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GROWTH OF WEIGHTED VOLUME AND SOME APPLICATIONS

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Abstract. We define cut-off functions in order to allow higher growth for Dirichlet energy. Our results are generalizations of the classical Cheng-Yau's growth conditions of parabolicity. Finally we give some applications to the function theory of Kähler and quaternionic-Kähler manifolds.

1. Introduction

Let M denote a complete non-compact n-dimensional Riemannian manifold and $\mathrm{Vol}(B_{x_0}(R))$ denote the volume of the geodesic ball with center at x_0 and radius R. If the center is not relevant, we write simply $B(R)$. The volume growth is a geometric condition which is useful in the function theory of M. For instance, Cheng and Yau [1] showed that if the volume growth of M satisfies $\mathrm{Vol}(B(R)) \le CR^2$ for some constant $C > 0$, then M must be parabolic, i.e. every superharmonic positive function on M must be constant. The growth condition in Cheng and Yau's result has been improved to $\mathrm{Vol}(B(R)) \le K(R)$, with $K(R)$ a function such that $R/K(R)$ is not L^1-integrable (cf. [4, 5, 9, 20]). We refer to [6] for other definitions and more properties of parabolicity.

Let us now consider another definition of parabolicity. The non-compact manifold M is parabolic if considering the exhaustion by balls $\{B(R_i)\}_{i\in\mathbb{N}}$ of M, the sequence $\int_{B(R_i)} |\nabla \phi_{R_i}|^2 dv$ goes to zero when i approaches infinity, where ϕ_{R_i} are cut-off functions supported on the ball $B(R_i)$, harmonic on the annulus $B(R_i) - B(r_0)$ and identically one on the boundary sphere $\partial B(r_0)$, with $r_0 < R_i$, $i \in \mathbb{N}$.

Looking for a generalization of the above definition, consider a non-negative continuous function P on M. We want to find sufficient conditions in order to ensure that the integral of the form $I_R := \int_M P|\nabla\phi_R|^2 dv$ goes to zero for large R, where ϕ_R is a cut-off function (a non-negative, bounded, smooth function with compact support) supported on the ball $B_{x_0}(R)$. That kind of integral appears often in computations using integration by parts and its asymptotic vanishing could help to show that some related integrals of the form $\int_{B(R)} |Q|^2\ \phi_R^2 dv$ converges

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also to zero as $R \to \infty$, implying that Q is identically zero, where Q is a useful expression having a geometric meaning.

Consequently, in order to proof that $I_R \to 0$ as $R \to \infty$ we choose a suitable cut-off function and use the condition on the growth of the P-weighted volume $\int_{B_{x_0}(R)} P\, dv$. Thus it is interesting to give the next definition motivated by the definition of parabolicity and consider manifolds where the property $I_R \to 0$ as $R \to \infty$ holds. For other definitions on weighted parabolicity we refer to [7], where the weight P is taken positive.

Definition 1.1. Let $P \in C^0(M)$ be a non-negative function. A complete Riemannian manifold M is said to be P-weighted parabolic if for any compact subset $K \subset M$ and each $\epsilon > 0$, there exists a cut-off function $\phi_R \in C^\infty(M)$ with $0 \leq \phi_R \leq 1$, $\phi_R \equiv 1$ in a neighbourhood of K and $\int_M P\, |\nabla \phi_R|^2 dv < \epsilon$ as $R \to \infty$.

Note that parabolicity corresponds to taking the constant function $P \equiv 1$. Using the above definition we get the following result, the main result of this paper.

Theorem 1.2. *Let M be a complete Riemannian manifold and $P \in C^0(M)$, non-negative and not identically zero. We define a function K by $K(R) := \int_{B_{x_0}(R)} P dv$ for some point x_0 such that $P(x_0) \neq 0$.*

(i) *If the function K satisfies*

$$\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty \quad as \quad R \to \infty\,, \tag{1}$$

then M is P-weighted parabolic.

(ii) *If $K'(r) > 0$ almost everywhere and $\int_{r_0}^{R} \frac{1}{K'(r)} dr \to \infty$ as $R \to \infty$, then M is P-weighted parabolic.*

The volume growth of orders $O(R^2)$ and $O(R^2 ln(R))$ can be usually obtained with standard cut-off functions, see Remark 3.2. These cases are included in the condition that $r/K(r)$ is not L^1-integrable of our theorem. A cut-off function with similar properties appeared in [19] in a geometric application. Our result was obtained independently, avoids the condition of $r/K(r)$ being non-increasing and makes explicit the method which can be used in other applications.

In Section 2, we recall some of the various concepts related to parabolicity and its weighted generalization. The main result is proved in Section 3. Finally in Section 4, we use our cut-off function to improve some known results on Kähler and quaternionic-Kähler manifolds.

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2. Preliminaries

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. As usual $B_{x_0}(R)$ denotes the geodesic ball centered at point x_0 of radius $R > 0$, ∇f denotes the gradient of a function f and $\Delta = dd^* + d^* d$ is the Hodge Laplacian.

The function u is said to be harmonic if Laplace's equation holds, i.e.

$$\Delta u = 0\,.$$

When M is a complex manifold we say that a function u is pluriharmonic if $\overline{\partial}\partial u = 0$. Where $\partial\colon A^{p,q}(M) \to A^{p+1,q}(M)$, $\overline{\partial}\colon A^{p,q} \to A^{p,q+1}(M)$ are given in local coordinates by $\partial\omega = dz^k \wedge \nabla_{\frac{\partial}{\partial z_k}}\omega$, $\overline{\partial}\omega = d\overline{z}^k \wedge \nabla_{\frac{\partial}{\partial \overline{z}^k}}\omega$, and $A^{p,q}(M)$ denotes the space of (p,q)-forms on (M,g).

Definition 2.1 ([13])**.** A manifold M is parabolic if for any the sequence of functions $\{f_i\}$ defined on the annulus $B(R_i) - B(R_0)$, satisfying $\Delta f_i = 0$ on $B(R_i) - B(R_0)$, with boundary conditions

$$f_i = \begin{cases} 1\,, & \text{on} \quad \partial B(R_0) \\ 0\,, & \text{on} \quad \partial B(R_i\,, \end{cases}$$

we have that $\int_M \|\nabla f_i\|^2 \to 0$ as $R_i \to \infty$ for any fixed R_0.

It is known that when $\mathrm{Vol}(B_{x_0}(R))$ satisfies $\int^\infty \frac{R}{\mathrm{Vol}(B_{x_0}(R))} dR = \infty$, then the manifold is parabolic (cf. [6]).

We are more interested in the following definition of parabolicity:

Definition 2.2 ([16])**.** A Riemannian manifold M is called parabolic if for each compact subset $K \subset M$ and each $\epsilon > 0$, there exists a smooth cut-off function $\phi \in C^\infty(M)$ with $0 \le \phi \le 1$, $\phi \equiv 1$ on a neighborhood of K and $\int_M |\nabla\phi|^2 dv < \epsilon$.

Definitions 2.1 and 2.2 are related by the concept of *capacity.* We denote by $\mathrm{Cap}(K,\Omega)$ the capacity of a compact set K inside a domain $\Omega \subset M$. It is defined as

$$\mathrm{Cap}(K,\Omega) = \inf_{u \in L(K,\ \Omega)} \int_\Omega \|\nabla u\|^2 dv\,,$$

where $L(K,\Omega)$ is the set of Lipschitz functions u on M with a compact support in $\overline{\Omega}$ such that $u|_K = 1$, $0 \le u \le 1$. If $\Omega = M$, we write $\mathrm{Cap}(K) = \mathrm{Cap}(K,M)$. It is known that

$$\mathrm{Cap}(K) = \lim_{i\to\infty} \mathrm{Cap}(K,\Omega_i)\,, \tag{2}$$

for any exhaustion sequence $\{\Omega_i\}$, $K \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset M$ and $\bigcup \Omega_i = M$.

Taking $\Omega_i = B_{x_0}(R_i)$, with $R_i \to \infty$, we have a harmonic function h_i that is the solution to the Dirichlet problem in $\Omega_i - K$:

$$\Delta h_i = 0\,, \qquad h_i|_{\partial\Omega_i} = 0\,, \qquad h_i|_{\partial K} = 1\,. \tag{3}$$

Then from Definition 2.1, we have $\lim_{i\to\infty} \int_{\Omega_i} \|\nabla h_i\|^2 dv = 0$, which implies $\mathrm{Cap}(B_{x_0}(R_0)) = 0$. From this we conclude that $\mathrm{Cap}(K) = 0$ for any compact set K, that is we have Definition 2.2 (see [6], Theorem 5.1, (6)).

For the converse, we take a sequence of harmonic functions h_i satisfying the boundary conditions (3). By Definition 2.2 we have that $\mathrm{Cap}(B_{x_0}(R_0)) = 0$, which, using (2), implies by that $\lim_{i\to\infty} \int_{\Omega_i} \|\nabla h_i\|^2 dv = 0$, i.e. we have Definition 2.1.

Now we give our definition of P-weighted parabolicity. Observe that when $P \equiv 1$, it coincides with Definition 2.2 of parabolicity.

Definition 2.3. Let M be a complete Riemannian manifold and P a non-negative continuous function on M. Then M is said *P-weighted parabolic* if for each geodesic ball $B_{x_0}(R)$ and each $\epsilon > 0$, there exist a cut-off function $\phi_R \in C^\infty(M)$ with $0 \leq \phi_R \leq 1$, $\phi_R \equiv 1$ in the ball $B_{x_0}(R)$ and such that $\int_M P|\nabla\phi_R|^2 dv \to 0$ as $R \to \infty$.

Example 2.4. A manifold M is called rotationally symmetric at a point $p \in M$ if the isotropy subgroup at p of the isometry group of M is $O(n)$. By Greene and Wu [3], the rotationally symmetric manifolds are recognized as model spaces for comparison in Riemannian geometry because of the simplicity of their geometric structure. A rotationally symmetric manifold is diffeomorphic to $\mathbb{R}^n$ or to the sphere S^n. The Riemannian metric on M in polar coordinates has the form

$$ds^2 = dr^2 + g^2(r)d\theta^2 \,,$$

with $g(0) = 0$ and $g'(0) = 1$. For instance, when $g(r) = r$, we get the Euclidean space $M = \mathbb{R}^n$, and when $g(r) = \sin r$, we get the round sphere $M = S^n$. On a rotationally symmetric manifold M it is known that the condition

$$\int^\infty \frac{1}{\mathrm{Vol}(S_R^{n-1})} dR = \infty$$

is equivalent to M being parabolic [6], where S_R^{n-1} denotes any $(n-1)$-sphere of radius R.

Given a non-negative continuous function P and defining $K(R) = \int_{B_{x_0}(R)} PdV$ as before, it is easy to see that the condition

$$K'(R) \leq CR \ln R \tag{4}$$

for some constant C, implies that $K(R) \leq CR^2 \ln R$ and then $\int^\infty \frac{R}{K(R)} dR = \infty$. Therefore M is P-weighted parabolic by our main Theorem 1.2.

Let us now assume that the function P depends on the radial coordinate, that is $P = P(r)$. We compute,

$$K(R) = \int_{B_{x_0}(R)} P(r)dV = \int_0^R \int_{S^{n-1}} P(r)g^{n-1}(r)d\Omega dr = \omega_{n-1} \int_0^R P(r)g^{n-1}(r)dr \,,$$

where $\omega_{n-1} = \int_{S^{n-1}} d\Omega$ is the volume of the $(n-1)$-sphere of radius one. It follows that

$$K'(R) = \omega_{n-1} P(R) g^{n-1}(R) \,.$$

Using (4), we conclude that if $P(R)g^{n-1}(R) \leq CR \ln R$, then $\int^\infty \frac{1}{K'(R)} dR = \infty$, that is M is P-weighted parabolic.

Therefore, we have the following:

Proposition 2.5. *Let M be a rotationally symmetric manifold with a metric $ds^2 = dr^2 + g^2(r)d\theta^2$. If the radial function $P(r)$ satisfies $P(r)g^{n-1}(r) \leq Cr \ln r$, then M is $P(r)$-weighted parabolic.*

For instance the Euclidean space $\mathbb{R}^3$ (i.e. $g(r) = r$) is not parabolic but it is $P(r)$-weighted parabolic for any $P(r) \leq C \frac{\ln r}{r}$.

3. Proof of the main theorem

In this section we prove the main result of this paper following some ideas of Section 7 in [6].

Proof of Theorem 1.2. For item (i), since $P(x_0) \neq 0$, by continuity of P we have that $K(r) > 0$ for any $r > 0$.

For $0 < r_0 < R$, define the cut-off function ϕ_R such that

$$\phi_R(x) = \begin{cases} 1, & 0 \leq x \leq r_0 \\ 0, & x \geq R\,. \end{cases}$$

Denoting by ρ the distance function with respect to the point x_0, $\rho(\cdot) = \mathrm{dist}(x_0, \cdot)$, choose ϕ_R on the interval (r_0, R) such that

$$\phi_R'(\rho) = -a_R \frac{\rho}{K(\rho)}\,,$$

where $a_R := \left(\int_{r_0}^R \frac{s}{K(s)} ds\right)^{-1}$. Hence $\phi_R(\rho) = a_R \int_\rho^R \frac{s}{K(s)} ds$, and the function ϕ_R verifies $\phi_R(r_0) = a_R \int_{r_0}^R \frac{s}{K(s)} ds = a_R a_R^{-1} = 1$ and $\phi_R(R) = 0$. Let us divide the interval $[r_0, R]$ into N subintervals of equal length. Denoting by $t_k := r_0 + k\frac{R-r_0}{N}$, $k = 0, \cdots, N-1$, we compute

(5)

$$\begin{aligned} I_R &:= \int_{B_{x_0}(R)-B_{x_0}(r_0)} |\nabla \phi_R|^2 P dv = \sum_{k=0}^{N-1} \int_{B_{x_0}(t_{k+1})-B_{x_0}(t_k)} |\nabla \phi_R|^2 P dv \\ &\leq \sum_{k=0}^{N-1} \frac{a_R^2 s_k^2}{(K(s_k))^2} \int_{B_{x_0}(t_{k+1})-B_{x_0}(t_k)} P dv = \sum_{k=0}^{N-1} \frac{a_R^2 s_k^2}{(K(s_k))^2} (K(t_{k+1}) - K(t_k))\,, \end{aligned}$$

where $s_k \in [t_n, t_{n+1}]$ gives the maximum of $|\nabla \phi_R|$ in the annulus $B_{x_0}(t_{k+1}) - B_{x_0}(t_k)$. The last expression in (5) can be interpreted as a Riemann sum which converges to $\int_{r_0}^R \frac{a_R^2 \rho^2}{(K(\rho))^2} dK(\rho)$ as $N \to \infty$. Using integration by parts, we get

$$\begin{aligned} \int_{r_0}^R \frac{a_R^2 \rho^2}{(K(\rho))^2} dK(\rho) &= -a_R^2 \int_{r_0}^R \rho^2 d\Big(\frac{1}{K(\rho)}\Big) \\ &= -a_R^2 \frac{\rho^2}{K(\rho)}\Big|_{r_0}^R + a_R^2 \int_{r_0}^R 2\frac{\rho}{K(\rho)} d\rho \leq 2a_R^2 a_R^{-1} = 2a_R\,. \end{aligned}$$

By the growth assumption (1), the expression $a_R \to 0$ as $R \to \infty$, which means that $I_R \to 0$ as $R \to \infty$. This concludes the proof of item (i).

For item (ii), using the notations of (i), we choose the cut-off function ϕ_R in the interval (r_0, R) so that

$$\phi_R'(\rho) = -\frac{a_R}{K'(\rho)}\,,$$

where $a_R = \left(\int_{r_0}^{R} \frac{ds}{K'(s)}\right)^{-1}$. Hence, $\phi_R(\rho) = a_R \int_{\rho}^{R} \frac{ds}{K'(s)}$. The same computation as in (5) gives

$$I_R \leq \int_{r_0}^{R} \frac{a_R^2}{K'(\rho)^2} dK(\rho) \,.$$

Moreover $\int_{r_0}^{R} \frac{a_R^2}{K'(\rho)^2} dK(\rho) \leq a_R^2 \int_{r_0}^{R} \frac{K'(\rho)}{K'(\rho)^2} d\rho = a_R^2 a_R^{-1} = a_R$. As before, by the growth assumption (1), we have $a_R \to 0$ as $R \to \infty$, and then $I_R \to 0$ as $R \to \infty$. The conclusion follows. □

Note that in the case $K'(r) > 0$ almost everywhere, item (i) is a consequence of item (ii) using the fact that the condition $\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty$, $R \to \infty$ implies $\int_{r_0}^{R} \frac{1}{K'(r)} dr \to \infty$, $R \to \infty$ (see [15]). For $P \equiv 1$, we get the well known sufficient condition of parabolicity:

Corollary 3.1. *Let M be a complete Riemannian manifold such that* $\mathrm{Vol}(B(R)) = K(R)$*, where $\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty$ as $R \to \infty$, then M is parabolic.*

Remark 3.2. One standard cut-off function is:

$$\phi_R(x) = \begin{cases} 1, & x \in B_{x_0}(e^R) \\ 2 - \frac{\log \rho(x)}{R}, & x \in B_{x_0}(e^{2R}) - B_{x_0}(e^R) \\ 0, & x \in M - B_{x_0}(e^{2R}) \,, \end{cases}$$

where $\rho(x)$ denotes the distance from a fixed point x_0. Then $|\nabla \phi_R(x)| = \frac{1}{R\rho(x)}$. That cut-off function allows a growth of order $O(R^2)$ as in the classical Cheng-Yau theorem on parabolicity. On the other hand, the following less-known (to our knowledge) cut-off function improves the order to $O(R^2 \ln(R))$. Put

$$\phi_R(x) = \begin{cases} 1, & x \in B_{x_0}(e^{e^R}) \\ 2 - \frac{\ln(\ln \rho(x))}{R}, & x \in B_{x_0}(e^{e^{2R}}) - B_{x_0}(e^{e^R}) \\ 0, & x \in M - B_{x_0}(e^{e^{2R}}) \,, \end{cases}$$

then $|\nabla \phi_R(x)| = \frac{1}{R\rho(x) \ln \rho(x)}$. The growth orders can be slightly improved following the pattern but they are all contained in our condition (1).

4. Some applications

Recall that the differential form Ω is *parallel* if $\nabla_X \Omega = 0$ for any vector field $X \in TM$, where as usual ∇ denotes the Levi-Civita connection.

A *quaternionic Kähler manifold* is a Riemannian manifold (M, g) with a rank 3 vector bundle $V \subset \mathrm{End}(TM)$ satisfying:

(1) In any coordinate domain U of M there exists a local basis $\{I, J, K\}$ of V such that

$$I^2 = J^2 = K^2 = -1 \,, \quad IJ = -JI = K \,, \quad JK = -KJ = I \,, \quad KI = -IK = J \,,$$

and $g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y)$ for all $X, Y \in TM$.

(2) If $\phi \in \Gamma(V)$, then $\nabla_X \phi \in \Gamma(V)$ for all $X \in TM$.

If for any nonzero tangent vector X, we have $u_{X,X} + u_{IX,IX} + u_{JX,JX} + u_{KX,KX} = 0$, then the function u is said *quaternionic harmonic*. Here we used the notation $u_{X,X} = \nabla du(X,X)$.

First we recall the following theorem which illustrates the kind of results that can be improved with our main Theorem 1.2.

Theorem 4.1 ([10, Theorem 3.1])**.** *Let M be a complete Riemannian manifold with a parallel p-form Ω. Assume that u is a harmonic function with its Dirichlet integral over geodesic balls centered at x_0 of radius R satisfying the growth condition*

$$\int_{B_{x_0}(R)} |\nabla u|^2 dv = o(R^2)$$

as $R \to \infty$. Then u satisfies

$$d * (du \wedge \Omega) = 0. \tag{6}$$

Note that the vanishing of $d * (du \wedge \Omega)$ implies the vanishing of ∇du and then u is harmonic. The result can be interpreted as an generalization to the non-compact case of the fact that exterior multiplication with a parallel form on a Riemannian manifold commutes with the Laplacian. For others generalizations of expression (6) and further consequences see [2].

Using our main Theorem, we improve the growth of Dirichlet integral to a function $K(r)$ verifying condition (1). We suppose that the harmonic function u is non-constant. Otherwise, the conclusion holds trivially.

Proposition 4.2. *Let M be a complete Riemannian manifold with parallel p-form Ω. Assume that u is a non-constant harmonic function and define the growth function K by $K(r) := \int_{B_{x_0}(r)} |\nabla u|^2 dv$ for some x_0 with $|\nabla u|(x_0) \neq 0$. If the function $K(\cdot)$ verifies that*

$$\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty, \quad R \to \infty,$$

then $d * (du \wedge \Omega) = 0$.

Proof. We use the same cut-off function ϕ_R as in our main theorem. The same computations done in the proof of Theorem 3.1 of [10] leads to

$$\int_M \phi_R^2 |d * (du \wedge \Omega)|^2 dv \leq C \int_M |\nabla \phi_R|^2 | * (du \wedge \Omega)|^2 dv \leq C \int_M |\nabla \phi_R|^2 |du|^2 dv \,, \tag{7}$$

where C is a positive constant.

Because of the growth assumption, the manifold is P-weighted parabolic with $P = |\nabla f|^2$. Thus the right hand side of (7) tends to zero as $R \to \infty$ which implies that $d * (du \wedge \Omega) = 0$.

□

The importance of the condition $d * (du \wedge \Omega) = 0$ for quaternionic Kähler manifolds can be seen in Lemma 3.1 of [10] where it is proved that a function u satisfying $d * (du \wedge \Omega) = 0$, for Ω the parallel 4-form determined by the quaternionic Kähler structure, is quaternionic harmonic. In our case, the growth hypothesis of Proposition 4.2 implies the same conclusion:

Corollary 4.3. *Let M^{4n} be a complete quaternionic Kähler manifold. Assume that u is a harmonic function with its Dirichlet integral satisfying the growth condition $\int_{B_{x_0}(R)} |\nabla u|^2 dv < K(R)$, with $K(R)$ verifying the growth condition* (1)*, then u is quaternionic harmonic.*

Now we present the following result on Kähler manifolds which motivated our research. Consider a smooth function f and a volume measure $e^{-f} dv$, i.e. a weight of the form $P = e^{-f}$. Then, a function u is called f-harmonic if

$$\Delta_f u := (dd_f^* + d_f^* d)u = 0\,,$$

where the co-differential d_f^* verifies $d_f^* = d^* + \iota_{\nabla f}$ and $\iota_{\nabla f}$ denotes the interior product with the vector field ∇f. The original result (cf. [14], Theorem 1.2) supposed a growth of order $O(R^2)$. In fact, this is a Liouville-type theorem because in [14] it was proven that if, in addition, f is proper, then u is constant.

Proposition 4.4. *Let $(M,\ g)$ be a Kähler manifold and $f \in C^\infty(M)$ with ∇f real holomorphic. If u is an f-harmonic function on M such that the growth function $K(r) = \int_{B_{x_0}(r)} |\nabla u|^2 e^{-f} dv$ verifies $\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty$ as $R \to \infty$, then u is pluriharmonic.*

Proof. By the growth assumption, for $\theta := \partial u$, we have

$$\int_{B_{x_0}(r)} |\theta|^2 e^{-f} dv \le CK(r)\,,$$

with $\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty$ as $R \to \infty$, and $C > 0$. By Theorem 1.2, M is P-weighted parabolic with $P = |\theta|^2 e^{-f}$. We repeat the steps (1.3) and (1.4) as in the proof of Theorem 1.2 in [14]. Thus θ is f-harmonic and we get

$$\int_M |d\theta|^2 \phi_R e^{-f} + \int_M |d_f^*\theta|^2 \phi_R e^{-f} \le C' \int_M |\theta|^2 |\nabla \phi_R|^2 e^{-f}\,. \tag{8}$$

By P-weighted parabolicity the right-hand side goes to zero as $R \to \infty$, and therefore (8) implies that $d\theta = d_f^*\theta = 0$. The conclusion follows by observing that $\bar\partial\partial u = d\partial u = d\theta = 0$. □

Using the proof of Proposition 4.4, we can also apply our main theorem to one result of Lam on L^2 harmonic 1-forms (c.f. [11, Theorem 5]), where the 1-form Θ verifies $\int_M |\Theta|^2 < +\infty$.

Proposition 4.5. *Let M be a complete Riemannian manifold with a parallel p-form Ω. Assume that Θ is a harmonic 1-form satisfying $\int_{B_{x_0}(R)} |\Theta|^2 = K(R)$, where $\int_{r_0}^{R} \frac{r}{K(r)} dr \to \infty$ as $R \to \infty$. Then Θ satisfies $d * (\Theta \wedge \Omega) = 0$.*

Proof. Since Θ is a harmonic 1-form and under the growth condition on $\int_{B_{x_0}(R)} |\Theta|^2$, one shows that Θ is closed and co-closed, that is $d\Theta = d^*\Theta = 0$. Indeed, following the steps mentioned in the last proposition we get the estimate (8), without the factors e^{-f}. Then, the same computation as in the original proof of Theorem 5 in [11] gives $*d*(\Theta \wedge \Omega) = (-1)^{n-1} d*(\Theta \wedge *\Omega)$ and the conclusion follows by replacing the cut-off function ϕ by the cut-off function ϕ_R in our main theorem. □

Remark 4.6. The argument above can be generalized to k-forms. In fact, Vieira [21] showed that on a smooth metric measure space $(M, g, e^{-f}dv)$, a L^2_f-integrable f-harmonic k-form Θ is closed and co-closed (i.e. $d^*_f\Theta = 0$). Using our cut-off function, instead of L^2_f-integrability we can allow the growth condition $\int_{B(R)} |\Theta|^2 e^{-f} dv \leq K(R)$, with $K(R)$ verifying condition (1), and conclude as in the proof of Proposition 4.4 that Θ is closed and co-closed.

Remark 4.7. Li showed in [12] a Lemma analogous to Proposition 4.4 (in the unweighted case) for a harmonic map u from a complete Kähler manifold M into a Riemannian manifold N with hermitian negative curvature as defined in [17] and having an energy growth of order $o(R^2)$. When N is also Kähler with strongly seminegative curvature and M is compact, the result was due originally to Siu [18] (see also [17] for a generalization). In the e^{-f}-weighted case, the argument of Siu was generalized by Munteanu-Wang [14] for f-harmonic maps between Kähler manifolds, the target having strongly seminegative curvature, with ∇f real holomorphic and energy growth of order $O(R^2)$.

The consequences of those results are Liouville-type theorems and deep rigidity results which do not depend on our technical improvements. See for instance Chapter 8 of [8] for an introduction on harmonic maps, other related results and more bibliography. Nevertheless, our cut-off function may be useful for other applications having geometrical or topological consequences.

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