weak approximate identity if, $|\varphi(aa_\alpha) - \varphi(a)| \longrightarrow 0$, for each $a \in A$ and $\varphi \in \Delta(A)$.

Note that the approximate identity and Δ-weak approximate identity of a Banach algebra can be different. Jones and Lahr proved that if $S = \mathbb{Q}^+$ the semigroup algebra $l^1(S)$ has a bounded Δ-weak approximate identity, but it does not have any bounded or unbounded approximate identity (see [4]).

Definition 2.2. Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. We say that A is Δ-weak φ-amenable if, there exists a $m \in A^{**}$ such that
(i) $m(\varphi) = 0$;
(ii) $m(\psi \cdot a) = \psi(a)$ $(a \in A, \psi \in \Delta(A) \setminus \{\varphi\})$.

Definition 2.3. Let A be a Banach algebra. We say that A is Δ-weak character amenable if it is Δ-weak φ-amenable for every $\varphi \in \Delta(A) \cup \{0\}$.

Lemma 2.4. *Let A be a Banach algebra such that $0 < |\Delta(A)| \le 2$. Then A is Δ-weak character amenable.*

Proof. If A has only one character, the proof is easy. Let $\Delta(A) = \{\varphi, \psi\}$, where $\varphi \neq \psi$. Hence, by the proof of Theorem 3.3.14 of [5], there exists a $a_0 \in A$ with $\varphi(a_0) = 0$ and $\psi(a_0) = 1$. Put $m = \widehat{a_0}$. Then $m(\varphi) = \widehat{a_0}(\varphi) = \varphi(a_0) = 0$ and for every $a \in A$, we have

$$m(\psi \cdot a) = \widehat{a_0}(\psi \cdot a) = \psi \cdot a(a_0) = \psi(aa_0) = \psi(a)\,.$$

So, A is Δ-weak φ-amenable. A Similar argument shows that A is Δ-weak ψ-amenable. Therefore A is Δ-weak character amenable. $\square$

The proof of the following theorem is omitted, since it can be proved in the same direction as Theorem 2.2 of [7].

Theorem 2.5. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Then A is Δ-weak φ-amenable if and only if there exists a net $(a_\alpha)_\alpha \subseteq \ker(\varphi)$ such that $|\psi(aa_\alpha) - \psi(a)| \longrightarrow 0$, for each $a \in A$ and $\psi \in \Delta(A) \setminus \{\varphi\}$.*

Example 2.6. (i) Let A be a Banach algebra with a bounded approximate identity. By Theorem 2.5, A is Δ-weak 0-amenable.

(ii) Let $S = \mathbb{Q}^+$. Then the semigroup algebra $l^1(S)$ has a bounded Δ-weak approximate identity (see [4]). So, Theorem 2.5, implies that $l^1(S)$ is Δ-weak 0-amenable.

Example 2.7. Let X be a Banach space and let $\varphi \in X^* \setminus \{0\}$ with $\|\varphi\| \leq 1$. Define a product on X by $ab = \varphi(a)b$ for all $a, b \in X$. With this product X is a Banach algebra which is denoted by $A_\varphi(X)$ (see [11]). Clearly, $\Delta(A_\varphi(X)) = \{\varphi\}$. Therefore by Lemma 2.4, $A_\varphi(X)$ is Δ-weak φ-amenable.

Example 2.8. Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Suppose that A is a φ-amenable and has a bounded right approximate identity. By Corollary 2.3 of [6], $\ker(\varphi)$ has a bounded right approximate identity. Let $(e_\alpha)_\alpha$ be a bounded right approximate identity for $\ker(\varphi)$. If there exists $a_0 \in A$ with $\varphi(a_0) = 1$ and $\lim_\alpha \big|\psi(a_0 e_\alpha) - \psi(a_0)\big| = 0$ for all $\psi \in \Delta(A) \setminus \{\varphi\}$, then A is Δ-weak φ-amenable. For seeing this suppose that m is w^*-$\lim_\alpha(\widehat{e_\alpha})$. Now, we have

$$m(\varphi) = \lim_\alpha \widehat{e_\alpha}(\varphi) = \lim_\alpha \varphi(e_\alpha) = 0\,,$$

and for every $\psi \in \Delta(A) \setminus \{\varphi\}$ and $a \in \ker(\varphi)$,

$$m(\psi \cdot a) = \lim_\alpha \widehat{e_\alpha}(\psi \cdot a) = \lim_\alpha \psi \cdot a(e_\alpha) = \lim_\alpha \psi(a e_\alpha) = \psi(a)\,.$$

Let $a \in A$. Then $a - \varphi(a)a_0 \in \ker(\varphi)$ and for every $\psi \in \Delta(A) \setminus \{\varphi\}$, we have

$$m\big(\psi \cdot (a - \varphi(a)a_0)\big) = \psi\big(a - \varphi(a)a_0\big)\,.$$

Therefore $m(\psi \cdot a) = \psi(a)$. So A is Δ-weak φ-amenable.

For $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A\widehat{\otimes}B)^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in A$ and $b \in B$. Then, with this notion,

$$\Delta(A\widehat{\otimes}B) = \{\varphi \otimes \psi : \varphi \in \Delta(A), \psi \in \Delta(B)\}\,.$$

Theorem 2.9. *Let A and B be Banach algebras and let $\varphi \in \Delta(A) \cup \{0\}$ and $\psi \in \Delta(B) \cup \{0\}$. Then $A\widehat{\otimes}B$ is Δ-weak $(\varphi \otimes \psi)$-amenable if and only if A is Δ-weak φ-amenable and B is Δ-weak ψ-amenable.*

Proof. Suppose that $A\widehat{\otimes}B$ is Δ-weak $(\varphi \otimes \psi)$-amenable. So, there exists $m \in (A\widehat{\otimes}B)^{**}$ such that

$$m(\varphi \otimes \psi) = 0\,, \quad m\big((\varphi' \otimes \psi') \cdot (a \otimes b)\big) = (\varphi' \otimes \psi')(a \otimes b)\,,$$

for all $a \otimes b \in A\widehat{\otimes}B$ and $(\varphi' \otimes \psi') \in \Delta(A\widehat{\otimes}B) \setminus \{\varphi \otimes \psi\}$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$, and define $m_\psi \in A^{**}$ by $m_\psi(f) = m(f \otimes \psi)$ $(f \in A^*)$. Then $m_\psi(\varphi) = m(\varphi \otimes \psi) = 0$ and for every $a \in A$ and $\varphi' \in \Delta(A) \setminus \{\varphi\}$, we have

$$\begin{aligned} m_\psi(\varphi' \cdot a) &= m(\varphi' \cdot a \otimes \psi) = m(\varphi' \cdot a \otimes \psi \cdot b_0) \\ &= m\big((\varphi' \otimes \psi) \cdot (a \otimes b_0)\big) = \varphi' \otimes \psi(a \otimes b_0) \\ &= \varphi'(a)\,. \end{aligned}$$

Thus A is Δ-weak φ-amenable. By a similar argument one can prove that B is Δ-weak ψ-amenable.

Conversely, assume that A is Δ-weak φ-amenable and B is Δ-weak ψ-amenable. By Theorem 2.5, there are bounded nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\ker(\varphi)$ and $\ker(\psi)$, respectively, such that $|\varphi'(a a_\alpha) - \varphi'(a)| \longrightarrow 0$ and $\big|\psi'(b b_\beta) - \psi'(b)\big| \longrightarrow 0$ for all $a \in A$, $b \in B$, $\varphi' \in \Delta(A) \setminus \{\varphi\}$ and $\psi' \in \Delta(B) \setminus \{\psi\}$. Consider the bounded net

$((a_\alpha \otimes b_\beta))_{(\alpha,\beta)}$ in $A\widehat{\otimes}B$. Let $\|a_\alpha\| \le M_1$, $\|b_\beta\| \le M_2$ and let $F = \sum_{i=1}^N c_i \otimes d_i \in A\widehat{\otimes}B$. For every $\varphi' \in \Delta(A) \setminus \{0\}$ and $\psi' \in \Delta(B) \setminus \{0\}$, we have

$$\begin{aligned}
&\big|\varphi' \otimes \psi'(F \cdot (a_\alpha \otimes b_\beta)) - \varphi' \otimes \psi'(F)\big| \\
&= \Big|\sum_{i=1}^N \Big[\big(\varphi'(c_i a_\alpha) - \varphi'(c_i)\big)\psi'(d_i b_\beta) + \varphi'(c_i)\big(\psi'(d_i b_\beta) - \psi'(d_i)\big)\Big]\Big| \\
&\le \sum_{i=1}^N M_2\|d_i\|\|\psi'\|\big|\varphi'(c_i a_\alpha) - \varphi'(c_i)\big| + \sum_{i=1}^N \|\varphi'\|\|c_i\|\big|\psi'(d_i b_\beta) - \psi'(d_i)\big| \\
&\longrightarrow 0\,.
\end{aligned}$$

Now let $G \in A\widehat{\otimes}B$, so there exist sequences $(c_i)_i \subseteq A$ and $(d_i)_i \subseteq B$ such that $G = \sum_{i=1}^\infty c_i \otimes d_i$ with $\sum_{i=1}^\infty \|c_i\|\|d_i\| < \infty$. Let $\varepsilon > 0$ be given, we choose $N \in \mathbb{N}$ such that $\sum_{i=N+1}^\infty \|c_i\|\|d_i\| < \varepsilon/4M_1M_2\|\varphi'\|\|\psi'\|$. Put $F = \sum_{i=1}^N c_i \otimes d_i$. Since $|\varphi' \otimes \psi'(F \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)| \longrightarrow 0$, it follows that there exists (α_0, β_0) such that $|\varphi' \otimes \psi'(F \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)| < \varepsilon/2$ for all $(\alpha,\beta) \ge (\alpha_0,\beta_0)$. Now for such a (α,β), we have

$$\begin{aligned}
&\big|\varphi' \otimes \psi'(G \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)\big| \\
&= \Big|\varphi' \otimes \psi'(F \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F) + \sum_{i=1+N}^\infty \big(\varphi'(c_i a_\alpha)\psi'(d_i b_\beta) - \varphi'(c_i)\psi'(d_i)\big)\Big| \\
&\le \varepsilon/2 + 2M_1M_2\|\varphi'\|\|\psi'\| \sum_{i+N}^\infty \|c_i\|\|d_i\| \le \varepsilon/2 + \varepsilon/2 = \varepsilon\,.
\end{aligned}$$

Hence $\big|\varphi' \otimes \psi'(G \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)\big| \longrightarrow 0$. Also, clearly $\big|\varphi' \otimes \psi'(G \cdot a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)\big| \longrightarrow 0$ for $\varphi' = 0$ and $\psi' = 0$ and it is easy to see that $((a_\alpha \otimes b_\beta))_{(\alpha,\beta)} \subset \ker(\varphi \otimes \psi)$. Therefore $A\widehat{\otimes}B$ is Δ-weak $(\varphi \otimes \psi)$-amenable, again by Theorem 2.5. □

Corollary 2.10. *Let A and B be Banach algebras. Then $A\widehat{\otimes}B$ is Δ-weak character amenable if and only if both A and B are Δ-weak character amenable.*

By using above corollary and Theorem 2.9, we can proof following proposition.

Proposition 2.11. *Let A and B be Banach algebras. Then $A\widehat{\otimes}B$ is Δ-weak character amenable if and only if $B\widehat{\otimes}A$ is Δ-weak character amenable.*

3. Δ-weak character amenability of $A \times_\theta B$

Let A and B be Banach algebras with $\Delta(B) \neq \emptyset$ and $\theta \in \Delta(B)$. Then the set $A \times B$ equipped with the multiplication

$$(a_1,b_1)\cdot(a_2,b_2) = (a_1a_2 + \theta(b_2)a_1 + \theta(b_1)a_2, b_1b_2) \quad (a_1,a_2 \in A, b_1,b_2 \in B)\,,$$

and the norm $\|(a,b)\| = \|a\| + \|b\|$ $(a \in A, b \in B)$, is a Banach algebra which is called the θ-Lau product of A and B and is denoted by $A \times_\theta B$. Lau product was introduced by Lau [8] for certain class of Banach algebras and was extended by Monfared [9] for the general case.

We note that the dual space $(A \times_\theta B)^*$ can be identified with $A^* \times B^*$, via

$$\langle (f,g),(a,b)\rangle = \langle a,f\rangle + \langle b,g\rangle \quad (a \in A, f \in A^*, b \in B, g \in B^*)\,.$$

Moreover, $(A \times_\theta B)^*$ is a $(A \times_\theta B)$-bimodule with the module operations given by

$$(f,g)\cdot(a,b) = \big(f\cdot a + \theta(b)f, f(a)\theta + g\cdot b\big)\,, \tag{3.1}$$

and

$$(a,b)\cdot(f,g) = \big(a.f + \theta(b)f, f(a)\theta + b\cdot g\big)\,, \tag{3.2}$$

for all $a \in A$, $b \in B$, $f \in A^*$ and $g \in B^*$.

Proposition 3.1. *Let A be a unital Banach algebra and B be a Banach algebra and $\theta \in \Delta(B)$. Then $A \times_\theta B$ has a Δ-weak approximate identity if and only if B has a Δ-weak approximate identity.*

Proof. Let $((a_\alpha, b_\alpha))_\alpha$ be a Δ-weak approximate identity for $A \times_\theta B$. For every $\psi \in \Delta(B)$ and $b \in B$ we have,

$$\big|\psi(bb_\alpha) - \psi(b)\big| = \big|(0,\psi)\big((0,b)(a_\alpha,b_\alpha)\big) - (0,\psi)(0,b)\big| \longrightarrow 0\,.$$

Then $(b_\alpha)_\alpha$ is a Δ-weak approximate identity for B.

Conversely, let e_A be the identity of A and $(b_\beta)_\beta$ be a Δ-weak approximate identity for B. We claim that $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta$ is a Δ-weak approximate identity for $A \times_\theta B$. In fact for every $a \in A, b \in B$ and $\varphi \in \Delta(A)$, we have

$$\begin{aligned}
&\big|(\varphi,\theta)\big((a,b)(e_A - \theta(b_\beta)e_A, b_\beta)\big) - (\varphi,\theta)(a,b)\big| \\
&\quad = \big|(\varphi,\theta)\big(a + \theta(b)e_A - \theta(bb_\beta)e_A, bb_\beta)\big) - (\varphi,\theta)(a,b)\big| \\
&\quad = 0\,.
\end{aligned}$$

Also for every $a \in A$, $b \in B$ and $\psi \in \Delta(B)$, we have

$$\big|(0,\psi)\big((a,b)(e_A - \theta(b_\beta)e_A, b_\beta)\big) - (0,\psi)(a,b)\big| = \big|\psi(bb_\beta) - \psi(b)\big| \longrightarrow 0\,.$$

Therefore $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta$ is a Δ-weak approximate identity for $A \times_\theta B$. $\square$

Theorem 3.2. *Let A be a unital Banach algebra and B be a Banach algebra and $\theta \in \Delta(B)$. Then $A \times_\theta B$ is Δ-weak character amenable if and only if both A and B are Δ-weak character amenable.*

Proof. Suppose that $A \times_\theta B$ is Δ-weak character amenable. Let $\varphi \in \Delta(A) \cup \{0\}$. Then there exists $m \in (A \times_\theta B)^{**}$ such that $m(\varphi,\theta) = 0$ and $m(h.(a,b)) = h(a,b)$ for all $(a,b) \in A \times_\theta B$ and $h \in \Delta(A \times_\theta B)$, where $h \neq (\varphi,\theta)$. Let e_A be the identity of A and define $m_\psi \in A^{**}$ by $m_\psi(f) = m(f, f(e_A)\theta)(f \in A^*)$. For every $a \in A$ and $\varphi' \in \Delta(A) \setminus \{\varphi\}$, we have

$$\begin{aligned}
m_\psi(\varphi'\cdot a) &= m\big(\varphi'\cdot a, (\varphi'\cdot a)(e_A)\theta\big) \\
&= m\big(\varphi'\cdot a, \varphi'(a)\theta\big) \\
&= m\big((\varphi',\theta)\cdot(a,0)\big) \\
&= (\varphi',\theta)(a,0) \\
&= \varphi'(a)\,.
\end{aligned}$$

Also $m_\psi(\varphi) = m(\varphi,\theta) = 0$. Thus A is a Δ-weak φ-amenable. Therefore A is Δ-weak character amenable.

Let $\psi \in \Delta(B)\cup\{0\}$. From the Δ-weak character amenability of $A\times_\theta B$ it follows that there exists a $m \in (A\times_\theta B)^{**}$ such that $m(0,\psi) = 0$ and $m(h\cdot(a,b)) = h(a,b)$ for all $(a,b) \in A\times_\theta B$ and $h \in \Delta(A\times_\theta B)$, where $h \neq (0,\psi)$. Define $m_\varphi \in B^{**}$ by $m_\varphi(g) = m(0,g)$. So $m_\varphi(\psi) = m(0,\psi) = 0$ and

$$m_\varphi(\psi'\cdot b) = m(0,\psi'\cdot b) = m\big((0,\psi')\cdot(0,b)\big) = (0,\psi')(0,b') = \psi'(b)\,,$$

for all $b \in B$ and $\psi' \in \Delta(B)\setminus\{\psi\}$. Therefore B is Δ-weak character amenable.

Conversely, let A and B be Δ-weak character amenable. We show that for every $h \in \Delta(A\times_\theta B)$, $A\times_\theta B$ is Δ-weak h-amenable. To see this we first assume that $h = (0,\psi)$, where $\psi \in \Delta(B)$. Since B is Δ-weak character amenable, by Theorem 2.5 there exists a net $(b_\beta)_\beta \subseteq \ker\psi$ such that $|\psi'(bb_\beta) - \psi'(b)| \longrightarrow 0$, for all $b \in B$ and $\psi' \in \Delta(B)$, where $\psi' \neq \psi$. Consider the bounded net $((e_A-\theta(b_\beta)e_A, b_\beta))_\beta \subseteq A\times_\theta B$. A similar argument as in the proof of Proposition 3.1, shows that

$$\big|(\varphi,\theta)\big((a,b)(e_A-\theta(b_\beta)e_A,b_\beta)\big) - (\varphi,\theta)(a,b)\big| \longrightarrow 0\,,$$

and

$$\big|(0,\psi)\big((a,b)(e_A-\theta(b_\beta)e_A,b_\beta)\big) - (0,\psi)(a,b)\big| \longrightarrow 0\,,$$

for all $\varphi \in \Delta(A), \psi \in \Delta(B)$ and $a \in A, b \in B$. Also one can easily check that $((e_A-\theta(b_\beta)e_A, b_\beta))_\beta \subseteq \ker h$. So, by Theorem 2.5, $A\times_\theta B$ is Δ-weak $(0,\psi)$-amenable.

Now let $h = (\varphi,\theta)$, where $\varphi \in \Delta(A)$. Since A is Δ-weak φ-amenable, by Theorem 2.5 there exists a net $(a_\alpha)_\alpha \subseteq \ker\varphi$ such that $|\varphi'(aa_\alpha) - \varphi'(a)| \longrightarrow 0$, for all $a \in A$ and $\varphi' \in \Delta(A)$, where $\varphi' \neq \varphi$. Also since B is Δ-weak θ-amenable again by Theorem 2.5, there exists a net $(b_\beta)_\beta \subseteq \ker(\theta)$ such that $|\psi'(bb_\beta) - \psi'(b)| \longrightarrow 0$, for all $b \in B$ and $\psi' \in \Delta(B)$, where $\psi' \neq \theta$. Consider the bounded net $((a_\alpha, b_\beta))_{(\alpha,\beta)} \subseteq A\times_\theta B$. It is easy to see that $((a_\alpha,b_\beta))_{(\alpha,\beta)} \subseteq \ker(\varphi,\theta)$. For every $a \in A$, $b \in B$ and $\psi' \in \Delta(B)$, we have

$$\big|(0,\psi')\big((a,b)(a_\alpha,b_\beta)\big) - (0,\psi')\big(a,b\big)\big| = \big|\psi'(bb_\beta) - \psi'(b)\big| \longrightarrow 0\,,$$

and for every $\varphi' \in \Delta(A)$,

$$\begin{aligned}
&\big|(\varphi',\theta)\big((a,b)(a_\alpha,b_\beta)\big) - (\varphi',\theta)\big((a,b)\big)\big| \\
&\quad= \big|\varphi'(aa_\alpha) + \theta(b_\beta)\varphi'(a) + \theta(b)\varphi'(a_\alpha) + \theta(bb_\beta) - \varphi'(a) - \theta(b)\big| \\
&\quad= \big|\varphi'(aa_\alpha) + \theta(b)\varphi'(a_\alpha) - \varphi'(a) - \theta(b)\big| \\
&\quad\leq \big|\varphi'(aa_\alpha) - \varphi'(a)\big| + |\theta(b)|\big|\varphi'(a_\alpha e_A) - \varphi'(e_A)\big| \longrightarrow 0\,.
\end{aligned}$$

So, Theorem 2.5, yields that $A\times_\theta B$ is Δ-weak (φ,θ)-amenable. Therefore $A\times_\theta B$ is Δ-weak character amenable. □

4. Δ-weak character amenability of abstract Segal algebras

We start this section with the basic definition of abstract Segal algebra; see [2] for more details. Let $(A,\|\cdot\|_A)$ be a Banach algebra. A Banach algebra $(B,\|\cdot\|_B)$ is an abstract Segal algebra with respect to A if:

(i) B is a dense left ideal in A;

(ii) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for all $b \in B$;
(iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for all a, $b \in B$.

Several authors have studied various notions of amenability for abstract Segal algebras; see, for example, [1, 12].

To prove our next result we need to quote the following lemma from [1].

Lemma 4.1. *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A. Then $\Delta(B) = \{\varphi|_B : \varphi \in \Delta(A)\}$.*

Theorem 4.2. *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A. If B is Δ-weak character amenable, then so is A. In the case that B^2 is dense in B and B has a bounded approximate identity the converse is also valid.*

Proof. Let $\varphi \in \Delta(A)$. Since B is Δ-weak character amenable, by Lemma 4.1 B is Δ-weak $\varphi|_B$-amenable. Now from the Theorem 2.5, it follows that there exists a bounded net $(b_\alpha)_\alpha$ in $\ker(\varphi|_B)$ such that

$$\big|\psi|_B(bb_\alpha) - \psi|_B(b)\big| \longrightarrow 0\,,$$

for all $b \in B$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi|_B$. Let $\psi \in \Delta(A)$ and $a \in A$. From the density of B in A it follows that there exists a net $(b_i)_i \subseteq B$ such that $\lim_i b_i = a$. So

$$\Big|\psi(ab_\alpha) - \psi(a)\Big| = \lim_i \Big|\psi|_B(b_i b_\alpha) - \psi|_B(b_i)\Big| \longrightarrow 0\,.$$

Then Theorem 2.5 implies that A is Δ-weak φ-amenable. Therefore A is Δ-weak character amenable.

Conversely, suppose that A is Δ-weak character amenable. Let $\varphi|_B \in \Delta(B)$. By Theorem 2.5, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $\big|\psi(aa_\alpha) - \psi(a)\big| \longrightarrow 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Let $(e_i)_i$ be a bounded approximate identity for B with bound $M > 0$. Set $b_\alpha = \lim_i(e_i a_\alpha e_i)$, for all α. From the fact that B^2 is dense in B and the continuity of φ, we infer that $b_\alpha \subseteq \ker(\varphi|_B)$. Moreover, for every $b \in B$ and $\psi|_B \in \Delta(B)$, with $\psi \neq \varphi$, we have

$$\begin{aligned}\Big|\psi|_B(bb_\alpha) - \psi|_B(b)\Big| &= \lim_i \big|\psi|_B(be_i a_\alpha e_i) - \psi|_B(b)\big| \\ &= \lim_i \big|\psi|_B(be_i^2 a_\alpha) - \psi|_B(b)\big| \\ &= \big|\psi|_B(ba_\alpha) - \psi|_B(b)\big| \longrightarrow 0\,.\end{aligned}$$

Hence, B is Δ-weak $\varphi|_B$-amenable by Theorem 2.5. Therefore B is Δ-weak character amenable. □

5. Δ-weak character amenability of module extension Banach algebras

Let A be a Banach algebra and X be a Banach A-bimodule. The l^1-direct sum of A and X, denoted by $A \oplus_1 X$, with the product defined by

$$(a, x)(a', x') = (aa', a \cdot x' + x \cdot a') \qquad (a, a' \in A, x, x' \in X)\,,$$

is a Banach algebra that is called the module extension Banach algebra of A and X.

Using the fact that the element $(0,x)$ is nilpotent in $A\oplus_1 X$ for all $x\in X$, it is easy to verify that

$$\Delta(A\oplus_1 X)=\{\tilde{\varphi}:\varphi\in\Delta(A)\}\,,$$

where $\tilde{\varphi}(a,x)=\varphi(a)$ for all $a\in A$ and $x\in X$.

Theorem 5.1. *Let A be a Banach algebra and X be a Banach A-bimodule. Then $A\oplus_1 X$ is Δ-weak character amenable if and only if A is Δ-weak character amenable.*

Proof. Suppose that A is Δ-weak character amenable. Let $\tilde{\varphi}\in\Delta(A\oplus_1 X)$. By Theorem 2.5, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $\big|\psi(aa_\alpha)-\psi(a)\big|\longrightarrow 0$, for all $a\in A$ and $\psi\in\Delta(A)$, with $\psi\neq\varphi$. Choose a bounded net $(a_\alpha,0)_\alpha$ in $A\oplus_1 X$. Clearly, $(a_\alpha,0)_\alpha\subseteq\ker(\tilde{\varphi})$. For every $a\in A$, $x\in X$ and $\tilde{\psi}\in\Delta(A\oplus_1 X)$, we have

$$\begin{aligned}\big|\tilde{\psi}\big((a,x)(a_\alpha,0)\big)-\tilde{\psi}(a,x)\big| &= \big|\tilde{\psi}\big(aa_\alpha,x\cdot a_\alpha\big)-\tilde{\psi}(a,x)\big|\\ &= \big|\psi(aa_\alpha)-\psi(a)\big|\longrightarrow 0\,.\end{aligned}$$

So, Theorem 2.5 implies that $A\oplus_1 X$ is Δ-weak $\tilde{\varphi}$-amenable. Therefore $A\oplus_1 X$ is Δ-weak character amenable.

For the converse, let $\varphi\in\Delta(A)$. Again by Theorem 2.5 there exists a bounded net $(a_\alpha,x_\alpha)_\alpha$ in $\ker(\tilde{\varphi})$ such that $\big|\tilde{\psi}\big((a,x)(a_\alpha,x_\alpha)\big)-\tilde{\psi}(a,x)\big|\longrightarrow 0$, for all $a\in A$, $x\in X$ and $\tilde{\psi}\in\Delta(A\oplus_1 X)$, with $\tilde{\psi}\neq\tilde{\varphi}$. So,

$$\begin{aligned}\big|\psi(aa_\alpha)-\psi(a)\big| &= \big|\tilde{\psi}(aa_\alpha,a\cdot x_\alpha+x\cdot a_\alpha)-\tilde{\psi}(a,x)\big|\\ &= \big|\tilde{\psi}\big((a,x)(a_\alpha,x_\alpha)\big)-\tilde{\psi}(a,x)\big|\longrightarrow 0\,,\end{aligned}$$

for all $a\in A$ and $\psi\in\Delta(A)$. Moreover, $\varphi(a_\alpha)=\tilde{\varphi}(a_\alpha,x_\alpha)=0$, for all α. Thus $(a_\alpha)_\alpha\subseteq\ker(\varphi)$. By Theorem 2.5, A is Δ-weak φ-amenable. Therefore A is Δ-weak character amenable. □

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