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BOUNDARY VALUE PROBLEMS FOR HADAMARD-CAPUTO IMPLICIT FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper, the authors establish sufficient conditions for the existence of solutions to implicit fractional differential inclusions with nonlocal conditions. Both of the cases of convex and nonconvex valued right hand sides are considered.

1. Introduction

In this paper, we are concerned with the existence of solutions to the boundary value problem for implicit fractional differential inclusions

$$ {}^{C}_{H}D^{\alpha}y(t) \in F(t, y(t), {}^{C}_{H}D^{\alpha}y(t)) , \quad \text{for a.e. } t \in J = [1, T],\ 0 < \alpha \leq 1, \tag{1} $$

$$ \sum_{1}^{m} a_k y(t_k) = y_1, \tag{2} $$

where ${}^{C}_{H}D^{\alpha}$ is the Hadamard-Caputo fractional derivative, $F\colon [1,T]\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $y_1 \in \mathbb{R}$, $a_k \in \mathbb{R}$, $k = 1, 2, \dots, m$, and $1 < t_1 < t_2 < \dots < t_m < T$.

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. There are numerous applications in viscoelasticity, electrochemistry, electromagnetism, etc. For basic details of the fractional calculus including some applications and recent results, we recommend the monographs of Kilbas *et al.* [22], Podlubny [24], and the papers of Agarwal *et al.* [5, 6], Momani *et al.* [23], Guerraiche *et al.* [19, 20], and the references therein.

The study of problems with nonlocal conditions was initiated by Byszewski [13] where he proved the existence and uniqueness of mild and classical solutions to nonlocal Cauchy problems. As remarked by Byszewski [13, 14], nonlocal conditions can be more useful than the standard initial condition in describing certain physical

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phenomena. Implicit fractional differential equations have been studied, for example, by Benchohra and Souid in [9, 10, 11].

In this paper, we present existence results for the problem (1)–(2) in case the right hand side is convex valued by using a fixed point theorem of Bohnnenblust-Karlin type, and for the case where the right hand side is nonconvex valued, we use a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [16]. An example is given in Section 4 to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. We let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}\,,$$

and we let $L^1(J;\mathbb{R})$ be the space of Lebesgue integrable functions $y\colon J \to R$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|\,dt\,.$$

Also, we let $AC(J,\mathbb{R})$ denote the space of functions $y\colon J \to \mathbb{R}$ that are absolutely continuous, $AC^1(J,\mathbb{R})$ be the space of functions $y\colon J \to \mathbb{R}$ that are absolutely continuous and have an absolutely continuous first derivative, and if $\delta = t\dfrac{d}{dt}$, then

$$AC^n_\delta(J,\mathbb{R}) = \{y\colon J \longrightarrow \mathbb{R} \mid \delta^{n-1}y(t) \in AC(J,\mathbb{R})\}\,.$$

For any Banach space $(X, \|\cdot\|)$, we set:
$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$;
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$;
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$;
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is *convex (closed)* valued if $G(x)$ is convex (closed) for all $x \in X$. We say that G is bounded on bounded sets if $G(B) = U_{x\in B}G(x)$ is bounded in X for all $B \in P_p(X)$.

The mapping G is *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set $N \subset X$ containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. In addition, G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$. It is well known that if a multivalued map G is completely continuous with nonempty compact values, then in fact G is u.s.c if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, and $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). The map G has a *fixed point* if there is $x \in X$ such that $x \in G(X)$, and the set of fixed points of G will be denoted by $Fix\,G$. We say that a multivalued map $G\colon J \to P_{cl}(\mathbb{R})$ is *measurable* if for every $y \in \mathbb{R}$, the function $t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$ is measurable.

Definition 2.1. A multi-valued maps $F\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(1) $t \to F(t,u,v)$ is measurable for each u, $v \in \mathbb{R}$, and
(2) $u \to F(t,u,v)$ is upper semicontinuous for almost all $t \in J$.
Furthermore, a Carathéodory function is called L^1-Carathéodory if
(3) for each $\rho > 0$, there exists $\phi_\rho \in L^1([0,T],\mathbb{R}^+)$ such that $\|F(t,u,v)\| = \sup\{|v| : v \in F(t,u,v)\} < \phi_\rho(t)$ for all $|v|$, $|u| < \rho$.

Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. The function $H_d\colon P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\}$$

is known as the Hausdorff-Pompeiu metric.

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is:
(1) Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) < \gamma d(x,y) \text{ for each } x,\ y \in X;$$

(2) a contraction if it is γ-Lipschitz with $\gamma < 1$.

The following fixed point result for contraction multivalued maps is due to Covitz and Nadler [16].

Lemma 2.3. *Let (X,d) be a complete metric space. If $N\colon X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

It will be convenient to have the following compactness criteria available in one of our proofs.

Theorem 2.4. (Kolmogorov compactness criterion [17]) *Let $\Omega \subseteq L^p(J,\mathbb{R})$, $1 \leq p \leq +\infty$. If*
(i) *Ω is bounded in $L^p(J,\mathbb{R})$, and*
(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$,*
then Ω is relatively compact in $L^p(J,\mathbb{R})$, where

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)\,ds\,.$$

For additional details on multivalued maps see, for example, the monographs of Aubin and Cellina [7], Aubin and Frankowska [8], Castaing and Valadier [15], or Deimling [17].

In what follows, log denotes the natural logarithm.

Definition 2.5 ([22])**.** The Hadamard fractional integral of order α for a function $h\colon [1,+\infty) \to \mathbb{R}$ is defined as

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds\,, \quad \alpha > 0\,,$$

provided the integral exists.

Definition 2.6 ([4])**.** Let $r \geq 0$ and $n-1 < \alpha < n$, where $n = [\alpha]+1$, and $h \in AC_\delta^n[1,+\infty)$. The Caputo-Hadamard fractional derivative of order α is defined by

$$\begin{aligned}({}_C^H D^\alpha h)(t) &= \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n h(s) \frac{ds}{s} \\ &= {}_H I^{n-\alpha}(\delta^n h)(t)\,.\end{aligned}$$

where $\delta = t\frac{d}{dt}$ and $\delta^n = \delta(\delta^{n-1})$.

Lemma 2.7 ([4])**.** *Let $h \in AC_\delta^n[1,+\infty)$ and $r > 0$. Then*

$${}^H I^r ({}_c^H D^r h)(t) = h(t) - \sum_{i=0}^{n-1} \frac{\delta^i y(1)}{i!} (\log t)^i\,.$$

Proposition 2.8 ([25])**.** *Let α, $\beta > 0$. Then we have:*

(1) *For $I^\alpha \colon L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$, if $f \in L^1(J,\mathbb{R})$, then*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t)\,.$$

(2) *If $f \in L^p(J,\mathbb{R})$, $1 < p < \infty$, then $\|I^\alpha f(t)\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f(t)\|_{L^p}$.*
(3) *The fractional integration operator I^α is linear.*
(4) *The fractional order integral operator I^α maps $L^1(J,\mathbb{R})$ into itself.*
(5) *If $\alpha = n$, then I_0^α is n-fold integration.*
(6) *The Caputo and Riemann-Liouville fractional derivative are linear.*
(7) *The Caputo derivative of a constant is equal to zero.*

Theorem 2.9 (Bohnenblust-Karlin 1950 [12])**.** *Let X be a Banach space, $K \in P_{cl,cv}(X)$, the operator $G \colon K \to P_{cl,cv}(K)$ be upper semicontinuous, and the set $G(K)$ be relatively compact in X. Then G has a fixed point in K.*

We define the set of all measurable selections of F that belong to the Banach space $L^1([1,T],\mathbb{R})$ by

$$S_{F,y}^1 = \{v \in L^1([1,T],\mathbb{R}) : v(t) \in F(t, y(t), {}_H^C D^\alpha y(t)) \ \text{ a.e. } \ t \in [1,T]\}\,.$$

3. Main results

Let us start by defining what we mean by a solution to the problem (1)–(2).

Definition 3.1. A function $y \in AC([1,T],\mathbb{R})$ is a solution of (1)–2 if there exists a function $x \in L^1([1,T],\mathbb{R})$ with $x(t) \in F(t, y(t), {}_H^C D^\alpha y(t))$ for a.e. $t \in [1,T]$ such that ${}_H^C D^\alpha y(t) = x(t)$ and the function y satisfies the conditions (2).

We assume that $\sum_{k=1}^m a_k \neq 0$ and set

$$a = \frac{1}{\sum_{k=1}^m a_k}\,.$$

In order to discuss the existence of solutions to the nonlocal problem (1)–(2), we need the following auxiliary lemma giving an equivalent integral formulation for our problem.

Lemma 3.2. *The nonlocal problem* (1)–(2) *is equivalent to the integral equation*

$$y(t) = ay_1 - \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s} \tag{3}$$

where x is the solution of the functional integral equation

$$x(t) \in F\Big(t, ay_1 - \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}, x(t)\Big). \tag{4}$$

Proof. Let ${}^C_H D^\alpha y(t) = x(t)$ in equation (1), i.e.,

$$x(t) \in F(t, y(t), x(t)) \tag{5}$$

and

$$y(t) = c_1 + {}^H I^\alpha x(t) = c_1 + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}. \tag{6}$$

Letting $t = t_k$ in (6), we obtain

$$y(t_k) = c_1 + \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t_k}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},$$

and so

$$\sum_{k=1}^m a_k y(t_k) = \sum_{k=1}^m a_k c_1 + \sum_{k=1}^m a_k \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t_k}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}. \tag{7}$$

Applying (2) to (7),

$$y_1 = \sum_{k=1}^m a_k c_1 + \sum_{k=1}^m a_k \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t_k}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},$$

and hence

$$c_1 = a\left(y_1 - \sum_{k=1}^m a_k \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t_k}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}\right). \tag{8}$$

Substituting (8) into (6) and (5), we obtain (3) and (4).

To complete the proof, we need to show that equation (3) satisfies the nonlocal problem (1)–(2). Differentiating (3), we obtain

$${}^C_H D^\alpha y(t) = x(t) \in F(t, y(t), {}^C_H D^\alpha y(t)).$$

Letting $t = t_k$ in (3) gives

$$y(t_k) = ay_1.$$

Then,

$$\sum_{k=1}^m a_k y(t_k) = \sum_{k=1}^m a_k a y_1 = y_1.$$

This completes the proof of the equivalence between the nonlocal problem (1)–(2) and the integral equation (3). □

In our main results we will make use of the following conditions.

(H1) $F\colon J\times\mathbb{R}\times\mathbb{R}\to P_{cv,cl}(\mathbb{R})$ is a Carathéodory multi-valued map.
(H2) There exist a positive function $b\in L^1(J)$ and constants $b_i>0$, $i=1,2$, such that

$$\|F(t,u_1,u_2)\|_{\mathcal{P}}=\sup\{|f|:f\in F(t,u_1,u_2)\}\le |b(t)|+b_1|u_1|+b_2|u_2|\,.$$

(H3) There exist constants l_1, $l_2>0$ such that

$$H_d(F(t,x,z),F(t,\bar{x},\bar{z}))<l_1|x-\bar{x}|+l_2|z-\bar{z}|$$

for every x, $\bar{x}$, z, $\bar{z}\in\mathbb{R}$.
(H4) $F\colon J\times\mathbb{R}\times\mathbb{R}\to P_{cp}(\mathbb{R})$ has the property that $F(\cdot,u_1,u_2)\colon J\to P_{cp}(\mathbb{R})$ is measurable and integrably bounded for each u_1, $u_2\in\mathbb{R}$.

Our first result is based on the Bohnenblust-Karlin fixed point theorem.

Theorem 3.3. *Assume that conditions (H1)–(H3) are satisfied. If*

$$\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}<1\,, \tag{9}$$

then the problem (1)–(2) *has at least one solution.*

Remark 3.4. Note that for an L^1-Carathéodory multifunction $F:J\times\mathbb{R}\times\mathbb{R}\to P_{cp}(\mathbb{R})$, the set $S^1_{F,y}$ is not empty.

Proof. We transform the problem (1)–(2) into a fixed point problem by defining the multivalued operator

$$N\colon L^1(J,\mathbb{R})\to\mathcal{P}(L^1(J,\mathbb{R}))$$

by

$$(Nx)(t)=\left\{h\in L^1(J,\mathbb{R}):h(t)=\left\{\begin{array}{l} ay_1-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{t_k}\Big(\log\frac{t}{s}\Big)^{\alpha-1}v(s)\frac{ds}{s}\\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{t}\Big(\log\frac{t}{s}\Big)^{\alpha-1}v(s)\frac{ds}{s}\end{array}\right.\right\} \tag{10}$$

where $v\in S^1_{F,x}$. Clearly, from Lemma 3.2, the fixed points of N are solutions to (1)–(2). We shall show that N satisfies the assumptions of Bohnenblust-Karlin fixed point theorem.

Let

$$r\ge\frac{|ay_1||T-1|+\dfrac{2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|a\|_{L^1}+\dfrac{2b_1|ay_1|(\log T)^{\alpha}}{\Gamma(\alpha+1)}}{1-\Big(\dfrac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}+\dfrac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Big)}$$

and consider the bounded set

$$B_r=\{x\in L^1(J,\mathbb{R}):\|x\|_{L^1}\le r\}\,.$$

The proof will be given in several steps.

Step 1: *$N(x)$ is convex for each $y \in L^1(J,\mathbb{R})$.* Let h_1, h_2 belong to $N(y)$; then there exist selections v_1, $v_2 \in S^1_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) = ay_1 &- \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v_i(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v_i(s) \frac{ds}{s} \end{aligned}$$

for $i = 1, 2$. Let $0 \le d \le 1$. Then, for each $t \in J$,

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) = ay_1 &- \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s}\Big)^{\alpha-1} (dv_1 + (1-d)v_2)(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} (dv_1 + (1-d)v_2)(s) \frac{ds}{s}\,. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(x)\,.$$

Step 2: *$N(B_r)$ is relatively compact.* First we show that $N(B_r)$ is bounded. Let $y \in B_r$. For each $h \in N(x)$ and $t \in J$, by (H2) we have

$$\begin{aligned} \|h\|_{L^1} &= \int_1^T |h(t)|\, dt \\ &= \int_1^T \Big| ay_1 - \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v(s) \frac{ds}{s} \Big|\, dt \\ &\le |ay_1||T-1| + 2\int_1^T \Big| \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v(s) \frac{ds}{s} \Big|\, dt \\ &\le |ay_1||T-1| + 2\int_1^T \Big(\frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} |v(s)| \frac{ds}{s} \Big)\, dt \\ &\le |ay_1||T-1| + 2\int_1^T \Big(\frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} |a(t)| \frac{ds}{s} \Big)\, dt \\ &\quad + 2b_1 \int_1^T \Big(\frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} |ay_1| \Big)\, dt \\ &\quad + 4b_1 \int_1^T \Big(\frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} \Big| \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} x(s) \frac{ds}{s} \Big| \Big)\, dt \\ &\quad + 2b_2 \int_1^T \Big(\frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} |x(s)| \frac{ds}{s} \Big)\, dt \\ &\le |ay_1||T-1| + \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}\|a\|_{L^1} + \frac{2b_1|ay_1|(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}\|x\|_{L^1} \\ &\quad + \frac{2b_2(\log T)^\alpha}{\Gamma(\alpha+1)}\|x\|_{L^1} \\ &\le |ay_1||T-1| + \frac{2(\log T)^\alpha}{\Gamma(\alpha+1)}\|a\|_{L^1} + \frac{2b_1|ay_1|(\log T)^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

$$+\Big(\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Big)r \le r.$$

The above inequalities show that

$$\|N(x)\| = \sup\{\|h\|_{L^1} : h \in N(x)\}\,,$$

which means that $N(B_r) \subset B_r$ and B_r is bounded; that is, $N(B_r)$ is bounded.

Next, we show that $(Nx)_\tau \to (Nx)$ uniformly in $L^1(J,\mathbb{R})$ for each $x \in B_r$. Let $x \in B_r$ and $h \in N(x)$; we then have

$$\begin{aligned}
\|h_\tau - h\|_{L^1} &= \int_1^T |h_\tau(t) - h(t)|\,dt \\
&= \int_1^T \Big|\frac{1}{\tau}\int_t^{t+\tau} h(s)ds - h(t)\Big|\,dt \\
&= \int_1^T \Big(\frac{1}{\tau}\int_t^{t+\tau} |h(s) - h(t)|\,ds\Big)dt \\
&= \int_1^T \Big(\frac{1}{\tau}\int_t^{t+\tau} \Big| -a\sum_{k=1}^m a_k{}^H I^\alpha v(s)|_{s=s_k} + {}^H I^\alpha v(s) \\
&\quad + a\sum_{k=1}^m a_k{}^H I^\alpha v(t)|_{t=t_k} - {}^H I^\alpha v(t)\Big|\,ds\Big)\,dt \\
&= \int_1^T \frac{1}{\tau}\int_t^{t+\tau} \Big| -a\sum_{k=1}^m a_k{}^H I^\alpha v(s)|_{s=s_k} + {}^H I^\alpha v(s) \\
&\quad + a\sum_{k=1}^m a_k{}^H I^\alpha v(t)|_{t=t_k} - {}^H I^\alpha v(t)\Big|\,ds\,dt.
\end{aligned}$$

Since $v(s) \in L^1(J,\mathbb{R})$ by Proposition 2.8 (4), it follows that ${}^H I^\alpha v(s) \in L^1(J,\mathbb{R})$. Hence,

$$\begin{aligned}
&\frac{1}{\tau}\int_t^{t+\tau}\Big|-a\sum_{k=1}^m a_k{}^H I^\alpha v(s)|_{s=s_k} + a\sum_{k=1}^m a_k{}^H I^\alpha v(t)|_{t=t_k} + {}^H I^\alpha v(s) - {}^H I^\alpha v(t)\Big|\,ds \\
&\qquad \le \frac{1}{\tau}\int_t^{t+\tau}\Big|-a\sum_{k=1}^m a_k{}^H I^\alpha v(s)|_{s=s_k} + a\sum_{k=1}^m a_k{}^H I^\alpha v(t)|_{t=t_k}\Big|\,ds \\
&\qquad\quad + \frac{1}{\tau}\int_t^{t+\tau} |{}^H I^\alpha v(s) - {}^H I^\alpha v(t)|\,ds\,.
\end{aligned}$$

Therefore,

$$(Nx)_\tau \to (Nx) \quad \text{uniformly as} \quad \tau \to 0\,.$$

As a consequence of the Kolmogorov compactness criteria, we see that $N(B_r)$ is relatively compact.

Step 3: *N has a closed graph.* Let $x_n \to x_*$, $h_n \in N(x_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies there exists $v_n \in S^1_{F,x_n}$

such that, for each $t \in J$,

$$h_n(t) = ay_1 - \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v_n(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v_n(s) \frac{ds}{s}\,.$$

We must show that there exists $v_* \in S^1_{F,x_*}$ such that, for each $t \in J$,

$$h_*(t) = ay_1 - \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v_*(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s}\Big)^{\alpha-1} v_*(s) \frac{ds}{s}\,.$$

Since $F(t,\cdot,\cdot)$ is upper semicontinuous, for every $\epsilon > 0$ there exist $n_0(x)$ such that $n \geq n_0(x)$ implies $v_n \in F(t,y(t),x(t)) \subset F(t,y_*(t),x_*(t)) + \epsilon B(0,1)$ a.e. $t \in J$.

Since F has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_* \quad \text{as} \quad m \to \infty\,,$$

and so

$$v_* \in F\big(t,y_*(t),x_*(t)\big) \quad \text{for} \quad t \in J\,.$$

For every $w(t) \in F(t,y_*(t),x_*(t))$, we have

$$|v_{n_m} - v_*| \leq |v_{n_m} - w(t)| + |w(t) - v_*|$$

and so

$$|v_{n_m} - v_*| \leq d(v_{n_m}(t), F(t,y_*(t),x_*(t)))\,.$$

By an analogous relation obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$\begin{aligned}
|v_{n_m} - v_*| &\leq H_d(F(t,y_{n_m}(t),x_{n_m}(t)), F(t,y_*(t),x_*(t))) \\
&\leq l_1|y_{n_m} - y_*| + l_2|x_{n_m} - x_*| \\
&\leq l_1|I^\alpha(x_* - x_{n_m})|_{t=t_k} + I^\alpha(x_{n_m} - x_*)| + l_2|x_{n_m} - x_*| \\
&\leq 2l_1|I^\alpha(x_{n_m} - x_*)| + l_2|x_{n_m} - x_*|\,.
\end{aligned}$$

Therefore,

$$|h_{n_m}(t) - h_*(t)| \leq \frac{2}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} |v_{n_m} - v_*| \frac{ds}{s}\,,$$

so

$$\|h_{n_m}(t) - h_*(t)\|_{L^1} \leq \Big(\frac{4l_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2l_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Big) \|x_{n_m}(t) - x_*(t)\|_{L^1}\,.$$

Then,

$$\|h_{n_m}(t) - h_*(t)\|_{L^1} \to 0 \quad \text{as} \quad m \to \infty.$$

Therefore, by the Bohnenblust-Karlin fixed point theorem, Theorem 2.9 above, N has a fixed point x in $B_r \subset L^1(J,\mathbb{R})$ that in turn is a solution of the nonlocal problem (1)–(2). This completes the proof of the theorem. □

We next present a result for the problem (1)–(2) with a nonconvex valued right hand side. Our considerations are based on the fixed point result in Lemma 2.3 for contraction multivalued maps given by Covitz-Nadler.

Theorem 3.5. *Assume that conditions (H3)–(H4) are satisfied. If*

$$\frac{4l_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2l_2(\log T)^{\alpha}}{\Gamma(\alpha+1)} < 1\,, \tag{11}$$

then the problem (1)–(2) *has at least one solution* $y \in L^1(J,\mathbb{R})$.

Remark 3.6. For each $y \in L^1(J,\mathbb{R})$, the set $S^1_{F,y}$ is nonempty since, by (H4), F has a measurable selection (see [15, Theorem III.6]).

Proof of Theorem 3.5. We shall show that N given by (10) satisfies the assumptions of the Covitz and Nadler fixed point theorem. The proof will be given in two steps.

Step 1: $N(x) \in P_{cl}(L^1(J,\mathbb{R}))$ *for all* $x \in L^1(J,\mathbb{R})$. Let $\{h_n\}_{n\geq 0} \in N(x)$ be such that $h_n \to h \in L^1(J,\mathbb{R})$. Then there exists $\{v_n\} \in S^1_{F,y}$ such that, for each $t \in J$,

$$h_n(t) = ay_1 - \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\,.$$

From (H3) and the fact that F has compact values, we may pass to a subsequence if necessary to obtain that v_n converges to v in $L^1(J,\mathbb{R})$, and hence $v \in S^1_{F,y}$. Thus, for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = ay_1 - \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}$$

so $\tilde{h} \in N(x)$.

Step 2: *There exists* $\gamma < 1$ *such that* $H_d(N(y), N(\bar{y})) < \gamma\|y-\bar{y}\|_\infty$ *for all* $y, \bar{y} \in C(J,\mathbb{R})$. Let $y, \bar{y} \in C(J,\mathbb{R})$ and $h_1 \in N(y)$. Then there exists $v_1 \in F(t, y(t), x(t))$ such that

$$h_1(t) = ay_1 - \frac{1}{\Gamma(\alpha)}\int_1^{t_k}\left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\frac{ds}{s}$$

for $t \in J$. From (H3) it follows that

$$H_d(F(t,y(t),x(t)), F(t,\bar{y}(t),\bar{x}(t)) \leq l_1|y(t)-\bar{y}(t)| + l_2|x(t)-\bar{x}(t)|\,.$$

Hence, there exists $w \in F(t,\bar{y}(t),\bar{x}(t))$ such that

$$|v_1(t) - w| \leq l_1|y(t)-\bar{y}(t)| + l_2|x(t)-\bar{x}(t)|, \quad t \in J\,.$$

Consider $U\colon J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l_1|y(t)-\bar{y}(t)| + l_2|x(t)-\bar{x}(t)|\}\,.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\bar{y}(t),\bar{x}(t)))$ is measurable, there exists a measurable selection $v_2(t)$ for V. So $v_2 \in F(t,\bar{y}(t),\bar{x}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \leq l_1|y(t)-\bar{y}(t)| + l_2|x(t)-\bar{x}(t)|$$

$$\leq 2l_1{}^H I^\alpha |x(t) - \bar{x}(t)| + l_2|x(t) - \bar{x}(t)|$$

for $t \in J$. For this v_2, set

$$h_2(t) = ay_1 - \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s} \Big)^{\alpha-1} v_2(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s} \Big)^{\alpha-1} v_2(s) \frac{ds}{s} .$$

Then, for $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s} \Big)^{\alpha-1} |v_1(t) - v_2(t)| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^{t_k} \Big(\log \frac{t}{s} \Big)^{\alpha-1} |v_2(t) - v_1(t)| \frac{ds}{s} \\ &\leq \frac{2}{\Gamma(\alpha)} \int_1^{t} \Big(\log \frac{t}{s} \Big)^{\alpha-1} |v_1(t) - v_2(t)| \frac{ds}{s} . \end{aligned}$$

Thus,

$$\|h_1 - h_2\|_{L^1} \leq \Big(\frac{4l_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2l_2(\log T)^{\alpha}}{\Gamma(\alpha+1)} \Big) \|x(s) - \bar{x}(s)\|_{L^1} .$$

For an analogous relation obtained by interchanging the roles of x and $\bar{x}$, it follows that

$$H_d(N(x), N(\bar{x})) \leq \Big(\frac{4l_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2l_2(\log T)^{\alpha}}{\Gamma(\alpha+1)} \Big) \|x(s) - \bar{x}(s)\|_{L^1} .$$

By (11), N is a contraction, and so by Lemma 2.3, N has a fixed point x that is a solution to (1)–(2). This completes the proof of the theorem. □

4. An example

We conclude this paper with an example to illustrate our main result. We apply Theorem 3.3 to the implicit fractional differential inclusion

$$(12) \qquad {}^C_H D^\alpha y(t) \in F(t, y(t), {}^C_H D^\alpha y(t)) , \quad \text{for a.e.} \quad t \in J = [1, e],\ 0 < \alpha \leq 1 ,$$

$$(13) \qquad \sum_1^m a_k y(t_k) = 1,$$

where

$$F(t, y(t), {}^C_H D^\alpha y(t)) = \{ v \in \mathbb{R} : f_1(t, y(t), {}^C_H D^\alpha y(t)) \leq v \leq f_2(t, y(t), {}^C_H D^\alpha y(t)) \}$$

and $f_1,\ f_2 : J \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for $t \in [1, e]$, $f_1(t, \cdot, \cdot)$ is lower semi-continuous (i.e., the set $\{ y \in \mathbb{R} : f_1(t, y(t), {}^c D^\alpha y(t)) > \mu_1 \}$ is open for each $\mu_1 \in \mathbb{R}$), and assume that for each $t \in [1, e]$, $f_2(t, \cdot, \cdot)$ is upper semi-continuous (i.e., the set $\{ y \in \mathbb{R} : f_2(t, y(t), {}^c D^\alpha y(t)) < \mu_2 \}$ is open for each $\mu_2 \in \mathbb{R}$). For example, if we have

$$\max(f_1(t, y(t), x(t)), f_2(t, y(t), x(t))) \leq \frac{t}{9} + \frac{1}{16}|y(t)| + \frac{1}{16}|x(t)| , \quad \text{for} \quad t \in J ,$$

then we would have $T = e$, $a(t) = \frac{t}{9}$, and $b_1 = b_2 = \frac{1}{16}$. In that case it is easy to see that

$$\Big(\frac{4b_1(\log T)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2b_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Big) = \Big(\frac{1}{4\Gamma(2\alpha+1)} + \frac{1}{8\Gamma(\alpha+1)}\Big) \leq 1\,.$$

Since all the conditions of Theorem 3.3 are satisfied, the problem (12)–(13) would have at least one solution y on $[1, e]$.

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