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(GENERALIZED) FILTER PROPERTIES OF THE AMALGAMATED ALGEBRA

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Abstract. Let R and S be commutative rings with unity, $f\colon R \to S$ a ring homomorphism and J an ideal of S. Then the subring $R \bowtie^f J := \{(a, f(a)+j) \mid a \in R \text{ and } j \in J\}$ of $R \times S$ is called the amalgamation of R with S along J with respect to f. In this paper, we determine when $R \bowtie^f J$ is a (generalized) filter ring.

1. Introduction

Throughout this paper, let R and S be two commutative rings with identity, J be a non-zero proper ideal of S, and $f\colon R \to S$ be a ring homomorphism.

D'Anna, Finocchiaro, and Fontana in [10] and [11] have introduced the following subring (with standard component-wise operations)

$$R \bowtie^f J := \{(r, f(r)+j) \mid r \in R \text{ and } j \in J\}$$

of $R \times S$, called the *amalgamated algebra* (or *amalgamation*) of R with S along J with respect to f. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [13]). Moreover, several classical constructions such as Nagata's idealization (cf. [16, page 2]), the $R+XS[X]$ and the $R+XS[\![X]\!]$ constructions can be studied as particular cases of this construction (see [10, Example 2.5 and Remark 2.8]). Recently, many properties of amalgamations investigated in several papers (e.g. [3], [4], [6], [20], etc.) and the construction has proved its worth providing numerous (counter)examples in commutative ring theory.

In [9], Cuong et al. introduced the notion of filter regular sequence as an extension of regular sequence, and via this notion, they studied f*-modules*, as an extension of (generalized) Cohen-Macaulay modules. This structure is a well-known structure in commutative algebra and have applications in algebraic geometry. Then, in [17], Nhan extended this notion to generalized regular sequence, which in turn,

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leads to the introduction of *generalized f-modules* in [18]. We have the following implications:

$$\text{Gorenstein ring} \Longrightarrow \text{Cohen-Macaulay ring} \Longrightarrow \text{generalized Cohen-Macaulay ring} \Longrightarrow f\text{-ring} \Longrightarrow \text{generalized } f\text{-ring}.$$

It has already investigated that when $R \bowtie^f J$ is one of the three first in the above list ([2], [4], [5], [6]). In this paper, we investigate when it is one of the two last properties.

The proofs for the two case is almost the same, but for f-modules easier. Therefore we deal with case of generalized f-modules in details, and the same proof with minor modifications works in the case of f-modules. We provide a sketch of proof for this case and leave details for the reader.

2. Results

Let us first fix some notation which we shall use throughout the paper: As mentioned above, R and S are two commutative rings with identity, J is an ideal of the ring S, and $f\colon R \to S$ is a ring homomorphism. In the sequel, we consider contractions and extensions with respect to the natural embedding $\iota_R\colon R \to R \bowtie^f J$ defined by $\iota_R(x) = (x, f(x))$, for every $x \in R$.

Let I be an ideal of R, and M be a finitely generated R-module such that $M \neq IM$. We shall refer to the length of a maximal M-sequence contained in I as the depth of M in I, and we shall denote this by $\operatorname{depth}(I, M)$. It will be convenient to use $\operatorname{depth} M$ to denote $\operatorname{depth}(\mathfrak{m}, M)$ when $(R, \mathfrak{m})$ is a local ring.

(Generalized) f-modules are defined in the context of Noetherian local rings for finitely generated modules. Thus we always assume that $(R, \mathfrak{m})$ is a Noetherian local ring and J is finitely generated as an R-module. We will also assume that $J \subseteq \operatorname{Jac}(S)$. When this is the case, $(R \bowtie^f J, \mathfrak{m}'^f)$ is also a Noetherian local ring (see [10, Proposition 5.7] and [12, Corollary 2.7]).

The notion of *M-generalized regular sequence* of M is defined as a sequence $x_1, \dots, x_n$ of elements in $\mathfrak{m}$ such that, for all $i = 1, \dots, n$, $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \dots, x_{i-1})M)$ satisfying $\dim R/\mathfrak{p} > 1$. The length of a maximal generalized regular sequence of M in I is called the *generalized depth of M in I* and denoted by $\operatorname{g-depth}(I, M)$. In this paper, we use the following characterization for $\operatorname{g-depth}(I, M)$ by the support of local cohomology module $H^i_I(M)$:

Lemma 2.1. *Let I be an ideal of R, and M be a finitely generated R-module. Then the following equality holds.*

$$\operatorname{g-depth}(I, M) = \min\{r \mid \textit{there exists } \mathfrak{p} \in \operatorname{Supp}_R(H^r_I(M)) \textit{ such that } \dim R/\mathfrak{p} > 1\}.$$

Proof. If $\dim M/IM > 1$, then the assertion holds by [17, Proposition 4.5]. If $\dim M/IM \leq 1$, then by definition, $\operatorname{g-depth}(I, M) = \infty$. The other side is also infinite since $\operatorname{Supp}_R(H^r_I(M)) \subseteq \operatorname{Supp}(M) \cap \operatorname{Supp}(R/I) = \operatorname{Supp}(M/IM)$. □

The following lemma, which has the key role in the proof of Theorem 2.4, links the g-depth of $R \bowtie^f J$ in the extension ideal $\mathfrak{a}^e$ to the g-depth of R and J in the prime ideal $\mathfrak{a}$.

Lemma 2.2. *Let* $\mathfrak{a} \in \operatorname{Spec}(R)$*. Then the following holds:*

$$\text{g-depth}(\mathfrak{a}^e, R \bowtie^f J) = \min\{\text{g-depth}(\mathfrak{a}, R), \text{g-depth}(\mathfrak{a}, J)\}\,.$$

Proof. We first show that the existence of some $\mathcal{P} \in \operatorname{Supp}_{R\bowtie^f J}\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)$ with the property $\dim R \bowtie^f J/\mathcal{P} > 1$ is equivalent to the existence of some $\mathfrak{p} \in \operatorname{Supp}_R\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)$ with the property $\dim R/\mathfrak{p} > 1$. To achieve this, first we note that, by [11, Lemma 3.6], the extension $\iota_R : R \to R \bowtie^f J$ is integral since we assume that J is finitely generated as an R-module. Therefore, for any $\mathcal{P} \in \operatorname{Spec}(R \bowtie^f J)$, we have $\dim R \bowtie^f J/\mathcal{P} > 1$ if and only if $\dim R/\mathcal{P}^c > 1$. Next, let $\mathcal{P} \in \operatorname{Supp}_{R\bowtie^f J}\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)$, say $\alpha/1$ is a non-zero element of $\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)_{\mathcal{P}}$. If $r \in R$ such that $r\alpha = 0$, then $f(r) \in \mathcal{P}$, i.e. $r \in \mathcal{P}^c$. We have thus proved $\mathcal{P}^c \in \operatorname{Supp}_R\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)$.
Suppose conversely that $\mathfrak{p} \in \operatorname{Supp}_R\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)$. Then, for some ideal $\mathcal{I}$ of $R \bowtie^f J$, with the property $R \bowtie^f J/\mathcal{I} \subseteq H^r_{\mathfrak{a}^e}(R \bowtie^f J)$, we have $\mathfrak{p} \in \operatorname{Supp}_R\left(R \bowtie^f J/\mathcal{I}\right)$. From this we have $\mathcal{I}^c \subseteq \mathfrak{p}$. By lying over property, there exists $\mathcal{P} \in \operatorname{Spec}(R \bowtie^f J)$ such that $\mathcal{I} \subseteq \mathcal{P}$ and $\mathcal{P}^c = \mathfrak{p}$, hence that $\mathcal{P} \in \operatorname{Supp}_{R\bowtie^f J}\left(R \bowtie^f J/\mathcal{I}\right) \subseteq \operatorname{Supp}_{R\bowtie^f J}\left(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\right)$. This completes the proof of our claim. Now we have:

$$\begin{aligned}
\text{g-depth}(\mathfrak{a}^e, R \bowtie^f J) &= \min\{r \mid \exists \mathcal{P} \in \operatorname{Supp}_{R\bowtie^f J}\big(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\big)\,; \dim R \bowtie^f J/\mathcal{P} > 1\}\\
&= \min\{r \mid \exists \mathfrak{p} \in \operatorname{Supp}_R\big(H^r_{\mathfrak{a}^e}(R \bowtie^f J)\big)\,; \dim R/\mathfrak{p} > 1\}\\
&= \min\{r \mid \exists \mathfrak{p} \in \operatorname{Supp}_R\big(H^r_{\mathfrak{a}}(R \bowtie^f J)\big)\,; \dim R/\mathfrak{p} > 1\}\\
&= \min\{r \mid \exists \mathfrak{p} \in \operatorname{Supp}_R(H^r_{\mathfrak{a}}(R) \oplus H^r_{\mathfrak{a}}(J))\,; \dim R/\mathfrak{p} > 1\}\\
&= \min\{\text{g-depth}(\mathfrak{a}, R), \text{g-depth}(\mathfrak{a}, J)\}\,.
\end{aligned}$$

The first and last equality hold by Lemma 2.1, while the second one holds by the above observation. The third equality follows by the Independence Theorem of local cohomology [7, Theorem 4.2.1], and the forth equality obtained using the R-module isomorphism $R \bowtie^f J \cong R \oplus J$ [10, Lemma 2.3]. □

Generalized f-modules were introduced in [18] as modules for which every system of parameters is a generalized regular sequence. A ring is called a *generalized f-ring* if it is a generalized f-module over itself. For more details we refer the reader to [17] and [18]. We define a finitely generated R-module M to be *maximal generalized f-module* if $\text{g-depth}(\mathfrak{p}, M) = \dim(R) - \dim(R/\mathfrak{p})$, for any $\mathfrak{p} \in \operatorname{Supp} M$ satisfying $\dim R/\mathfrak{p} > 1$. This definition has stem in the following proposition.

Proposition 2.3. *Assume that* M *is a finitely generated* R*-module such that* $\dim M > 1$*. Then the following statements are equivalent:*

(1) M *is a generalized f-module.*
(2) $\text{g-depth}(\mathfrak{p}, M) = \dim(M) - \dim(R/\mathfrak{p})$ *for each* $\mathfrak{p} \in \operatorname{Supp} M$ *satisfying* $\dim R/\mathfrak{p} > 1$.
(3) $\text{g-depth}(I, M) = \dim(M) - \dim(R/I)$ *for any proper ideal* I *of* R *satisfying* $I \supseteq \operatorname{Ann}(M)$ *and* $\dim R/I > 1$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) is by [18, Proposition 2.5]. The proof of (2) $\Rightarrow$ (3) is similar to the proof of [14, Remark 4.2], using [17, Proposition 4.3 (ii)] and [18, Proposition 2.5]. $\square$

We use the above proposition to investigate when $R \bowtie^f J$ is a generalized f-ring, which is one of our main results. Recall that a finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ is called a *maximal Cohen-Macaulay R-module* if $\operatorname{depth} M = \dim R$. In the sequel, when we consider J as a module, we always consider it as an R-module via the homomorphism $f : R \to S$. In particular, by $\operatorname{Supp} J$ we mean $\operatorname{Supp}_R J$.

Theorem 2.4. *The following statements are equivalent:*

(1) *$R \bowtie^f J$ is a generalized f-ring.*
(2) *R is a generalized f-ring and J is a maximal generalized f-module.*
(3) *R is a generalized f-ring and $J_\mathfrak{p}$ is maximal Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Supp}(J)$ satisfying $\dim R/\mathfrak{p} > 1$.*

Proof. We first assume that $\dim J > 1$. The process of proof shows that the opposite assumption, $\dim J \leq 1$, leads to trivial cases.
(1) $\Rightarrow$ (2) Assume that $R \bowtie^f J$ is a generalized f-ring and pick $\mathfrak{p} \in \operatorname{Spec}(R)$ satisfying $\dim R/\mathfrak{p} > 1$. By [11, Lemma 3.6], $\iota_R : R \to R \bowtie^f J$ is an integral extension. Hence, by lying over property, $\mathfrak{p} = \mathfrak{p}^{ec}$, hence that $\dim R \bowtie^f J/\mathfrak{p}^e = \dim R/\mathfrak{p} > 1$. Now, by Proposition 2.3 and Lemma 2.2, we have:

$$\begin{aligned} \dim R - \dim R/\mathfrak{p} &= \dim R \bowtie^f J - \dim R \bowtie^f J/\mathfrak{p}^e \\ &= \text{g-depth}(\mathfrak{p}^e, R \bowtie^f J) \\ &\leq \text{g-depth}(\mathfrak{p}, R) \\ &\leq \dim R - \dim R/\mathfrak{p}\,. \end{aligned}$$

Again we use Proposition 2.3 to see that R is a generalized f-ring, and a similar argument will show that J is a maximal generalized f-module.

(2) $\Rightarrow$ (1) Suppose that R is a generalized f-ring and J is a maximal generalized f-module. Then, from Lemma 2.2 and Proposition 2.3, we deduce that $\text{g-depth}(\mathfrak{p}^e, R \bowtie^f J) = \text{g-depth}(\mathfrak{p}, R)$, for any $\mathfrak{p} \in \operatorname{Spec}(R)$. Now, let $\mathcal{P} \in \operatorname{Spec}(R \bowtie^f J)$ and $\dim R \bowtie^f J/\mathcal{P} > 1$. Then $\dim R/\mathcal{P}^c > 1$ and, by Lemma 2.2 and Proposition 2.3, we have:

$$\begin{aligned} \dim R \bowtie^f J - \dim R \bowtie^f J/\mathcal{P} &= \dim R - \dim R/\mathcal{P}^c \\ &= \text{g-depth}(\mathcal{P}^c, R) \\ &= \text{g-depth}(\mathcal{P}^{ce}, R \bowtie^f J) \\ &\leq \text{g-depth}(\mathcal{P}, R \bowtie^f J) \\ &\leq \dim R \bowtie^f J - \dim R \bowtie^f J/\mathcal{P}\,. \end{aligned}$$

Thus inequalities are equality, and another appeal to Proposition 2.3 gives the desired conclusion.

(2) $\Rightarrow$ (3) Let $\mathfrak{p} \in \operatorname{Supp}(J)$ with the property $\dim R/\mathfrak{p} > 1$. In order to show that $J_\mathfrak{p}$ is maximal Cohen-Macaulay, observe that [17, Proposition 4.4] together

with our assumptions yields the following inequalities:

$$\operatorname{depth} J_{\mathfrak{p}} \geq \text{g-depth}(\mathfrak{p}, J) = \dim R - \dim R/\mathfrak{p} \geq \dim R_{\mathfrak{p}} \geq \operatorname{depth} J_{\mathfrak{p}}.$$

(3) $\Rightarrow$ (2) Let $\mathfrak{p} \in \operatorname{Supp}(J)$ satisfying $\dim R/\mathfrak{p} > 1$. Then, using [17, Proposition 4.4] and [8, Proposition 1.2.10(a)], we get a prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$ such that $\mathfrak{q} \in \operatorname{Supp}(J)$, $\dim R/\mathfrak{q} > 1$, and $\text{g-depth}(\mathfrak{p}, J) = \operatorname{depth} J_{\mathfrak{q}}$. The following inequalities complete the proof:

$$\begin{gathered}\text{g-depth}(p, J) = \operatorname{depth} J_{\mathfrak{q}} = \dim R_{\mathfrak{q}} \geq \text{g-depth}(\mathfrak{q}, R) = \\ \dim R - \dim R/\mathfrak{q} \geq \dim R - \dim R/\mathfrak{p} \geq \text{g-depth}(p, J)\,.\end{gathered}$$

□

Recall that if $f := id_R$ is the identity homomorphism on R, and I is an ideal of R, then $R \bowtie I := R \bowtie^{id_R} I$ is called the amalgamated duplication of R along I. The next corollary deals with this case.

Corollary 2.5. *$R \bowtie I$ is a generalized f-ring if and only if R is a generalized f-ring and I is maximal generalized f-module if and only if R is a generalized f-ring and $I_{\mathfrak{p}}$ is maximal Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Supp}(I)$ satisfying $\dim R/\mathfrak{p} > 1$.*

Let M be an R-module. Nagata (1955) considered a ring extension of R called the the *idealization* of M in R , denoted here by $R \ltimes M$ [16, page 2]. As in [10, Remark 2.8], if $S := R \ltimes M$, $J := 0 \ltimes M$, and $\iota : R \to S$ be the natural embedding, then $R \bowtie^{\iota} J \cong R \ltimes M$. It is easy to check that, as R-modules, $0 \ltimes M \cong M$. The following corollary shows when the idealization is generalized f-ring.

Corollary 2.6. *If M is a finitely generated R-module, then $R \ltimes M$ is a generalized f-ring if and only if R is a generalized f-ring and M is a maximal generalized f-module if and only if R is a generalized f-ring and $M_{\mathfrak{p}}$ is maximal Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Supp} M$ satisfying $\dim R/\mathfrak{p} > 1$.*

In the remaining part of the paper we investigate when $R \bowtie^f J$ is an f-ring. The arguments are the same as the ones in the case of generalized f-ring. But, for the readers convenience, we give brief proofs and refer the reader to previous arguments.

The notion of *M-filter regular sequence* is defined as a sequence $x_1, \ldots, x_n$ of elements in $\mathfrak{m}$ such that $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \ldots, x_{i-1})M) \setminus \{\mathfrak{m}\}$ and for all $i = 1, \ldots, n$. The *filter depth*, $\text{f-depth}(I, M)$, of I on M is defined as the length of any maximal M-filter regular sequence in I. Here, we use the following characterization for $\text{f-depth}(I, M)$ (see [15, Theorem 3.1] and [14, Theorem 3.10]):

$$\text{f-depth}(I, M) = \inf\{r \mid H^r_I(M) \text{ is not an Artinian } R\text{-module}\}\,.$$

The following lemma expresses $\text{f-depth}(\mathfrak{p}^e, R \bowtie^f J)$, the f-depth of extension of a prime ideal $\mathfrak{p}$ of R in $R \bowtie^f J$. For the proof, we use the elementary fact that being Artinian as an $R \bowtie^f J$-module is the same as being Artinian as an R-module.

Lemma 2.7. *Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the following holds:*

$$\text{f-depth}(\mathfrak{p}^e, R \bowtie^f J) = \min\{\text{f-depth}(\mathfrak{p}, R), \text{f-depth}(\mathfrak{p}, J)\}.$$

Proof. By [14, Theorem 3.10] (and arguments similar to Lemma 2.2), we have:

$$\begin{aligned}\text{f-depth}(\mathfrak{p}^e, R \bowtie^f J) &= \inf\{r \mid H^r_{\mathfrak{p}^e}(R \bowtie^f J) \text{ is not Artinian } R \bowtie^f J\text{-module}\}\\ &= \inf\{r \mid H^r_{\mathfrak{p}^e}(R \bowtie^f J) \text{ is not Artinian } R\text{-module}\}\\ &= \inf\{r \mid H^r_{\mathfrak{p}}(R \bowtie^f J) \text{ is not Artinian } R\text{-module}\}\\ &= \inf\{r \mid H^r_{\mathfrak{p}}(R) \oplus H^r_{\mathfrak{p}}(J) \text{ is not Artinian } R\text{-module}\}\\ &= \min\{\text{f-depth}(\mathfrak{p}, R), \text{f-depth}(\mathfrak{p}, J)\}\,.\end{aligned}$$

□

In [9], the authors introduced *f-modules* as modules for which every system of parameters is a filter regular sequence. The ring R is called an *f-ring* if it is an f-module over itself. This structure is a well-known structure in commutative algebra and have applications in algebraic geometry. For more details we refer the reader to [9], [14], and [21]. We define an R-module M to be *maximal f-module* if $\text{f-depth}(\mathfrak{p}, M) = \dim(R) - \dim(R/\mathfrak{p})$, for any $\mathfrak{p} \in \operatorname{Supp} M \setminus \{\mathfrak{m}\}$. This definition has stem in the following Proposition [14, Theorem 4.1 and Remark 4.2]:

Proposition 2.8. *For a finitely generated R-module M, the following statements are equivalent:*

(1) *M is an f-module*
(2) *for any $\mathfrak{p} \in \operatorname{Supp} M \setminus \{\mathfrak{m}\}$, $\text{f-depth}(\mathfrak{p}, M) = \dim(M) - \dim(R/\mathfrak{p})$*
(3) *for any proper ideal I of R with the property $I \supseteq \operatorname{Ann}(M)$ and $\sqrt{I} \neq \mathfrak{m}$, $\text{f-depth}(I, M) = \dim(M) - \dim(R/I)$*

We use the above proposition to investigate when $R \bowtie^f J$ is f-ring, which is our final result.

Theorem 2.9. *The following statements are equivalent:*

(1) *$R \bowtie^f J$ is an f-ring.*
(2) *R is an f-ring and J is a maximal f-module.*
(3) *R is an f-ring and $J_{\mathfrak{p}}$ is maximal Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Supp}(J) \setminus \{\mathfrak{m}\}$.*

Proof. (1) ⇒ (2) Assume that $R \bowtie^f J$ is an f-ring and pick $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. As before, the extension $\iota_R : R \to R \bowtie^f J$ is integral, and so $\mathfrak{p} = \mathfrak{p}^{ec}$. Thus $\sqrt{\mathfrak{p}^e} \neq \mathfrak{m}'^f$ and $\dim R \bowtie^f J/\mathfrak{p}^e = \dim R/\mathfrak{p}$. Then Proposition 2.8 gives the desired conclusion, just as in the proof of Theorem 2.4.

(2) ⇒ (1) Suppose that R is an f-ring and J is a maximal f-module, and let $\mathcal{P} \in \operatorname{Spec}(R \bowtie^f J) \setminus \{\mathfrak{m}'^f\}$. Then $\mathcal{P}^c \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ and Proposition 2.8 gives the desired conclusion, as in the case of Theorem 2.4.

(2) ⇔ (3) The proof of this part is the same as the proof in Theorem 2.4, using the following equality instead of [17, Proposition 4.4]:

$$\text{f-depth}(\mathfrak{p}, J) = \min\{\operatorname{depth}(\mathfrak{p}R_{\mathfrak{q}}, J_{\mathfrak{q}}) \mid \mathfrak{q} \in \operatorname{Supp}(J/\mathfrak{p}J) \setminus \{\mathfrak{m}\}\}\,.$$

For the proof the equality, see the proof of [14, Theorem 3.10]. □

Corollary 2.10 (cf. [19, Theorem 3.5])**.** *$R \bowtie I$ is an f-ring if and only if R is an f-ring and I is maximal f-module if and only if R is an f-ring and $I_\mathfrak{p}$ is maximal Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Supp}(I) \setminus \{\mathfrak{m}\}$.*

Corollary 2.11. *If M is a finitely generated R-module, then $R \ltimes M$ is an f-ring if and only if R is an f-ring and M is a maximal f-module if and only if R is an f-ring and $M_\mathfrak{p}$ is maximal Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$.*

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