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FOREWORD TO PROCEEDINGS OF EQUADIFF 15

The Conference on Differential Equations and Their Applications – abbreviated as Equadiff – is one of the oldest active series of mathematical conferences in the world. The tradition of the Czechoslovak Equadiff dates back to 1962 when Equadiff 1 took place in Prague. The subsequent Czechoslovak Equadiff conferences are held since then periodically in Prague, Bratislava, and Brno every four years (with few exceptions). The Western Equadiff conferences are organized in various cities in Western Europe, starting in Marseille in 1970 and with the last meeting in Leiden in 2019.

The last Equadiff was held in Brno in summer 2022 as the 15th conference within the Czechoslovak Equadiff series, and hence it bears the name Equadiff 15. The conference was rescheduled to the year 2022 from the original date in July 2021 due to an unstable pandemic situation in the world. The proceedings from all previous Czechoslovak Equadiff conferences are available via the Czech Digital Mathematics Library at

`https://dml.cz/handle/10338.dmlcz/700001`.

The conference Equadiff 15 was organized by joint efforts of the Faculty of Science of Masaryk University (and its Department of Mathematics and Statistics) with the Faculty of Civil Engineering of Brno University of Technology, the Institute of Mathematics of the Czech Academy of Sciences, and the Brno branch of The Union of Czech Mathematicians and Physicists. The conference took place at the campus of the Faculty of Economics and Administration of Masaryk University from July 11 till July 15, 2022. More than 250 participants from 37 countries from all over the world attended the 241 talks of the conference, including 6 plenary talks, 17 invited talks, 124 talks in 33 organized minisymposia, 75 contributed talks, and 19 posters.

The proceedings of Equadiff 15 cover the theory of differential equations in a broad sense, including their theoretical aspects, numerical methods, and applications. The proceedings contain 29 scientific articles written by participants of Equadiff 15. The papers are divided into three sections according to the program of the conference:

- ordinary differential equations (15 papers),
- partial differential equations (9 papers),
- numerical analysis and applications (5 papers).

Each manuscript underwent a rigorous refereeing process to ensure its scientific quality. This issue contains the contributions from section Ordinary differential equations.

We would like to take this opportunity to express our special thanks to all the participants for their active contributions to the success of the Equadiff 15 conference. Our gratitude and appreciation belong to the members of the Scientific Committee who ensured the high standards of the scientific activities of the conference, to the organizers and supporting PhD students for their efforts towards the realization of the conference, to the administration of the Faculty of Economics and Administration of Masaryk University for providing the venue for the conference and for their organizational support, to the management and employees of the Accommodation and Catering Services of Masaryk University for their help with the organization and realization of the catering during the conference, to the workers of the Botanical Garden of the Faculty of Science of Masaryk University for providing the flower decoration, and to the director of the Department of Mathematics and Statistics of the Faculty of Science of Masaryk University for financial support. We also thank to Ilona Lukešová from the Editorial Office of Archivum Mathematicum for her extensive editorial work on these proceedings.

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Zuzana Došlá
Jan Chleboun
Pavel Krejčí
Martin Kružík
Šárka Nečasová
Roman Šimon Hilscher
(The editors)

The guest editors

Zuzana Došlá: Masaryk University, Faculty of Science, Kotlářská 2, 611 37 Brno, Czech Republic, `dosla@math.muni.cz`

Jan Chleboun: Czech Technical University in Prague, Faculty of Civil Engineering, Thákurova 7, 166 29 Praha 6, Czech Republic, `jan.chleboun@cvut.cz`

Pavel Krejčí: Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, `pavel.krejci@cvut.cz`

Martin Kružík: Institute of Information Theory and Automation of the Czech Academy of Sciences, Pod vodárenskou věží 4, 182 00 Praha 8, Czech Republic, `kruzik@utia.cas.cz`

Šárka Nečasová: Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, `matus@math.cas.cz`

Roman Šimon Hilscher: Masaryk University, Faculty of Science, Kotlářská 2, 611 37 Brno, Czech Republic, `hilscher@math.muni.cz`

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TOPOLOGICAL ENTROPY AND DIFFERENTIAL EQUATIONS

Jan Andres and Pavel Ludvík

Abstract. On the background of a brief survey panorama of results on the topic in the title, one new theorem is presented concerning a positive topological entropy (i.e. topological chaos) for the impulsive differential equations on the Cartesian product of compact intervals, which is positively invariant under the composition of the associated Poincaré translation operator with a multivalued upper semicontinuous impulsive mapping.

1. Introduction and some preliminaries

The main aim of this short note is two-fold: (i) to describe briefly the recent state of the study of a topic at the title, (ii) to indicate the investigation of topological entropy for differential equations with multivalued impulses.

The first definition of topological entropy was given in 1965 by Adler, Konheim and McAndrew for (single-valued) continuous maps in compact topological spaces (see [1]). Another definition was introduced in 1971 by Bowen for uniformly continuous maps in not necessarily compact metric spaces (see [10]), who proved the equivalence of his definition with the one in [1] in compact metric spaces.

Definition 1 (cf. [10])**.** Let (X,d) be a metric space, K be a compact subset of X and $f\colon X\to X$ be a uniformly continuous map. A set $S\subset K$ is called (n,ε)-*separated* with respect to f, for a positive integer n and $\varepsilon>0$, if for every pair of distinct points $x,y\in S$, $x\neq y$, there is at least one k with $0\leq k<n$ such that $d(f^k(x),f^k(y))>\varepsilon$. Then, denoting the number of different orbits of length n by

$$s(n,\varepsilon,f,K):=\max\{\#S:\ S\subset X \text{ is an } (n,\varepsilon)\text{-separated set with respect to } f\}\,,$$

the *topological entropy* $h(f)$ of f is defined as

$$h(f):=\sup_{K\subset X,\ K\text{ is compact}}\ \lim_{\varepsilon\to 0}\left[\limsup_{n\to\infty}\frac{1}{n}\log s(n,\varepsilon,f,K)\right]\,. \tag{1}$$

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The topological entropy in the sense of Definition 1 is, besides other things, a topological invariant, but not a homotopy invariant. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then according to [10, Theorem 15]

$$h(f) = \sum_{|\lambda_i|>1} \log|\lambda_i|\,, \tag{2}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f.

Definition 1 can be applied to differential equations via the associated Poincaré translation operators along their trajectories (see Section 2 below).

In this way, the upper and lower estimates of topological entropy were obtained for linear systems of ordinary differential equations in e.g. [9, 12, 23, 24].

For nonlinear systems, the obtained results can be characterized as either generic in $\mathbb{R}^2$ (lower estimates) by means of the Artin braid group theory (see e.g. [15, 19, 20, 27]) or rather implicit for higher-dimensional systems (upper and lower estimates, especially in $\mathbb{R}^3$) (see e.g. [16], [21], and the references therein) or just numerical (see e.g. [26]).

For systems with impulses, the results are rare. For nonlinear impulsive systems (see e.g. [2, 5, 8]), and for multivalued or discontinuous impulsive systems (see e.g. [3, 4, 7, 13, 14]).

Of course, if the systems or impulses are multivalued, then Definition 1 is insufficient, and must be appropriately changed. A suitable definition with this respect seems to be the following one by Kelly and Tennant (see [17]).

Definition 2 (cf. [17])**.** Let (X, d) be a compact metric space and $\varphi\colon X \to \mathcal{K}(X)$, where $\mathcal{K}(X) := \{K \subset X :\ K \text{ is a non-empty compact subset}\}$, be an upper semicontinuous map (i.e. φ has a closed graph $\Gamma_\varphi := \{(x, y) \in X \times X : y \in \varphi(x)\}$.

Let the *space of n-orbits* of φ be denoted as

$$\mathrm{Orb}_n(\varphi) := \{(x_1, \dots, x_n) \in X^n : x_{i+1} \in \varphi(x_i), i = 1, \dots, n-1\}\,.$$

We say that $S \subset \mathrm{Orb}_n(\varphi)$ is (n, ε)-*separated* for φ, for a positive integer n and $\varepsilon > 0$, if for every pair of distinct n-orbits $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ there is at least one k with $1 \le k \le n$ such that $d(x_k, y_k) > \varepsilon$.

The *topological entropy* $h_{\mathrm{KT}}(\varphi)$ of φ is defined as

$$h_{\mathrm{KT}}(\varphi) := \lim_{\varepsilon\to 0}\left[\limsup_{n\to\infty} \frac{1}{n}\log(s(n, \varepsilon, \varphi))\right], \tag{3}$$

where $s(n, \varepsilon, \varphi)$ stands for the largest cardinality of an (n, ε)-separated subset of $\mathrm{Orb}_n(\varphi)$ for φ, i.e.

$$s(n, \varepsilon, \varphi) := \max\{\#S :\ S \subset \mathrm{Orb}_n(\varphi) \text{ is an } (n, \varepsilon)\text{-separated set for } \varphi\}\,.$$

One can readily check that, for single-valued continuous maps in compact metric spaces, Definition 2 reduces to Definition 1.

It will be convenient to prove the following lemma.

Lemma 3. *Let $\varphi : X \to \mathcal{K}(X)$ and $\psi : Y \to \mathcal{K}(Y)$ be upper semicontinuous maps with compact convex values in compact convex subsets X and Y of Banach spaces.*

Then the inequality

$$h_{\mathrm{KT}}(\varphi \times \psi) \geq \max\{h_{\mathrm{KT}}(\varphi), h_{\mathrm{KT}}(\psi)\}$$

holds for the Cartesian product $\varphi \times \psi : X \times Y \to \mathcal{K}(X \times Y)$, *where* $(\varphi \times \psi)(x, y) = \varphi(x) \times \psi(y)$, *for every* $(x, y) \in X \times Y$, *and* $X \times Y$ *is endowed with the maximum norm.*

Proof. According to the well known Kakutani-Fan's theorem (see e.g. [6, Corollary I.6.21]), there exist fixed points $\overline{x} \in \varphi(\overline{x})$ and $\overline{y} \in \psi(\overline{y})$ of φ and ψ, and subsequently $(\overline{x}, \overline{y}) \in (\varphi \times \psi)(\overline{x}, \overline{y})$ of $\varphi \times \psi$.

One can easily check that from Definition 2 it immediately follows (cf. [17, Theorem 4.2]):

$$\begin{aligned} h_{\mathrm{KT}}(\varphi \times \psi) &\geq \max\{h_{\mathrm{KT}}(\varphi \times \mathrm{id}\,|_{\overline{y}}), h_{\mathrm{KT}}(\mathrm{id}\,|_{\overline{x}} \times \psi)\} \\ &= \max\{h_{\mathrm{KT}}(\varphi), h_{\mathrm{KT}}(\psi)\}\,, \end{aligned}$$

which completes the proof. □

Remark 4. The inequality in Lemma 3 can be generalized to the equality

$$h_{\mathrm{KT}}(\varphi_1 \times \ldots \times \varphi_n) = \sum_{i=1}^{n} h_{\mathrm{KT}}(\varphi_i)\,,$$

where $\varphi_i : X_i \to \mathcal{K}(X_i)$ are upper semicontinuous maps in compact metric spaces X_i, $i = 1, \ldots, n$. Its proof is rather technical, but can be made quite analogously as in the single-valued case (see e.g. [25, Theorem 7.10]).

Let us note that in our papers [3, 4, 7] still formally another extension of Definition 1 was employed, matching with the Nielsen fixed point theory on tori $\mathbb{R}^n/\mathbb{Z}^n$. On the other hand, here we would like to follow rather the ideas from [8], where single-valued arguments were, however, exclusively applied on compact subsets in Euclidean spaces $\mathbb{R}^n$.

2. Some further preliminaries

Hence, as already pointed out, it will be also convenient to recall some properties of the Poincaré translation operators $T_\omega \colon \mathbb{R}^n \to \mathbb{R}^n$, associated with the differential equation

$$x' = F(t, x) \tag{4}$$

where $F \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $F(t, x) \equiv F(t + \omega, x)$, for some given $\omega > 0$, and is the *Carathéodory mapping*, i.e.

(i) $F(\cdot, x) \colon [0, \omega] \to \mathbb{R}^n$ is measurable, for every $x \in \mathbb{R}^n$,

(ii) $F(t, \cdot) \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous, for almost all (a.a.) $t \in [0, \omega]$.

Let, furthermore, (4) satisfy a uniqueness condition and all solutions of (4) entirely exist on the whole $(-\infty, \infty)$.

By a (*Carathéodory*) *solution* $x(\cdot)$ of (4), we understand a locally absolutely continuous function, i.e. $x \in AC_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, which satisfies (4) for a.a. $t \in \mathbb{R}$. For a continuous right-hand side F, we have obviously $x \in C^1(\mathbb{R}, \mathbb{R}^n)$.

The *Poincaré translation operator* $T_\omega : \mathbb{R}^n \to \mathbb{R}^n$ along the trajectories of (4) is defined as follows:

$$T_\omega(x_0) := \{x(\omega) : x(\cdot) \text{ is a solution of (4) such that } x(0) = x_0\}\,. \tag{5}$$

It is well known (see e.g. [18, 22]) that T_ω is an orientation-preserving homeomorphism such that $T_\omega^k = T_{k\omega}$, for every $k \in \mathbb{N}$. It is also isotopic to identity.

If $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$, then T_ω is a diffeomorphism of class C^1 such that $\det \mathrm{D}\, T_\omega^k(x_0) > 0$ holds for every $x_0 = T_\omega^k(x_0)$ and any $k \in \mathbb{N}$, where the mapping $\mathrm{D}\, T_\omega(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ denotes the Fréchet derivative of T_ω at $x_0 \in \mathbb{R}^n$, which is a linear map corresponding to the Jacobian matrix of T_ω at x_0.

For $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$, we can consider the variation equation of (4) with respect to an ω-periodic solution $x(\cdot)$ of (4), namely

$$x' = \mathrm{D}_x\, F(t, x(t))x\,, \tag{6}$$

where $\mathrm{D}_x\, F(t,x)$ is the Jacobian matrix of $F(t,x)$ with respect to x. It is a linear differential equation with ω-periodic continuous coefficients. If $W(t)$ is its fundamental matrix, then $\mathrm{D}\, T_\omega(x_0) = W(\omega)W(0)^{-1}$.

Now, consider the vector linear equation

$$x' = A(t)x\,, \tag{7}$$

$A(t) = \{a_{ij}(t)\}_{i,j=1}^n$, with ω-periodic measurable coefficients $a_{ij}(t)$, where $\left|\int_0^\omega a_{ij}(t)\,\mathrm{d}t\right| < \infty$, for all $i, j = 1, \dots, n$. It is well known that its solution $x(\cdot)$ with $x(0) = x_0$ can be expressed as $x(t) = W(t)x_0$, where $W(t)$ is the fundamental matrix of (7) such that $W(0) = W^{-1}(0)$ is a unit matrix. Thus, $T_\omega(x_0) = x(\omega) = W(\omega)x_0$. The operator $W(\omega)$ is called a *monodromy operator* and the eigenvalues of the related matrix are the *multiplicators* of (7). The same terminology is related to (6).

If, in particular, $A(t) \equiv A$ in (7) has constant coefficients $a_{ij}(t) \equiv a_{ij}$; $i, j = 1 \dots, n$, then the solutions $x(\cdot)$ of (7) take the simple form $x(t) = e^{At}x_0$, i.e. $W(t) = e^{At}$, and so

$$T_\omega(x_0) = e^{A\omega}x_0 = W(\omega)x_0\,.$$

The multiplicators μ_i of the monodromy matrix $W(\omega)$ can be therefore easily expressed as

$$\mu_i = e^{\lambda_i \omega}\,, \quad i = 1, \dots, n\,,$$

where λ_i, $i = 1, \dots, n$, are the eigenvalues of A. Hence, in order to have $|\mu_i| > 1$, for some $i \in \{1, \dots, n\}$, it is sufficient and necessary that $\operatorname{Re}\lambda_i > 0$ for such $i \in \{1, \dots, n\}$.

Because of a uniqueness condition, there is, for any $k \in \mathbb{N}$, an evident one-to-one correspondence between the $k\omega$-periodic solutions $x(\cdot)$ of (4), $x(t) \equiv x(t + k\omega)$ and $x(t) \not\equiv x(t + j\omega)$ for $j < k$, and k-periodic points x_0 of T_ω such that $x(0) = x_0$, i.e. $x_0 = T_\omega^k(x_0)$ and $x_0 \neq T_\omega^j(x_0)$ for $j < k$.

Now, consider the impulsive differential equation

$$\begin{cases} x' = F(t,x),\ t \neq t_j := j\omega\,, \quad \text{for } \omega > 0\,, \\ x(t_j^+) = I(x(t_j^-))\,,\ j \in \mathbb{Z}\,, \end{cases} \tag{8}$$

where $F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is as above in (4), and $I : \mathbb{R}^n \to K_0$ is a compact continuous impulsive mapping such that $K_0 := \operatorname{cl} I(\mathbb{R}^n)$ and $I(K_0) = K_0$. Its solutions will be also understood in the same Carathéodory sense, i.e. $x \in AC((t_j, t_{j+1}))$, $j \in \mathbb{Z}$.

One can readily check that there is again a one-to-one correspondence between $k\omega$-periodic solutions $x(\cdot)$ of (8) and k-periodic points $x_0 = x(0)$ of the composition $I \circ T_\omega$, for any $k \in \mathbb{N}$.

Therefore, we can naturally introduce the following definition of topological entropy for (4), (7) and (8).

Definition 5. We say that equation (4) (in particular (7)), resp. (8), has a *topological entropy* h if $h = h(T_\omega)$, resp. $h = h(I \circ T_\omega)$.

In view of Definitions 1, 5 and formulas (2), (5), equation (7) has a topological entropy $h = h(T_\omega) = h(W(\omega)\cdot\mathrm{id}) = \sum_{|\mu_i|>1} \log|\mu_i|$, where $W(\omega)$ is a fundamental matrix of (7) at $\omega > 0$ and μ_i, $i = 1,\dots,n$, are its eigenvalues. In particular, for $A(t) \equiv A$, we get that

$$h = h(e^{A\omega}\cdot\mathrm{id}) = \sum_{\operatorname{Re}\lambda_i>0} \log|e^{\lambda_i\omega}| = \omega\log e \sum_{\operatorname{Re}\lambda_i>0} \operatorname{Re}\lambda_i\,.$$

For $\omega = 1$, it is in accordance with the calculations in [9, 12, 23].

Similarly, equation (8), where $F(t,x) := A(t)x$ and I is a real $(n\times n)$-matrix, has a topological entropy

$$h = h(I\cdot W(\omega)\cdot\mathrm{id}) = \sum_{|\nu_i|>1} \log|\nu_i|\,,$$

where $W(\omega)$ is as above and ν_i, $i = 1,\dots,n$, are the eigenvalues of the product $I\cdot W(\omega)$.

3. Topological chaos for impulsive differential equations

In spite of the above arguments, observe that e.g. the scalar equation $x' = ax$ with $I = \mathrm{id}|_{\mathbb{R}}$ possesses for $a = \frac{\log 2}{\omega\log e}$ the positive entropy $h = h(T_\omega) = \log 2$, but does not admit any nontrivial (nonzero) periodic solution. In the spirit of the "criticism" in [11], since the dynamics of $x' = ax$ have not a complicated behaviour, Definitions 1 and 5 are not suitable for any sort of deterministic chaos. In other words, to speak about topological chaos determined by a positive entropy requires here to be restricted to compact subsets of $\mathbb{R}^n$, which are positively invariant under the compositions $I \circ T_\omega$.

For multivalued impulses, the situation becomes still more delicate, because Definition 1 must be replaced e.g. by Definition 2.

For the sake of simplicity, we will consider just the linear homogeneous diagonal system of differential equations with special multivalued upper semicontinuous impulses, namely

$$\begin{cases} x' = (\operatorname{diag}[a_1,\dots,a_n])\,x\,,\ x = (x_1,\dots,x_n)\,,\\ x_i(t_j^+) \in I_i(x_i(t_j^-))\,,\ i = 1,\dots,n, j\in\mathbb{Z}\,,\end{cases} \tag{9}$$

where $I_i : \mathbb{R} \to \mathcal{K}([0,1])$, $I_i(x_i) \equiv I_i(x_i+1)$, and

$$I_i|_{[0,1]}(x_i) := \begin{cases} [0,1]\,, & \text{for } x_i \in \{0,1\}\,, \\ x_i\,, & \text{otherwise (i.e. for } x_i \in (0,1))\,, \end{cases} \quad i = 1,\dots,n.$$

Definition 6. We say that system (9) has a *topological entropy* h if $h := h_{\mathrm{KT}}(I \circ T_1|_{[0,1]^n})$, where h_{KT} denotes the topological entropy in the sense of Definition 2, $I = (I_1,\dots,I_n)$ and $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ is the associated Poincaré translation operator along the trajectories of $x' = (\operatorname{diag}[a_1,\dots,a_n])x$, defined for $\omega = 1$ in (5).

Theorem 7. *The topological entropy h of equation* (9) *satisfies the inequality $h \geq (\log\sqrt{2})\sum_{a_i \geq 0} \operatorname{sgn}(1+a_i)$. In particular, if at least one coefficient a_i is nonnegative (i.e. $a_i \geq 0$, for some $i \in \{1,\dots,n\}$), then equation* (9) *exhibits on $[0,1]^n$ a topological chaos in the sense that $h > 0$.*

Proof. The associated Poincaré translation operator T_1 along the trajectories of $x' = (\operatorname{diag}[a_1,\dots,a_n])x$, defined for $\omega = 1$ in (5), takes the form $T_1(x) = (e^{a_1},\dots,e^{a_n})x = W(1)x$, where $\mu_i = e^{a_i}$, $i = 1,\dots,n$, are the multiplicators of $W(1)$.

Since

$$I \circ T_1(x) = (I_1(e^{a_1}x_1),\dots,I_n(e^{a_n}x_n))\ , \text{where } x = (x_1,\dots,x_n) \in [0,1]^n,$$

$I_i(e^{a_i}x_i) \equiv I_i(e^{a_i}(x_i + e^{-a_i}))$, and

$$I_i|_{[0,e^{-a_i}]}(e^{a_i}x_i) = \begin{cases} [0,1]\,, & \text{for } x_i \in \{0,e^{-a_i}\}\,, \\ x_i\,, & \text{otherwise (i.e. for } x_i \in (0,e^{-a_i}))\,, \end{cases}$$

$i = 1,\dots,n$, we obtain by means of the equality in Remark 4 (for a particular inequality, see Lemma 3) that

$$h := h_{\mathrm{KT}}(I \circ T_1|_{[0,1]^n}) = \sum_{i=1}^{n} h_{\mathrm{KT}}(I_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]}))\,.$$

Checking the proof of [17, Theorem 6.2], it is straightforward to realize that $h_{\mathrm{KT}}(I_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]})) \geq \log\sqrt{2}$, provided $a_i \geq 0$ for some $i \in \{1,\dots,n\}$. Summing up, $h = h_{\mathrm{KT}}(I \circ T_1|_{[0,1]^n}) \geq (\log\sqrt{2})\sum_{a_i\geq 0} \operatorname{sgn}(1+a_i)$, as claimed. □

Remark 8. If at least one component I_i, for some $i \in \{1,\dots,n\}$, of the impulsive mapping I is replaced by $\hat{I}_i : \mathbb{R} \to \mathcal{K}([0,1])$, where $\hat{I}_i(x_i) \equiv \hat{I}_i(x_i+1)$, and

$$\hat{I}_i|_{[0,1]}(x) := \begin{cases} [0,1]\,, & \text{for } x \in \{0,1\}\,, \\ \{0\}\,, & \text{otherwise (i.e. for } x_i \in (0,1))\,, \end{cases}$$

then (cf. Lemma 3) $h = h_{\mathrm{KT}}(\hat{I}_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]})) = \infty$, for any $a_i \in \mathbb{R}$, because (see [17, Theorems 5.4 and 7.1])

$$h_{\mathrm{KT}}(\hat{I}_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]}) \geq \frac{1}{2} h_{\mathrm{KT}}(\hat{I}_i^{\,2}(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]}) = h_{\mathrm{KT}}([0,1]|_{[0,1]}) = \infty\,,$$

where $[0,1]|_{[0,1]}$ denotes a constant multivalued mapping with values $[0,1]$.

Remark 9. In case of diagonalizable or weakly coupled systems, the situation becomes more complicated and requires a further technical elaboration. For the single-valued case, see e.g. [8].

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Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
17. listopadu 12, 771 46 Olomouc, Czech Republic
E-mail: `jan.andres@upol.cz`

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
17. listopadu 12, 771 46 Olomouc, Czech Republic
E-mail: `pavel.ludvik@upol.cz`

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GENERAL EXACT SOLVABILITY CONDITIONS FOR THE INITIAL VALUE PROBLEMS FOR LINEAR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Natalia Dilna

Abstract. Conditions on the unique solvability of linear fractional functional differential equations are established. A pantograph-type model from electrodynamics is studied.

1. Introduction

The fractional differential equations (FDEs) get a significant interest in modern literature on differential equations and are represented by numerous papers. Here referred to a few of them only [1, 2, 3, 4, 5, 6, 7, 8, 9].

The application scale of mentioned equations is quite broad. We want to accentuate the [9], where the authors made a complex overview of possible applications of FDE: the theories of differential, integral, and integro-differential equations, special functions of mathematical physics, and some present-day applications of fractional calculus, including fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics, and signal processing, and so on.

Conditions on the unique solvability of the boundary value problem for functional differential equations is a fundamental and non-trivial part of the study, and many publications are focused on them, for example, [10, 13, 14].

The main goal of our investigation is the exact conditions lookup of the unique solvability of the boundary value problem for the fractional functional differential equations (FFDEs). Some recent results [3, 4, 5, 6, 8] motivated us to continue in this direction.

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2. Problem formulation

We consider fractional functional differential problem

$$D_a^q u(t) = (lu)(t) + f(t)\,, \quad t \in [a,b] \tag{2.1}$$

$$u(a) = c\,, \tag{2.2}$$

where D_a^q is the Caputo fractional derivative of order q, $0 < q < 1$, with the lower limit zero, operator $l = (l_k)_{k=1}^n \colon AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is the bounded linear operator, function $f \in C([a,b],\mathbb{R}^n)$ and $c \in \mathbb{R}^n$.

The main goal of our investigations is to find exact conditions sufficient for the unique solvability of the initial value problem (2.2) for systems of the linear FFDEs (2.1) presented by isotone operators (see Definition 2.3). A pantograph-type model from electrodynamics is studied as well.

Here are used spaces:

- $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions $[a,b] \to \mathbb{R}^n$ with the norm $C([a,b],\mathbb{R}^n) \ni u \to \max_{t\in[a,b]} |u(t)|_\infty = \max_{t\in[a,b]} \operatorname{ess\,sup} |u(t)|$;
- $AC([a,b],\mathbb{R}^n)$ is the Banach space of absolutely continuous functions $[a,b] \to \mathbb{R}^n$ with the norm $AC([a,b],\mathbb{R}^n) \ni u \to \int_0^l \|u'(\xi)\| d\xi + \|u(0)\|$.

Definition 2.1. By a *solution* of linear boundary-value problem (2.1), (2.2) we understand an absolutely continuous vector-function $u : [a,b] \to \mathbb{R}^n$ possessing property (2.2) and satisfying FFDE (2.1) for almost all t from the interval $[a,b]$.

Definition 2.2 ([2])**.** For a function u given on the interval $[a,b]$ *the Caputo derivative* of fractional order q is defined by

$$D_a^q u(t) = \frac{1}{\Gamma(1-q)} \left(\frac{d}{dt}\right) \int_a^t (t-s)^{-q} \big(u(s) - u(a)\big)\, ds\,,\ 0 < q < 1,$$

where $\Gamma(q) : [0,\infty) \to \mathbb{R}$ is Gamma-function:

$$\Gamma(q) := \int_0^\infty t^{q-1} e^{-t}\, dt\,. \tag{2.3}$$

Definition 2.3 ([4])**.** For certain given $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{pmatrix} \tag{2.4}$$

an operator $l\colon AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is *σ-positive operator* if the fact that the relation

$$\sigma u(t) \geq 0\,, \quad t \in [a,b] \tag{2.5}$$

is true implies that

$$\sigma(lu)(t) \geq 0\,, \quad \text{for a.e.} \quad t \in [a,b]\,. \tag{2.6}$$

3. Auxiliary statements

Lemma 3.1 ([9, Lemma 2.21 and Lemma 2.22]). *Let $0 < q < 1$ and let $u(t) \in C([a,b],\mathbb{R}^n)$ or $u(t)$ belongs to the space of essentially bounded measurable functions $L_\infty([a,b],\mathbb{R}^n)$, then*

$$D_a^q I_a^q u(t) = u(t) \quad \text{almost everywhere on} \quad [a,b].$$

If $u(t) \in C^1([a,b],\mathbb{R}^n)$ or $u(t) \in AC([a,b],\mathbb{R}^n)$, then

$$I_a^q D_a^q u(t) = u(t) - u(a) \quad \text{almost everywhere on} \quad [a,b]\,,$$

where

$$I_a^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s)\, ds\,,$$

and Γ-function is defined by (2.3).

Taking into account Definition 2.1, Lemma 3.1 and relation (2.3) the next obvious Lemma is fulfilled.

Lemma 3.2. *The problem* (2.1), (2.2) *on $[a,b]$ is equivalent to the equation*

$$u(t) = u(a) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (lu)(s) ds + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)\, ds\,.$$

Lemma 3.3 ([12, the Fredholm alternative, Corollary from Theorem VI.14]). *The nonhomogeneous problem* (2.2) *for linear FFDE* (2.1) *is uniquely solvable if the corresponding homogeneous problem*

$$u(a) = 0 \tag{3.1}$$

for linear FFDE

$$D_a^q u(t) = (lu)(t)\,, \quad t \in [a,b]\,, \tag{3.2}$$

only has a trivial solution.

Let us fix $r \in \mathbb{N}$ and constants $\{h_1, h_2, \dots, h_r\} \in (0,+\infty)$ and introduce the sequence of functions

$$y_k(t) := \frac{\sum_{i=1}^r h_i}{\Gamma(q)} \int_a^t (t-s)^{q-1} (l y_{k-i})(s)\, ds\,, \quad k \geq r\,, \quad t \in [a,b]\,, \tag{3.3}$$

where $\{y_0, y_1, \dots y_{r-1}\} \in AC([a,b],\mathbb{R}^n)$ chosen so that

$$\sigma y_k(t) \geq 0\,, \quad t \in [a,b]\,, \quad k = 0,1,\dots,r-1\,, \tag{3.4}$$

and

$$y_k(a) = 0\,, \quad k = 0,1,\dots,r-1\,. \tag{3.5}$$

Remark 3.4. If $r = 1$ and $h_1 = 1$, equality (3.3) takes the form

$$y_k(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (l y_{k-1})(s)\, ds\,, \quad t \in [a,b]\,, \quad k \in \mathbb{N}\,, \tag{3.6}$$

and thus coincides with the sequence studied, e.g., in [4]. Formula (3.6) defines the standard iteration sequence used in studies of the uniqueness of the trivial solution of the integral fractional functional equation $y(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}(ly)(s)ds$, $t \in [a,b]$, which, because of Lemma 3.2, is equivalent to the homogeneous problem (3.1), (3.2).

Next, we will need the following technical Lemmas.

Lemma 3.5. *Suppose that the operator $l\colon AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is σ-positive. Then, for arbitrary absolutely continuous functions $\{y_k\}_{k=0}^{r-1} : [a,b] \to \mathbb{R}^n$ satisfying conditions* (3.4)*,* (3.5)*, the corresponding functions $y_r, y_{r+1}, \dots$ defined by formulae* (3.3) *also satisfy conditions* (3.4)*,* (3.5)*:*

$$\sigma y_k(t) \geq 0\,, \quad t \in [a,b]\,, \quad y_k(a) = 0\,, \quad k \geq r\,.$$

Proof of Lemma 3.5. In view of (3.3), we have

$$y_r(t) := \frac{\sum_{i=1}^r h_i}{\Gamma(q)} \int_a^t (t-s)^{q-1}(ly_{r-i})(s)\,ds\,, \quad t \in [a,b]\,. \tag{3.7}$$

Taking into account the σ-positivity of the operator l and the non-negativeness of the coefficients $h_1, h_2, \dots, h_r$ in formula (3.3) and condition (3.4) yields $\sigma(ly_{r-i})(t) \geq 0$, $t \in [a,b]$. By induction, it is easy to show that (3.7) is fulfilled for all $k \geq r$. The property $y_k(a) = 0$ for all $k = 0,1,2,\dots,m$ is obvious from condition (3.4) and formula (3.5). □

Lemma 3.6. *For arbitrary vectors $x_0, x_1, \dots, x_m$ from $\mathbb{R}^n$, and some constants $\{\theta_k\}_{k=1}^m \subset [0,+\infty)$, the equality*

$$\sum_{k=r}^{m} \theta_k \sum_{i=1}^{r} h_i x_{k-i} = \sum_{j=0}^{m-1} \mu_j x_j \tag{3.8}$$

is fulfilled, where

$$\mu_k = \sum_{\nu \in T_{r,m}(k)} \theta_{\nu+k} h_\nu\,, \quad k = 0,1,\dots,m-1\,, \tag{3.9}$$

$$\text{and} \quad T_{r,m}(k) = \{\nu \in \mathbb{N} |\ \nu \leq r \leq \nu + k \leq m\}\,, \quad r \in \mathbb{N}\,.$$

4. General theorem

Theorem 4.1. *Suppose that operator l is σ-positive. Assume also that for some integers r and m, $m \geq r \geq 1$, a real number $\rho \in (1,+\infty)$, some constants $\{\theta_k\}_{k=1}^m \subset [0,+\infty)$ and $\{h_i\}_{i=1}^r \subset [0,+\infty)$, and certain absolutely continuous vector-functions $y_0, y_1, \dots, y_{r-1}$ satisfying conditions* (3.4)*,* (3.5)*, and the relation*

$$\sigma \sum_{k=0}^{r} \theta_k y_k(t) > 0 \quad \text{for all} \quad t \in (a,b] \tag{4.1}$$

such that the functional differential inequality

$$\sigma\Big(\sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{r}\Big(\sum_{j \in T_{r,m}(k)} \theta_{j+k} h_j - \rho\theta_k\Big)(l_k y)(t) - \rho\theta_m (l_m y)(t)\Big) \geq 0 \tag{4.2}$$

is fulfilled for a.e. t from $[a,b]$, $r \in \mathbb{N}$, $m \geq r$ and y_k, $k \geq r$ defined by (3.3).

Then, the homogenous linear initial value problem (3.1), (3.2) *only has a trivial solution and the nonhomogeneous linear Cauchy problem* (2.1), (2.2) *is uniquely solvable for an arbitrary $c \in \mathbb{R}$ and an arbitrary function $f \in C([a,b],\mathbb{R}^n)$.*

The unique solution of the problem (2.2) *for the equation* (2.1) *is representable in the form of a uniformly convergent on $[a,b]$ functional series*

$$u(t) = f_c(t) + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}(lf_c)(s)\,ds \\ + \frac{1}{\Gamma(q)}\int_a^t (t-\cdot)^{q-1} l\Big(\frac{1}{\Gamma(q)}\int_a^{\cdot}(t-s)^{q-1}(lf_c)(\xi)d\xi\Big)(s)\,ds + \dots,$$

where $f_c(t) := c + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s)\,ds$.

If, furthermore, the inequality $\sigma\Big(c + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}f(s)\,ds\Big) \geq 0$, *is true a.e. on $[a,b]$, then the unique solution $u(\cdot)$ of the initial value problem* (2.2) *for FFDE* (2.1) *satisfy the condition* (2.5).

Proof. To prove Theorem 4.1 we need Theorem 4 from [4].

Theorem 4.2 ([4, Theorem 4])**.** *Assume that the linear operator $l = (l_k)_{k=1}^n$ in equation* (2.1) *is σ-positive. Suppose that there exist such a number $\rho > 1$ an function $y \in AC([a,b],\mathbb{R}^n)$ with properties*

$$y(a) = 0, \quad \sigma y(t) > 0 \quad \text{for} \quad t \in (a,b]\,, \tag{4.3}$$

and a certain integer $k \geq 0$ that the components of the function $(y_{k,\nu})_{\nu=1}^n$ of the respective function y_k are continuous.

Additionally, the following fractional functional differential inequality

$$\sigma\Big(D_a^q y(t) - \rho(ly)(t)\Big) \geq 0 \text{ for a.e. } t \in [a,b] \tag{4.4}$$

is fulfilled.

Then the assertion of Theorem 4.1 is true for the inhomogeneous (2.1), (2.2) *and homogeneous* (3.1), (3.2) *Cauchy problems.*

We consider certain absolutely continuous vector-functions $\{y_k\}_{k=0}^{r-1} : [a,b] \to \mathbb{R}^n$ and construct the corresponding sequence of the functions $\{y_k\}_{k=0}^{r-1} : [a,b] \to \mathbb{R}^n$ according to formula (3.3) for $m \geq r$. Next, we introduce the function

$$y(t) = \sum_{k=0}^{m} \theta_k y_k(t)\,, \quad t \in [a,b]\,, \tag{4.5}$$

with the coefficients $\{\theta_k\}_{k=0}^m \in [0,+\infty)$ determined by the assumptions of the theorem. Note that, in view of (3.4), assumption (4.1) implies that (4.3) holds. Let us show that, under our assumptions, function (4.5) satisfies inequality (4.4). Taking into account (4.5), the corresponding function

$$\omega(t) := D_a^q y(t) - \rho(ly)(t) \tag{4.6}$$

has the form $\omega(t) := \sum_{k=0}^{m} \theta_k \big(D_a^q y_k(t) - \rho(l_k y)(t)\big)$, whence

$$\omega(t) := \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=r}^{m} \theta_k D_a^q y_k(t) - \rho \sum_{k=0}^{m} \theta_k (l_k y)(t)\,, \quad m \geq r\,. \tag{4.7}$$

In view of formula (3.3) and Lemma 3.1 for the functions $y_r, y_{r+1}, \dots$, we have $D_a^q y_k(t) = \sum_{i=1}^{r} h_i(l y_{k-i})(t)$, $t \in [a,b]$, $k \geq r$, and, therefore, equality (4.7) can be rewritten

$$\omega(t) := \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=r}^{m} \theta_k \sum_{i=1}^{r} h_i(l y_{k-i})(t) - \rho \sum_{k=0}^{m} \theta_k (l y_k)(t)\,. \tag{4.8}$$

Taking into account Lemma 3.6, formula (4.8) can be rewritten as

$$\begin{aligned} \omega(t) &:= \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{m-1} \mu_k (l y_k)(t) - \rho \sum_{k=0}^{m} \theta_k (l y_k)(t) \\ &= \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{m-1} (\mu_k - \rho\theta_k)(l y_k)(t) - \rho\theta_m (l y_m)(t)\,, \end{aligned} \tag{4.9}$$

where $\mu_0, \mu_1, \dots \mu_{m-1}$ are given by relation (3.9). In view of (3.9), equality (4.9) is equivalent to the relation

$$\omega(t) = \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{m-1} \Big(\sum_{\nu \in T_{r,m}(k)} \theta_{\nu+k} h_\nu - \rho\theta_k \Big)(l y_k)(t) - \rho\theta_m (l y_m)(t)\,.$$

Hence, relation (4.2) guarantees that function (4.6) satisfies the condition $\sigma\omega \geq 0$ for a. e. $t \in [a,b]$, i. e., the fractional functional differential inequality (4.4) holds for the function y given by (4.5). It is obvious, that constructed in such way, y is a solution of the differential inequality (4.4).

To apply Theorem 4.2 we need to show that, under our assumptions, the solution mentioned possesses properties (3.4).

In view of Lemma 3.5, σ-positiveness of the operator l and non-negativity of all constants θ_k, $k = 0, 1, \dots, m$ the inequality

$$\sigma\theta_k y_k(t) \geq 0\,, \quad t \in [a,b]\,, \quad k = 0, 1, \dots, m\,, \tag{4.10}$$

is satisfied.

It follows from (4.10) that $\sigma\big(\sum_{k=0}^{m} \theta_k y_k(t) - \sum_{k=0}^{r-1} \theta_k y_k(t)\big) = \sigma \sum_{k=r}^{m} \theta_k y_k(t) \geq 0$, for $t \in [a,b]$, $k = 0, 1, \dots, m$, and, hence,

$$\sigma \sum_{k=0}^{m} \theta_k y_k(t) \geq \sigma \sum_{k=0}^{r-1} \theta_k y_k(t) \quad t \in [a,b]\,, \quad k = 0, 1, \dots, m\,. \tag{4.11}$$

Inequality (4.11) yields $\sigma y(t) = \sigma \sum_{k=0}^{m} \theta_k y_k(t) \geq \sigma \sum_{k=0}^{r-1} \theta_k y_k(t))$, $t \in [a,b]$ whence, under the assumption (4.1), we obtain $\sigma y(t) \geq \sigma \sum_{k=0}^{r-1} \theta_k y_k(t)) > 0$, $t \in (a,b]$ i.e., y satisfies condition (4.3). Thus, we have shown that function (4.5) satisfies the fractional functional differential inequality (4.4) and possesses properties (4.3) i. e., the assumptions of Theorem 4.2 are satisfied. Application of Theorem 4.2 leads us to the assertion required. □

Remark 4.3. Condition (4.2) appearing in the Theorem 4.1 presented are unimprovable in the sense that, generally speaking, that condition can not be assumed with $\rho = 1$. To check this, one can use, e.g., example 1 from [4].

5. Pantograph type model

Let us consider problem (2.1), (2.2) in view

$$D_0^q u(t) = \sum_{i=1}^{m} P_i(t)u(\lambda_i t) + f(t)\,, \quad t \in [0,1]\,, \quad u(0) = c\,, \tag{5.1}$$

where

$$(lu)(t) := \sum_{i=1}^{m} P_i(t)u(\lambda_i t)\,, \quad \text{and} \quad P_i(t) \quad := \begin{pmatrix} p^i_{11}(t) & p^i_{12}(t) & \ldots & p^i_{1n}(t) \\ p^i_{21}(t) & p^i_{22}(t) & \ldots & p^i_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p^i_{n1}(t) & p^i_{n2}(t) & \ldots & p^i_{nn}(t) \end{pmatrix} \tag{5.2}$$

have continuous components and $\lambda_i \in (0,1)$, $m \in \mathbb{N}$, $f \in C([0,1], \mathbb{R}^n)$.

Equation (5.1) is a famous equation called the pantograph type equation arising in electrodynamics [11]. The pantograph is a device used in electric trains to collect electric current from the overload lines.

Now let us establish exact conditions sufficient for the unique solvability of the initial value problem (5.1).

Theorem 5.1. *Suppose that*

$$\sigma P_i(t)\sigma \geq 0 \quad \textit{for almost all} \quad t \in [0,1], \quad 1 \leq i \leq m, \tag{5.3}$$

is fulfilled, where every P_i, $i = 1, \ldots, n$, are defined by (5.2) *and have continuous components, σ is defined by* (2.4)*, and assume that there exists a real number $\rho > 1$ such that the functional differential inequality*

$$\sigma(y_0(t) + y_1(t) - \rho y_2(t)) \geq 0 \tag{5.4}$$

is satisfied for almost all t from $[0,1]$ *and increasing functions y_0 and y_1 with properties* (3.4), (3.5), (4.1) *and*

$$y_2(t) = \frac{\rho+1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sum_{i=1}^{m} P_i(s)(y_0(\lambda_i s) + y_1(\lambda_i s))ds. \tag{5.5}$$

Then, the assertion of Theorem 4.1 is true for the problem (5.1)*.*

Proof. Let us consider the function y from (4.5) in view:

$$y(t) = \theta y_0(t) + \theta y_1(t) + \theta y_2(t), \quad t \in [0,1] \quad \theta \in (0, +\infty),$$

where y_2 defined by (5.5) and increasing $\{y_0, y_1\}$ chosen so that (3.4), (3.5) and (4.1) are fulfilled. Obviously, $D_0^q y_2(t) = (\rho+1) \sum_{i=1}^m P_i(s)\ (y_0(\lambda_i t)+y_1(\lambda_i t))$, where

$h_0 = h_1 = \rho + 1$. Let us consider (4.8) with $r = 2$, $m = 2$, then

$$\omega(t) = \theta D_0^q y_0(t) + \theta D_0^q y_1(t) + \theta \sum_{i=1}^{m} P_i(s)\big(y_0(\lambda_i t) + y_1(\lambda_i t) - \rho y_2(\lambda_i t)\big)\,. \tag{5.6}$$

By the σ-positivity of the operator (5.2) (see condition (5.3) and Lemma 9 from [4]), inequality (5.4) and properties (3.4), (3.5), (4.1) for increasing functions y_0, y_1 we get that continuous function $\omega(t)$ from (5.6) implies the condition (4.2) from Theorem 4.1. The application of that theorem to the initial-value problem (5.1) and corresponding homogeneous problem implies the assertions required. Theorem 5.1 is proved. □

Remark 5.2. It is shown above that condition (5.4), which was obtained from complicated formula (4.2), is much more simple for an application then the inequality (25) from [4].

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Institute of Mathematics, Slovak Academy of Sciences,
Štefánikova 49, 814 73 Bratislava, Slovakia
E-mail: dilna@mat.savba.sk

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SYSTEMS OF DIFFERENTIAL EQUATIONS MODELING NON-MARKOV PROCESSES

Irada Dzhalladova and Miroslava Růžičková

Abstract. The work deals with non-Markov processes and the construction of systems of differential equations with delay that describe the probability vectors of such processes. The generating stochastic operator and properties of stochastic operators are used to construct systems that define non-Markov processes.

1. Introduction concepts

In recent years, studies of the non-Markovian dynamics of open systems have become increasingly popular, with a diverse range of researchers involved. The theory of non-Markovian random processes is constantly developing and meets modern requirements. This interest arose from the fundamental problem of defining and quantifying memory effects in the quantum realm, how to use and develop applications based on them, and also because of the question of what are the ultimate limits for controlling the dynamics of open systems.

In addition, there are many important control problems that are not naturally formulated as Markov decision processes. For example, if the agent cannot directly observe the state of the environment, then it is more appropriate to use a partially observable Markov model of the decision process. Even with complete observability, the probability distribution over the next states may not depend only on the current state.

Some postulated problems and also models with non-Markov parameters using fractional dynamics, predictive control or stabilization are considered in [2].

In the presented work, constructions of certain non-Markovian random processes are proposed using stochastic operators, which are called generating operators. Naturally, other methods of constructing non-Markov random processes can be

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proposed. In any case, these processes will be determined by equations with a delayed argument.

In the constructions proposed by us, the stochastic operator plays an important role, so we will present its definition and some basic properties. For a deeper understanding of this term see works [1, 4].

Definition 1.1. Let on the probability space $(\Omega,\ \mathcal{F},\ P)$ be defined two random variables $x \equiv x(\omega)\colon \Omega \to \mathbb{R}$ and $y \equiv y(\omega)\colon \Omega \to \mathbb{R}$ with probability density functions $f_1(x)$ and $f_2(y)$ respectively. Then the operator $L\colon f_1(x) \to f_2(y)$,

$$f_2(y) = Lf_1(x)\,,$$

is said to be the *stochastic or generating operator*.

Theorem 1.2 ([1])**.** *Let on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be defined two random variables $x,\ y$ with probability density functions $f_1(x)$ and $f_2(y)$ respectively. Let $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable monotonically increasing function such that $\lim\limits_{x\to-\infty} g(x) = -\infty\,,\quad \lim\limits_{x\to+\infty} g(x) = +\infty$. Then*

$$f_1(x) = f_2(g(x))\frac{dg(x)}{dx}\,.$$

It should be noted that, in general, there is considered a set S of functions $f(x), x = (x_1,\, x_2, \dots, x_m) \in \mathbb{R}^m$ such that

$$f(x) \geq 0, \quad \int\limits_{\mathbb{R}^m} f(x)dx = 1\,,$$

and the operator L is mapping a set S to the itself.
If $f_1(x) \in S$ implies $f_2(y) = Lf_1(x) \in S$, then the operator L is stochastic.

A similar statement as Theorem 1.2 can also be formulated if f_1, f_2 are vector functions. For this we will use the following notation

$$\det\frac{Dg(x)}{Dx} := \begin{vmatrix} \dfrac{\partial g_1(x)}{\partial x_1} & \dfrac{\partial g_1(x)}{\partial x_2} & \dots & \dfrac{\partial g_1(x)}{\partial x_m} \\ \dfrac{\partial g_2(x)}{\partial x_1} & \dfrac{\partial g_2(x)}{\partial x_2} & \dots & \dfrac{\partial g_2(x)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_m(x)}{\partial x_1} & \dfrac{\partial g_m(x)}{\partial x_2} & \dots & \dfrac{\partial g_m(x)}{\partial x_m} \end{vmatrix} \neq 0\,,$$

where $g\colon \mathbb{R}^m \to \mathbb{R}^m$ is a continuously differentiable function.

Theorem 1.3 ([1])**.** *Let $g\colon \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable function for which there exists the inverse function $h\colon \mathbb{R}^m \to \mathbb{R}^m$ to g, i. e., $y = g(x),\ \ x = h(y)$, $\det\dfrac{Dh(y)}{Dy} \neq 0$, and $\lim\limits_{\|x\|\to\infty} \|g(x)\| = \infty,\ \lim\limits_{\|x\|\to\infty} \|h(x)\| = \infty$. If on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are defined two random variables $x\colon \Omega \to \mathbb{R}^m,\ y\colon \Omega \to \mathbb{R}^m$ with* *probability density functions* $f_1(x)$, $f_2(y)$, *respectively,* $x, y \in \mathbb{R}^m$, $f_1, f_2 \in S$, *such that* $y = g(x)$, $x = h(y)$, *then*

$$f_1(x) = f_2(g(x)) \left| \det \frac{Dg(x)}{Dx} \right|,$$

$$f_2(y) = f_1(h(y)) \left| \det \frac{Dh(y)}{Dy} \right|.$$

First, we show a possible construction of a differential equation determining some random process using a stochastic operator (for details see [5, 6]).
Let $L_\tau(t, \varepsilon)$ be a stochastic operator that depends on the parameter ε and is defined for an m-dimensional probability vector $P(t+\tau)$, $\tau < 0$, such that there exists the limit

$$\lim_{\varepsilon \to 0} L_\tau(t, \varepsilon) P(t+\tau) = P(t)\,.$$

In addition, for any continuous vector function P there exists an operator

$$A_\tau(t) P(t+\tau) = \lim_{\varepsilon \to \infty} \frac{L_\tau(t, \varepsilon) P(t+\tau) - P(t)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\partial L_\tau(t, \varepsilon)}{\partial \varepsilon} P(t+\tau)\,.$$

Then the difference equation

$$P(t+\varepsilon) = L_\tau(t, \varepsilon) P(t+\tau), \qquad \varepsilon > 0,\ \tau \leq 0\,,$$

determines the vector of probabilities of some random process $\xi(t) = \xi(t, \omega)$, $\omega \in \Omega$. This equation can be written in the form

$$\frac{P(t+\varepsilon) - P(t)}{\varepsilon} = \frac{L_\tau(t, \varepsilon) P(t+\tau) - P(t)}{\varepsilon}\,.$$

If $\varepsilon \to 0$ in this relation, assuming that the vector $P(t)$ is differentiable, we obtain a system of differential equations

$$\frac{dP(t)}{dt} = A_\tau(t) P(t+\tau)\,, \tag{1.1}$$

which describes some random process $\xi(t) = \xi(t, \omega), \omega \in \Omega$.
The operator $A_\tau(t)$ in system of differential equations (1.1) is so called the **generating (stochastic) operator**.

2. Main results

Using the properties of stochastic operators given in Theorems 1.2 and 1.3 (see [1] for other properties), we show some constructions of generating operators that can be used to construct systems of differential equations whose solutions are non-Markov stochastic processes.

Theorem 2.1. *Let* $A_\tau(t)$ *be a generating operator and let* $0 < \alpha(t) \leq c, c \in \mathbb{R}^+$. *Then* $\alpha(t) A_\tau(t)$ *is also a generating operator.*

Proof. The difference equation

$$P\left(t + \frac{\varepsilon}{\alpha(t)}\right) = L_\tau(t, \varepsilon) P(t+\tau), \qquad 0 < \alpha(t) \leq c$$

determines the probability vector of some random process. This equation we rewritten into the form

$$\frac{P(t+\varepsilon\alpha^{-1}(t)) - P(t)}{\varepsilon} = \frac{L_\tau(t,\varepsilon)P(t+\tau) - P(t)}{\varepsilon}$$

and if $\varepsilon \to 0$, we obtain the system of differential equations

$$\frac{dP(t)}{dt} = \alpha(t)A_\tau(t)P(t+\tau)\,,$$

which corresponds to equation (1.1). This proves the theorem. □

Theorem 2.2. *Let $A_\tau^{(1)}(t), A_\tau^{(2)}(t)$ be generating operators. Then $A_\tau^{(1)}(t) + A_\tau^{(2)}(t)$ is also a generating operator.*

Proof. Let $A_\tau^{(k)}(t), k = 1,2$ be generating operators such that

$$\lim_{\varepsilon\to\infty} \varepsilon^{-1}\Big(L_\tau^{(k)}(t,\varepsilon)P(t+\tau) - P(t)\Big) = A_\tau^{(k)}(t)P(t+\tau)\,.$$

The difference equation

$$P\left(t+\frac{\varepsilon}{2}\right) = \frac{1}{2}\sum_{k=1}^{2} L_\tau^{(k)}(t,\varepsilon)P(t+\tau)\,,$$

determines the probability vector of some random process. This equation we rewritten into the form

$$\varepsilon^{-1}\left(P\left(t+\frac{\varepsilon}{2}\right) - P(t)\right) = \frac{1}{2}\sum_{k=1}^{2}\varepsilon^{-2}\Big(L_\tau^{(k)}(t,\varepsilon)P(t+\tau) - P(t)\Big)\,.$$

If $\varepsilon \to 0$ than we obtain the system of differential equations

$$\frac{dP(t)}{dt} = \sum_{k=1}^{2} A_\tau^{(k)}(t)P(t+\tau)\,.$$

This proves the theorem. □

Theorem 2.3. *Let $A_\tau^{(k)}(t),\ k = 1,2,\dots,N$ be generating operators and let functions $\alpha_k(t),\ k = 1,2,\dots,N$ satisfy the conditions $0 < \alpha_k(t) \le c, c \in \mathbb{R}^+$. Then*

$$\sum_{k=1}^{N} \alpha_k(t)A_\tau^{(k)}(t)$$

is also a generating operator.

Proof. The proof follows from Theorems 2.1, 2.2.

□

Now, as a consequence of Theorem 2.2, we consider possible options for constructing systems of differential equations that describe the probability vector of various random processes.

Corollary 2.4. *Let $A_0(t)$ be an $m\times m$ matrix with elements $a_{js}(t), j, s = 1, 2, \ldots, m$ such that*

$$\sum_{j=1}^{m} a_{js}(t) = 0, \qquad a_{js}(t) \geq 0,\ j \neq s, \qquad a_{jj}(t) \leq 0\,, \quad j, s = 1, 2, \ldots, m\,. \tag{2.1}$$

If the elements $a_{js}(t), j, s = 1, 2, \ldots, m$ are bounded, then the operator

$$L_\tau(t, \varepsilon)P(t+\tau) \equiv P(t) + A_0(t)P(t)$$

will be stochastic for sufficiently small values of $\varepsilon > 0$, where the matrix $A_0(t)$ defines the generating operator, and the system of differential equations

$$\frac{dP(t)}{dt} = A_0(t)P(t)$$

determines the vector of probabilities of the finite-valued Markov process.

Corollary 2.5. *Let $\Pi(t)$ be an arbitrary stochastic matrix. The operator given by the equality*

$$L_\tau(t, \varepsilon) = (1-\varepsilon)P(t) + \varepsilon\Pi(t)P(t-\tau(t))\,, \qquad \tau(t) \geq 0\,,$$

is stochastic when $0 \leq \varepsilon \leq 1$. Then the system of linear differential equations

$$\frac{dP(t)}{dt} = \Pi(t)\big(P(t-\tau) - P(t)\big)\,, \qquad \tau(t) \geq 0$$

determines the vector of probabilities of some non-Markov random process.

Corollary 2.6. *Let elements $a_{js}(t), j, s = 1, 2, \ldots, m$ of matrix $A_0(t)$ satisfy (2.1) and $0 \leq \alpha_k(t) \leq c$, $c = \mathrm{const}$, $\tau_k(t) \geq 0$, $\Pi_k(t) \in L_\tau$, $k = 0, 1, 2, \ldots, N$. Then the system of differential equations*

$$\frac{dP(t)}{dt} = A_0(t)P(t) + \sum_{k=0}^{N} \alpha_k(t)\big(\Pi_k(t)P(t-\tau_k(t)) - P(t)\big) \tag{2.2}$$

determines the vector of probabilities of some non-Markov random process.

Corollary 2.7. *Let $\alpha(t,\tau) \geq 0$, $t \geq 0$, $\tau \geq 0$, $\int\limits_0^\infty \alpha(t,\tau)d\tau \leq c$, $c = \mathrm{const}$, $\Pi(t,\tau) \in L_\tau$, and elements $a_{js}(t), j, s = 1, 2, \ldots, m$ of matrix $A_0(t)$ satisfy (2.1). Then, if $N \to \infty$, system (2.2) yields the system of integro-differential equations*

$$\frac{dP(t)}{dt} = A_0(t)P(t) + \int_0^\infty \alpha(t,\tau)\big(\Pi(t,\tau)P(t-\tau) - P(t)\big)d\tau\,,$$

which determines the vector of probabilities of some non-Markov random process.

Corollary 2.8. *Let $F(t,x)$ be a vector of partial probability densities*

$$F(t,x) = \big(f_1(t,x), \ldots, f_n(t,x)\big), \quad f_k(t,x) \geq 0,\ k = 1, 2, \ldots, n,$$

$$\int\limits_{\mathbb{R}^m} \sum_{k=1}^{n} f_k(t,x)dx = 1\,,$$

and let $L_\tau(t,\varepsilon)$ be a stochastic operator that depends on $\varepsilon \ge 0$ and is defined for the vector $F(t+\tau,x)$ at $\tau \le 0$. We assume that $\lim\limits_{\varepsilon\to 0^+} L_\tau(t,\varepsilon)F(t+\tau,x) = F(t,x)$ and there exists an operator $A_\tau(t)$ such that

$$\begin{aligned} A_\tau(t)F(t+\tau,x) &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\big(L_\tau(t,\varepsilon)F(t+\tau,x) - F(t,x)\big) \\ &= \lim_{\varepsilon\to 0^+} \frac{\partial L_\tau(t,\varepsilon)}{\partial \varepsilon} F(t+\tau,x)\,. \end{aligned}$$

Then the operator equation $F(t+\varepsilon,x) = L_\tau(t,\varepsilon)F(t+\tau,x)$ determines the partial densities of the distribution of some non-Markov random process when $\varepsilon > 0$.

Remark 2.9. Assuming differentiation $F(t,x)$ with respect to t, if $\varepsilon \to 0^+$ in the equation

$$\varepsilon^{-1}\big(F(t+\varepsilon,x) - F(t,x)\big) = \varepsilon^{-1}\big(L_\tau(t,\varepsilon)F(t+\tau,x) - F(t,x)\big)\,,$$

we obtain a system of differential equations

$$\frac{\partial F(t,x)}{\partial t} = A_\tau(t)F(t+\tau,x) \tag{2.3}$$

that describes the partial distribution densities of some random process.

Theorem 2.10. *Let $A_\tau^{(k)}(t)$, $k = 1,2,\dots,N$ be generating operators and let functions $\alpha_k(t)$, $k = 1,2,\dots,N$ satisfy the conditions $0 < \alpha_k(t) \le c, c \in \mathbb{R}^+$. Then the system of differential equations*

$$\frac{\partial F(t,x)}{\partial t} = \sum_{k=0}^{N} \alpha_k(t)A_\tau^{(k)}(t)F(t+\tau,x)$$

determines the partial distribution densities of some random process.

Proof. It follows from Theorem 2.3, the operator

$$\alpha_\tau(t) = \sum_{k=0}^{N} \alpha_k(t)A_\tau^{(k)}(t)$$

is also a generating operator. The statement then follows from (2.3). □

Corollary 2.11. *Let Π be an arbitrary stochastic matrix. The generating operator*

$$A_\tau(t)F(t+\tau,x) = \Pi F(t+\tau,x) - F(t,x)\,, \qquad \tau(t) \ge 0$$

corresponds to the stochastic operator

$$L_\tau(t,\varepsilon)F(t+\tau,x) = \varepsilon\Pi F(t+\tau,x) + (1-\varepsilon)F(t,x)\,, \qquad \tau(t) \ge 0\,,$$

and the generating operator

$$A_\tau(t)F(t+\tau,x) = F\big(t+\tau,\Psi(t,x)\big)\left|\frac{D\Psi(t,x)}{Dx}\right| - F(t,x)\,, \qquad \tau(t) \ge 0$$

corresponds to the stochastic operator

$$A_\tau(t,x)F(t+\tau,x) = \varepsilon F\big(t+\tau,\Psi(t,x)\big)\left|\frac{D\Psi(t,x)}{Dx}\right| + (1-\varepsilon)F(t,x), \qquad \tau(t) \ge 0,$$

where $y=\Psi(t,x)$ is differentiable vector function defined for $x\in\mathbb{R}^m, t\geq 0$.

Corollary 2.12. *Let $\Pi_k, k=0,1,2,\dots,N$ be $n\times n$ stochastic matrices, and let $y=\Psi_k(t,x), k=0,1,2,\dots,N$ be differentiable vector functions which mutually uniquely map $\mathbb{R}^m$ to $\mathbb{R}^m$. Then the system of differential equations*

$$\frac{\partial F(t,x)}{\partial t}=\sum_{k=0}^{N}\alpha_k(t)\left(\Pi_k F\big(t-\tau_k(t),\Psi_k(t,x)\big)\left|\det\frac{D\Psi_k(t,x)}{Dx}\right|-F(t,x)\right),$$
$$\tau_k(t)\geq 0,\ k=0,1,2,\dots,N\,,$$

determines the partial densities of the distribution of some non-Markov process. When $\tau_k(t)\equiv 0, k=0,1,2,\dots,N$, the random process will be Markov.

Corollary 2.13. *Let $f(t,x)$ be differentiable vector function with respect to t,x and let $y=\Psi(t,x)$ be differentiable vector function with respect to x with projections $\phi_k(t,x), k=1,2,\dots,m$. If $\varepsilon\to 0$, then the stochastic operator*

$$L^{(1)}f(t,x)=f\big(t,x+\varepsilon\Psi(t,x)\big)\det\left(E+\varepsilon\frac{D\Psi(t,x)}{Dx}\right)$$

reduces to the generating operator

$$A^{(1)}f(t,x)=\operatorname{div}\big(f(t,x),\Psi(t,x)\big)=\sum_{k=1}^{m}\frac{\partial\big(f(t,x)\phi_k(t,x)\big)}{\partial x_k}\,.$$

Corollary 2.14. *Let the stochastic operator*

$$\begin{aligned}L^{(2)}f(t,x)&=\frac14 f\left(t,x+\sqrt{2\varepsilon}\Phi(t,x)\right)\det\left(E+\sqrt{2\varepsilon}\frac{D\Phi(t,x)}{Dx}\right)\\&+\frac14 f\left(t,x+\sqrt{2\varepsilon}\Phi(t,x)\right)\det\left(E-\sqrt{2\varepsilon}\frac{D\Phi(t,x)}{Dx}\right)\\&+\frac12 f\left(t,x+\frac{\varepsilon}{2}\Phi(t,x)\right)\det\left(E-\sqrt{2\varepsilon}\frac{D\Phi(t,x)}{Dx}\right)\end{aligned}$$

be given, where $\Phi(t,x)$ is a vector-function twice differentiable with respect to x, with projections

$$\varphi_k(t,x)=\operatorname{grad}\big(\varphi_k(t,x),\Phi(t,x)\big)=\sum_{s=1}^{m}\frac{\partial\varphi_k(t,x)}{\partial x_s}\varphi_k(t,s)\,,\ k=1,2,\dots,m\,.$$

Then the corresponding generating operator takes the form

$$
\begin{aligned}
A^{(2)}(t)f(t,x) &= \operatorname{div}\Big(\Phi(t,x)\operatorname{div}\big(f(t,x)\Phi(t,x)\big)\Big) = \frac{1}{2}f(t,x)\\
&= \big(\operatorname{div}\Phi(t,x)\big)^2 + \frac{1}{2}\Big(\operatorname{grad}\operatorname{div}\Phi(t,x)\Phi(t,x)\Big)\\
&\quad + \operatorname{div}\Phi(t,x)\Bigg(\operatorname{grad}f(t,x) + \frac{1}{2}\sum_{k,s=1}^{m}\frac{\partial^2 f(t,x)}{\partial x_k \partial x_s}\varphi_k(t,x)\varphi_s(t,x)\Bigg)\\
&\quad + \frac{1}{2}\sum_{k,s=1}^{m}\frac{\partial f(t,x)}{\partial x_k}\big(\operatorname{grad}\varphi_k(t,x)\Phi(t,x)\big)\,.
\end{aligned}
$$

3. Conclusion and further research direction

The paper shows possible procedures for constructing stochastic operators that can be used to construct differential equations with delay for analytically given non-Markov processes. In our opinion, in the situation developing in the modern world, when modeling with the study of decision making under uncertainty, non-Markovian processes will dominate.

There are still many unsolved questions in the field of the construction of non-Markovian models. Although non-Markov models describe events more realistically in many situations, there is a need to focus on building such models that will be even more personalized by incorporating domain knowledge.

The results presented here, in particular Theorem 2.3, make it possible to construct systems of differential equations with delay of the Kolmogorov-Feller type (see [3]), which can be used to construct moment equations for systems of differential equations, as well as differential equations with non-Markov coefficients.

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Department of Computer Mathematics and Information Security,
Kyiv National Economic University after named V. Hetman,
Kyiv 03068, Peremogy 54/1, Ukraine
E-mail: idzhalladova@gmail.com

Department of Analysis, Faculty of Mathematics,
University of Białystok, K. Ciołkowskiego 1M,
15-245 Białystok, Poland
E-mail: m.ruzickova@uwb.edu.pl

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STATIONARY SOLUTIONS OF SEMILINEAR SCHRÖDINGER EQUATIONS WITH TRAPPING POTENTIALS IN SUPERCRITICAL DIMENSIONS

Filip Ficek

Abstract. Nonlinear Schrödinger equations are usually investigated with the use of the variational methods that are limited to energy-subcritical dimensions. Here we present the approach based on the shooting method that can give the proof of existence of the ground states in critical and supercritical cases. We formulate the assumptions on the system that are sufficient for this method to work. As examples, we consider Schrödinger-Newton and Gross-Pitaevskii equations with harmonic potentials.

1. Introduction

The most common approach in the study of nonlinear Schrödinger equations (NLSE) is based on the variational methods. However, since these methods rely on some compactness results, they cease to work in energy-supercritical dimensions. From the application point of view, it does not seem to pose a great problem because such equations are usually used to describe various quantum-mechanical systems that are at most three-dimensional. An example of such NLSE is the Schrödinger-Newton-Hooke equation (SNH) that describes a self-gravitating quantum gas in a harmonic trap:

$$\begin{cases} i\,\partial_t \psi = -\Delta\psi + |x|^2\psi + v\psi\,, \\ \Delta v = |\psi|^2\,, \end{cases} \tag{1.1}$$

where ψ is the wavefunction and the nonlinearity is introduced by the gravitational potential v. In [2] the authors showed that this system can be also obtained as a nonrelativistic limit of the perturbations of the anti-de Sitter spacetime. This result connects it to one of the most important open problems in mathematical general relativity, the stability of anti-de Sitter spacetime [1], and gives a motivation to investigate Eq. (1.1) in higher dimensions.

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The literature regarding NLSE with trapping potentials (potentials that diverge to ∞ as $|x| \to \infty$) in supercritical dimensions is rather scarce. Existence of a whole family of stationary solutions (solutions satisfying the ansatz $\psi(t,x) = e^{-i\omega t}u(x)$, where ω is some real value called the frequency and u is a real function vanishing at infinity) of SNH was shown in [7]. The only other similar system that was investigated in supercritical dimensions was the Gross-Pitaevskii equation with a harmonic potential (GP) [3, 11, 12, 13, 14]:

$$i\,\partial_t\psi = -\Delta\psi + |x|^2\psi - |\psi|^2\psi\,. \tag{1.2}$$

The goal of this short paper is to formulate a common framework that can be used for various semilinear Schrödinger equations with trapping potentials in supercritical dimensions. In Section 2 we describe our approach, which is based on a reduction to the ordinary differential equation and application of the shooting method. We state the necessary assumptions and prove the existence of ground states for systems satisfying them. Section 3 shows how this result can be applied to SNH and GP equations. Finally, in Section 4 we outline the possible extensions and future prospects.

2. Main result

Since we are interested in stationary solutions, the initial Schrödinger equation is reduced to a nonlinear elliptic equation with a trapping potential. Ground states of such equations are usually spherically symmetric [10], letting us to write down the problem as an ODE, typically having the form

$$-u'' - \frac{d-1}{r}u' + V(r)u - F(r,u(r)) = \omega\,u\,, \tag{2.1}$$

where $r = |x|$, $u(r)$ is the solution we seek, V is the trapping potential (i.e. $\lim_{|x|\to\infty} V(x) = \infty$), and F denotes the nonlinearity. When looking for stationary solutions, one usually specifies some characteristics of the sought solution, e.g., its frequency or mass. Here we will be looking for ground states u with some fixed value in the center of the symmetry $u(0) = b > 0$. Regularity of the solution implies $u'(0) = 0$. Since u is a ground state, we also require that $\lim_{r\to\infty} u(r) = 0$ and $u(r) > 0$. Our final goal is to prove that there is such ω that there exists a solution u satisfying the conditions above and Eq. (2.1) – the ground state with frequency ω.

In principle, one could now try to employ the shooting method with ω as the shooting parameter. However, sometimes one has a better control on some other quantity related to ω that we will denote by c. Let us then rewrite the above equation as the following initial value problem

$$\begin{cases} u'' + \frac{d-1}{r}u' - V(r)u + F_c(r,u(r)) = 0\,, \\ u(0) = b > 0, \quad u'(0) = 0\,, \end{cases} \tag{2.2}$$

where F_c contains the nonlinearity and depends continuously on some parameter c. One can easily show that the singularity at $r = 0$ present in this equation does not pose any problem and all classical results regarding existence, uniqueness, and continuous dependence of the solutions still hold [9]. Hence, for any fixed value of c, we get some function $u_c(r)$ with its maximal domain $[0, R_c)$, where R_c may be infinite.

The proof of the existence of the ground states for SNH presented in [7] is following this line of action and then relies on the analysis of the behaviour of u_c as c changes. For the sake of generality, it may be convenient to perform here such analysis in isolation from the initial ODE-based context. We can just see the set of solutions of Eq. (2.2) for various c as the family of functions $\{u_c\}$ depending continuously on a single parameter c (their derivatives u_c' also depend continuously on c since they are solutions to the second order ODE). Assume then that this family satisfies the following six conditions.

(A1) There is a value of c such that the function u_c has r_0 at which $u_c(r_0) = 0$ while $u_c(r) > 0$ and $u_c'(r) < 0$ for $r \in (0, r_0)$.

(A2) The function u_0 is positive.

(A3) For any c, if at some point r_0 it holds $u_c(r_0) = u_c'(r_0) = 0$, then u is identically zero.

(A4) Functions u_c cannot have an inflection point while they are positive and decreasing.

(A5) It holds $u_c''(0) < 0$ for $c > 0$.

(A6) For any c, it either holds $\lim_{r\to R_c} u_c(r) = \infty$, $\lim_{r\to R_c} u_c(r) = -\infty$, or $\lim_{r\to R_c} u_c(r) = 0$, where R_c may be infinite.

As we prove now it leads to the existence of such c_0 that u_{c_0} is the ground state of our problem. It means that for a generic problem such as (2.2), it is enough to check whether the solutions satisfy these conditions to show that the ground state exists. This is the approach we employ in Section 3.

Theorem 2.1. *Let $\{u_c | c \geq 0\}$ be a family of at least twice differentiable functions with domains $[0, R_c)$ satisfying $u_c(0) = b > 0$ and $u_c'(0) = 0$. Let the values of u_c and u_c' depend pointwise-continuously on c. Then if this family satisfies* (A1)–(A6), *there exists c_0 such that u_{c_0} is a positive function on domain $[0, \infty)$ and decreasing to zero at infinity.*

Proof. Let us introduce a set of values of parameter c defined by the behaviour of u_c:

$$I = \{c \geq 0 \mid \exists\, r_0 > 0 : u_c(r_0) = 0 \text{ while } u_c(r) > 0 \text{ and } u_c'(r) < 0 \text{ for } r \in (0, r_0)\} .$$

Assumption (A1) tells us that this set is not empty, so $c_0 = \inf I$ is finite, while (A2) implies that $0 \notin I$. We claim that u_{c_0} is the sought function. The main tool in this proof will be the continuous dependence of u_c and u_c' on the parameter c.

Assume $c_0 > 0$ for now. If u_{c_0} crosses zero at some point, let us denote the first such occurrence by r_0. Then u_{c_0} must do it transversally due to (A3). It means that there exists U – a neighbourhood of c_0 such that for all values of c in it, u_c is also crossing zero. Additionally, since $c_0 = \inf I$, thanks to (A4) and (A5) the functions u_c for every c in U must be decreasing up to the crossing with zero (because no new stationary point may appear between $r = 0$ and the first crossing as c slightly changes). It means that $U \subset I$ and as a result c_0 cannot be the infimum of I. The fact that u_{c_0} cannot cross zero rules out the possibility that $\lim_{r\to\infty} u_{c_0}(r) = -\infty$.

Let us now assume that $\lim_{r\to\infty} u_{c_0}(r) = \infty$. Condition (A5) tells us that u_{c_0} is initially decreasing, so there must be a point where u_{c_0} has the first positive minimum. This time the continuous dependence of u'_{c_0} on c tells us that for some small neighbourhood of c_0 functions u_c also have such a minimum, thanks to (A4). It contradicts c_0 being the infimum of I, so it must hold $u'_{c_0}(r) < 0$. Similar analysis also applies to the case of $c_0 = 0$.

As u_{c_0} cannot diverge to any of the infinities, the trichotomy (A6) tells us that $\lim_{r\to\infty} u_{c_0}(r) = 0$. We additionally know that $u_{c_0}(r) > 0$ and $u'_{c_0}(r) < 0$, so u_{c_0} is a positive decreasing function. □

3. Applications

Now we briefly show how one can apply Theorem 2.1 to show the existence of the ground states in the cases of two different semilinear Schrödinger equations: (1.1) and (1.2). As already noted, we will be looking for solutions u with some fixed central value $u(0) = b > 0$.

3.1. **Schrödinger-Newton-Hooke equation.**

As the ground state of Eq. (1.1) we understand a stationary solution with both u and v tending to zero at infinity while u stays positive. Such solutions must be spherically symmetric as shown in [4]. It leads to a system of two ODEs for which the shooting method approach may seem problematic at the first glance since we do not know a priori the right value of $v(0)$. It means that in fact there are two shooting parameters: ω and $v(0)$. Even though there exist methods that may work in the case of such two-dimensional shooting [9], it is more convenient to get rid of ω completely by introducing $h(r) = \omega - v(r)$. As a result, one gets the equivalent system of equations

$$\begin{cases} u'' + \frac{d-1}{r}u' - r^2 u + hu = 0\,, \\ h'' + \frac{d-1}{r}h' + u^2 = 0\,. \end{cases} \tag{3.1}$$

This formal change of variables can be justified as long as $\lim_{r\to\infty} h(r)$ exists. Fortunately, this is the case as can be seen by rewriting the second line of Eq. (3.1) into

$$h'(r) = -\frac{1}{r^{d-1}} \int_0^r u(s)^2 s^{d-1}\, ds\,. \tag{3.2}$$

For stationary solutions, since u vanishes in infinity, this equation leads to $|h(r)| < Ar^2$, where A is some constant. As a result, for large r, the harmonic term in Eq. (3.1) dominates the nonlinear one and u decays exponentially. Then from Eq. (3.2) one sees that h converges to some finite value as needed. In the end we are left with Eq. (3.1) together with the initial conditions $u(0) = b$, $h(0) = c$, and $u'(0) = h'(0) = 0$. The analysis of this system, will lead us to the following result:

Proposition 3.1. *For any $b > 0$ there exists a value of ω such that system* (1.1) *has a ground state with $u(0) = b$.*

Proof. In this proof we show that for any $b > 0$ solutions to Eq. (3.1) with initial conditions $u(0) = b$, $h(0) = c$, and $u'(0) = h'(0) = 0$ form a one-parameter family $\{u_c\}$ that satisfies assumptions (A1)–(A6). A similar proof of this proposition has been presented in [7]. The main goal here is to recast it into the framework introduced by Theorem 2.1.

We start by investigating the behaviour of the solutions u_c for large values of c. Then it is convenient to introduce the rescaled variables $\widetilde{r} = \sqrt{c}r$, $\widetilde{u}_c(\widetilde{r}) = u_c(r)$, and $\widetilde{h}_c(\widetilde{r}) = h_c(r)/c$.

$$\begin{cases} \widetilde{u}_c'' + \frac{d-1}{\widetilde{r}}\widetilde{u}_c' - \frac{\widetilde{r}^2}{c^2}\widetilde{u}_c + \widetilde{h}_c\widetilde{u}_c = 0\,, \\ \widetilde{h}_c'' + \frac{d-1}{\widetilde{r}}\widetilde{h}_c' + \frac{1}{c^2}\widetilde{u}_c^2 = 0\,. \end{cases}$$

Taking the limit $c \to \infty$ removes two terms from this system and leaves us with equations that can be explicitly solved: $\widetilde{h}_\infty$ is just equal to 1, while $\widetilde{u}_\infty$ can be expressed with the Bessel function $J_{\frac{d}{2}-1}$ and oscillates indefinitely with decreasing amplitude. It implies that for large enough values of c the solution u_c is crossing zero and monotonically decreasing beforehand, resulting in (A1).

To prove that (A2) holds, let us assume otherwise: that u_0 crosses zero for the first time at some $R > 0$. Then multiplication of the first equation in (3.1) by $u_0(r)r^{d-1}$ and integrating over the interval $[0, R]$ leads to some identity. A similar identity can be obtained by multiplying by $u_0'(r)r^d$ and integrating over the same domain. Another two identities can be obtained in an analogous way from the second equation in (3.1) and combining all four of them yields (see [7] for the details)

$$\begin{aligned} (d-6)\int_0^R u_0'(r)^2 r^{d-1} dr + (d+2)\int_0^R r^2 u_0(r)^2 r^{d-1}\,dr + 2u_0'(R)^2 R^d \\ + h_0'(R)^2 R^d + (d-2)h_0(R)h_0'(R)R^{d-1} = 0\,. \end{aligned}$$

This Pohozaev-type identity for $d \geq 6$ (i.e., in critical and supercritical dimensions for SNH) consists of purely positive terms on its left-hand side because $h_0(0) = 0$ and $h_c(r)$ is decreasing for any c due to Eq. (3.2). We arrive at a contradiction.

Assumption (A3) clearly holds, while (A4) can be checked by a simple analysis of the system (3.1). Additionally, a proper examination of Eq. (3.1) in the limit $r \to 0$ gives $u_c''(0) = -bc/d$ and proves (A5).

Finally, (A6) can be obtained by observing that since h_c is decreasing, for sufficiently large values of r ($r > \sqrt{c}$), the term $-r^2 + h_c(r)$ is negative. When it happens, the first line of Eq. (3.1) tells us that u_c cannot have positive maxima, nor negative minima. It means that $u_c(r)$ must be monotone from some point on. Then, if $\lim_{r\to\infty} u_c(r)$ exists, it must be equal to zero because otherwise one can calculate the limit of

$$u_c'(r) = \frac{1}{r^{d-1}}\int_0^r \left[s^2 - h_c(s)\right] u_c(s)s^{d-1}\,ds$$

as $r \to \infty$ using the L'Hôpital's rule and get $\lim_{r\to\infty} |u_c'(r)| = \infty$. It contradicts the convergence of u_c resulting in trichotomy (A6).

Since all necessary assumptions are satisfied, Theorem 2.1 tells us that there exists such a value c_0 that u_{c_0} is a positive solution decaying to zero at infinity. It also is the ground state of the initial problem (1.1) with frequency that can be restored as $\omega = \lim_{r\to\infty} h_{c_0}(r)$. □

3.2. **Gross-Pitaevskii equation.** In the case of Eq. (1.2) the stationary solution ansatz and spherical symmetry assumption (justified by [10]) lead to the equation

$$u'' + \frac{d-1}{r}u' - r^2 u + u^3 + \omega u = 0\,. \tag{3.3}$$

Then one has the following result:

Proposition 3.2. *For any $b > 0$ there exists a value of ω such that the solution u to Eq.* (1.2) *with $u(0) = b$ is a ground state.*

Proofs of this proposition can be found in [3] and [14]. However, in both of these works the authors need to rely on some functional-analytic methods. Theorem 2.1 suggests a more elementary way of obtaining this result.

For Eq. (3.3) the frequency ω can be directly used as the shooting parameter c, so let $\omega = c$. Most of the assumptions needed for Theorem 2.1 can be checked in a similar way as for SNH. By considering the variables $\widetilde{r} = \sqrt{c}r$, $\widetilde{u}_c(\widetilde{r}) = u_c(r)$ and then taking the limit $c \to \infty$ in Eq. (3.3), one can prove (A1). Assumption (A2) can again be obtained with the use of the Pohozaev identity, see [3] for the details, but this time it holds for $d \geq 4$ (critical and supercritical dimensions for GP). One can also very simply get (A3), (A4), and (A5).

Unfortunately, assumption (A6) cannot be proven as simply as before, when one could just use the monotonicity of h. Here we can get a better view by introducing new variables $t = r^2/2$ and $w(t) = u(r)/r$ in which Eq. (3.3) becomes

$$\ddot{w} + \frac{d+2}{2t}\dot{w} + w(w^2-1) + \frac{d-1}{4t^2}w + \frac{\omega}{2t}w = 0\,.$$

Dots denote here the derivatives in t. This system can be interpreted as a description of the damped motion of a point particle in a potential changing its shape from unimodal with a minimum at $w = 0$ to W-like with minimas at $w = \pm 1$ and a maximum at $w = 0$. This physical picture suggests that the only possible long-time behaviours of the particle are either confinement in one of the two valleys and settling at $w = \pm 1$ or convergence to the maximum at $w = 0$. In particular, since the damping term behaves like t^{-1} it should be impossible for w to oscillate indefinitely [15]. However, the strict proof of this fact would require further work. After going back to the original variables, $w \to \pm 1$ would lead to $u \to \pm\infty$, while $w \to 0$ would give $u \to 0$, implying the trichotomy.

Combination of all these conditions would lead, via Theorem 2.1, to the existence of c_0 such that u_{c_0} is the ground state with frequency $\omega = c_0$.

4. Conclusions

An additional question one can ask regarding the obtained solutions is about their uniqueness, i.e., whether for a fixed $b > 0$ there is only one value of the shooting parameter c giving the ground state. At this point, no general method of proving this seems to be available. One must instead refer to the case by case analysis. For example, in the case of SNH the uniqueness of the ground state can be proved by methods presented either in [5] or [8] (the second approach was applied in [7]). However, for GP no similar result exists at this point [3] (even though numerical experiments suggest that the obtained ground states are also unique in this case).

The main advantage of the method presented over the other similar approaches [3, 5, 14], is that it can be easily expanded to cover also excited states – stationary solutions that decay to zero at infinity but are crossing zero. Let us just mention here that such states are not bound to be spherically symmetric, so by reduction to ordinary differential equations some solutions are usually lost. Then, to prove the existence of a solution crossing zero exactly once, one can define a set of shooting parameter values in a similar manner as before

$$\begin{aligned} I = \{c \geq 0 \,|\, \exists\, 0 < r_0 < \rho_1 < r_1 : u_c(r_0) = u_c(r_1) = 0 \\ \text{and } u_c'(\rho_1) = 0, \text{ while } u_c(r) > 0,\, u_c'(r) < 0 \text{ for } r \in (0, r_0), u_c(r) < 0, \\ u_c'(r) < 0 \text{ for } r \in (r_0, \rho_1), \text{and } u_c(r) < 0,\, u_c'(r) > 0 \text{ for } r \in (\rho_1, r_1)\}\,. \end{aligned}$$

This set is non-empty in the cases we covered here since in the limit $c \to \infty$ the solutions were oscillating. This time one needs some better control on the stationary points of the solutions than was needed for the ground state, in particular regarding the emergence of new stationary points from infinity as c changes. Then it is easy to show that $\inf I$ is the sought value of c. This idea can be further generalised to any number of crossings with zero by the appropriate choice of the set I. Existence of such a ladder of excited states in the case of SNH was shown using this method in [7].

One can also look for other systems that can be investigated with this approach. Let us point out that the ideas presented above can be easily applied to a broad range of problems with other trapping potentials (not necessarily harmonic) and simple nonlinearities (for example, $|u|^{p-1}u$ where $p > 1$). Some early work suggests that similar methods can also work in the case of systems of elliptic equations, such as considered in [6] but in the presence of some trapping potential.

Finally, this research is just a first step in the broader goal of understanding the dynamics of semilinear Schrödinger equations with trapping potentials in supercritical dimensions. Some of the results regarding the dependence of frequency ω on central value b suggest interesting changes in stability of the ground states in higher dimensions [3, 7, 11]. We plan to pursue this direction in the future work.

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Institute of Theoretical Physics, Jagiellonian University,
Łojasiewicza 11, 30-348 Kraków, Poland
E-mail: `ficekf@gmail.com`

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A NOTE ON THE OSCILLATION PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH $p(t)$-LAPLACIAN

Kōdai Fujimoto

Abstract. This paper deals with the oscillation problems on the nonlinear differential equation $(a(t)|x'|^{p(t)-2}x')'+b(t)|x|^{\lambda-2}x=0$ involving $p(t)$-Laplacian. Sufficient conditions are given under which all proper solutions are oscillatory. In addition, we give *a-priori* estimates for nonoscillatory solutions and propose an open problem.

1. Introduction

We consider the second-order nonlinear differential equation

$$(a(t)|x'|^{p(t)-2}x')' + b(t)|x|^{\lambda-2}x = 0, \quad t \ge t_0\,, \tag{1.1}$$

where $a(t)$, $b(t)$, and $p(t) > 1$ are positive continuous functions and $\lambda > 1$ is a constant. In addition, we assume

$$\limsup_{t\to\infty} a(t) < \infty, \tag{1.2}$$

that is, there exists $\alpha > 0$ such that $a(t) < \alpha$ for $t \ge t_0$.

Note that the differential operator in equation (1.1) is called $p(t)$-Laplacian. Such operator appears in mathematical models in a wide range of research fields such as nonlinear elasticity theory, electrorheological fluids, and image processing (see [2, 10, 14]). In recent years, increasing interest has been paid to the study of ordinary differential equations with $p(t)$-Laplacian. For example, we can find those results in [1, 4, 8, 9, 15, 16, 17, 18] and the references cited therein.

A function $x(t)$ is said to be a *solution* of equation (1.1) defined on (t_0, τ), if $x(t)$ and its quasiderivative

$$x^{[1]}(t) = a(t)|x'(t)|^{p(t)-2}x'(t)$$

are continuously differentiable, and $x(t)$ satisfies equation (1.1) on (t_0, τ). We study solutions of equation (1.1) which are defined on (t_0, τ); if $\tau < \infty$ then we suppose

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that $x(t)$ is nonextendable to the right, i.e.,

$$\limsup_{t\to\tau-}(|x(t)|+|x'(t)|)=\infty\,.$$

A nontrivial solution $x(t)$ of equation (1.1) is said to be a *singular solution of the first kind*, if there exists $T_x > t_0$ such that $x(t) \equiv 0$ for $t \geq T_x$. It is said to be a *singular solution of the second kind*, if $\tau < \infty$. It is said to be a *proper solution* if $x(t)$ is not singular. The existence of proper solutions for equation (1.1) can be referred to in [1]. A proper solution $x(t)$ of equation (1.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory.*

Let $p(t) \equiv p > 1$. Then equation (1.1) becomes the so-called generalized Emden-Fowler differential equation

$$(a(t)|x'|^{p-2}x')' + b(t)|x|^{\lambda-2}x = 0, \quad t \geq t_0 \tag{1.3}$$

with the classical p-Laplacian. It is known that the study of equation (1.3) originates from gas dynamics in astrophysics. Moreover, asymptotic behavior of solutions of equation (1.3) corresponds to the concentration of a substance disappearing according to an isothermal reaction in an finite slab of catalyst (see [13]). Hence, a lot of papers have been devoted to the study of equation (1.3) (see [3, 5, 6, 7, 11, 12, 13]). Especially, on the oscillation problems, the following theorem is proved in [12].

Theorem A. All proper solutions of equation (1.3) with $a(t) \equiv 1$ are oscillatory if

$$\int^{\infty} b(t)\,dt = \infty\,. \tag{1.4}$$

According to the proof of Theorem A, we can easily get the analogue for equation (1.1) when $\liminf_{t\to\infty} p(t) > 1$ under (1.2). Here, a natural question now arises: Are all proper solutions of equation (1.1) oscillatory when $\lim_{t\to\infty} p(t) = 1$? The purpose of this paper is to answer the question. To be precise, we give sufficient conditions under which all proper solutions of equation (1.1) are oscillatory. Our main result is stated as follows.

Theorem 1.1. *Assume* (1.2). *Suppose that there exists a constant* $c > 0$ *such that*

$$p(t) \geq 1 + \frac{c}{\log\log t}\,. \tag{1.5}$$

Then, all proper solutions of equation (1.1) *are oscillatory if* (1.4) *holds.*

Remark 1.2. Theorem 1.1 contains not only the case of $\lim_{t\to\infty} p(t) = 1$, but also the case of $\liminf_{t\to\infty} p(t) > 1$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we give some remarks and open problems.

2. Proof of the main theorem

In this section, we give the proof of Theorem 1.1. We begin with the following lemma.

Lemma 2.1. *Assume* (1.2) *and* (1.5). *Let* $y \in C^1[t_0, \infty)$ *be a function satisfying* $y(t) \neq 0$ *for* $t \geq t_0$. *Then, for any* $T \geq t_0$,

$$\limsup_{t\to\infty} \left\{ \frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)} + (\lambda - 1) \int_T^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\, ds \right\} \geq 0 \tag{2.1}$$

holds.

Proof. Suppose, toward a contradiction, that (2.1) is false. Then, there exist constants $k > 0$ and $T' > t_0$ such that

$$\frac{k|y(T')|^{\lambda-1}}{\alpha} < 1 \tag{2.2}$$

and

$$\frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)} + (\lambda - 1) \int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\, ds \leq -k\,,$$

that is,

$$k + (\lambda - 1) \int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\, ds \leq -\frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)} \tag{2.3}$$

for $t \geq T'$. Since the left-hand side of (2.3) is positive, we see that $y(t)y'(t) < 0$ for $t \geq T'$.

Dividing (2.3) by its left-hand side and multiplying by $-y'(t)/y(t)$, we get

$$-\frac{y'(t)}{y(t)} \leq \frac{a(t)|y'(t)|^{p(t)}/|y(t)|^{\lambda}}{k + (\lambda - 1) \int_{T'}^t \left\{a(s)|y'(s)|^{p(s)}/|y(s)|^{\lambda}\right\} ds},$$

and therefore, we obtain

$$-(\log|y(t)|)' \leq \frac{1}{\lambda - 1} \left(\log \left(k + (\lambda - 1) \int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\, ds \right) \right)'$$

for $t \geq T'$. Integrating the both sides of this inequality from T' to t, we have

$$\begin{aligned} &-\log|y(t)| + \log|y(T')| \\ &\leq \frac{1}{\lambda - 1} \left(\log \left(k + (\lambda - 1) \int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\, ds \right) - \log k \right) \end{aligned}$$

for $t \geq T'$. Hence, we obtain

$$k \left| \frac{y(T')}{y(t)} \right|^{\lambda-1} \leq k + (\lambda - 1) \int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\, ds \tag{2.4}$$

for $t \geq T'$.

From (2.3) and (2.4), we have

$$k \left| \frac{y(T')}{y(t)} \right|^{\lambda-1} \leq -\frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)}\,,$$

that is,

$$k|y(T')|^{\lambda-1} \leq -a(t)|y'(t)|^{p(t)-2}y'(t) \operatorname{sgn} y(t)$$

for $t \geq T'$. Using (1.2) and $y(t)y'(t) < 0$ for $t \geq T'$, we get

$$k|y(T')|^{\lambda-1} \leq a(t)|y'(t)|^{p(t)-1} \leq \alpha |y'(t)|^{p(t)-1}\,,$$

and therefore, we obtain

$$|y'(t)| \geq \left(\frac{k|y(T')|^{\lambda-1}}{\alpha}\right)^{1/(p(t)-1)} \tag{2.5}$$

for $t \geq T'$. We note that

$$c_0 := \frac{k|y(T')|^{\lambda-1}}{\alpha} < 1$$

from (2.2).

According to (1.5), we have

$$\frac{1}{p(t)-1} \leq \frac{\log\log t}{c},$$

and hence, we obtain

$$c_0^{1/(p(t)-1)} \geq c_0^{(\log\log t)/c} = (\log t)^{(\log c_0)/c} = \left(\frac{1}{\log t}\right)^{|\log c_0|/c}.$$

Together with (2.5), we get

$$|y'(t)| \geq \left(\frac{1}{\log t}\right)^{|\log c_0|/c}$$

for $t \geq T'$. In the case of $y(t) > 0$, this implies

$$y'(t) \leq -\left(\frac{1}{\log t}\right)^{|\log c_0|/c}.$$

Integrating the both sides of this inequality, we get

$$\lim_{t\geq\infty} y(t) - y(T') \leq -\int_{T'}^{\infty} \left(\frac{1}{\log t}\right)^{|\log c_0|/c} dt = -\infty\,.$$

This is a contradiction. In the case of $y(t) < 0$, as in the same manner in the previous case, we obtain

$$\lim_{t\to\infty} y(t) = \infty\,,$$

which is a contradiction. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose, toward a contradiction, that equation (1.1) has a nonoscillatory solution $x(t)$. Then, from Lemma 2.1, we have (2.1) with $y(t) = x(t)$. Without loss of generality, we may assume $x(t) \neq 0$ for $t \geq t_0$. Then, we can calculate

$$(|x(t)|^{\lambda-2}x(t))' = (\lambda-1)x'(t)|x(t)|^{\lambda-2}$$

and

$$\frac{(a(t)|x'(t)|^{p(t)-2}x'(t))'}{|x(t)|^{\lambda-2}x(t)} = \left(\frac{a(t)|x'(t)|^{p(t)-2}x'(t)}{|x(t)|^{\lambda-2}x(t)}\right)' + (\lambda-1)\frac{a(t)|x'(t)|^{p(t)}}{|x(t)|^{\lambda}}\,. \tag{2.6}$$

Dividing equation (1.1) by $|x(t)|^{\lambda-2}x(t)$, we have

$$\frac{(a(t)|x'(t)|^{p(t)-2}x'(t))'}{|x(t)|^{\lambda-2}x(t)} = -b(t)\,.$$

Integrating the both sides of this inequality from t_0 to t, we get

$$\int_{t_0}^{t} \frac{(a(s)|x'(s)|^{p(s)-2}x'(s))'}{|x(s)|^{\lambda-2}x(s)}\,ds = -\int_{t_0}^{t} b(s)\,ds\,.$$

Together with (2.6), we obtain

$$\begin{aligned}\frac{a(t)|x'(t)|^{p(t)-2}x'(t)}{|x(t)|^{\lambda-2}x(t)} &+ (\lambda-1)\int_{t_0}^{t}\frac{a(s)|x'(s)|^{p(s)}}{|x(s)|^{\lambda}}\,ds\\ &= \frac{a(t_0)|x'(t_0)|^{p(t_0)-2}x'(t_0)}{|x(t_0)|^{\lambda-2}x(t_0)} - \int_{t_0}^{t} b(s)\,ds \to -\infty\end{aligned}$$

as $t\to\infty$. This is a contradiction to (2.1). □

3. Discussion and remarks

From Theorem 1.1, we see that if (1.5) holds (that is to say, $p(t)$ tends to 1 more slowly than $1/\log\log t$) then there are no nonoscillatory solutions. On the other hand, the nonexistence of nonoscillatory solutions is not guaranteed when $p(t)$ tends to 1 so rapidly. Hence, in this section, we consider the case when (1.5) is false.

If a nonoscillatory solution $x(t)$ of equation (1.1) is eventually negative, then $-x(t)$ is an eventually positive solution of equation (1.1). Hence, when we discuss nonoscillatory solutions, we focus only on eventually positive solutions, and let us simply call them *positive solutions*.

Let $x(t)$ be a positive solution. Then, from equation (1.1), $x^{[1]}(t)$ is decreasing. Therefore, we see that the sign of $x^{[1]}(t)$ is eventually constant, that is, $x(t)$ has a monotonicity for large t. The following proposition shows the *a-priori* estimate for nonoscillatory solutions for equation (1.1).

Proposition 3.1. *Assume* (1.4)*. If there exists a nonoscillatory solution $x(t)$ of equation* (1.1)*, then $x(t)$ is decreasing to* 0 *as $t\to\infty$.*

Proof. We first suppose that $x(t)$ is nondecreasing. Integrating equation (1.1) from t_0 to ∞, we get

$$\begin{aligned}(x(t_0))^{\lambda-1}\int_{t_0}^{\infty} b(t)\,dt \le \int_{t_0}^{\infty} b(t)(x(t))^{\lambda-1}\,dt &= -\lim_{t\to\infty} x^{[1]}(t) + x^{[1]}(t_0)\\ &< x^{[1]}(t_0) < \infty\,,\end{aligned}$$

which is a contradiction.

We next suppose that there exists a constant $c_1 > 0$ such that $x(t)$ is decreasing to c_1. Then, we have

$$-x^{[1]}(t) > -x^{[1]}(t) + x^{[1]}(t_0) = \int_{t_0}^{t} b(s)(x(s))^{\lambda-1}\,ds \ge c_1^{\lambda-1}\int_{t_0}^{t} b(s)\,ds\,,$$

that is to say,

$$-x'(t) \geq \left(\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\,ds \right)^{1/(p(t)-1)}$$

for $t \geq t_0$. Integrating the both sides of this inequality from t_0 to ∞, we obtain

$$\int_{t_0}^{\infty} \left(\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\,ds \right)^{1/(p(t)-1)} dt \leq -\lim_{t\to\infty} x(t) + x(t_0) = -c_1 + x(t_0) < \infty\,.$$

On the other hand, from (1.2) and (1.4), we get

$$\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\,ds > 1$$

for t sufficiently large, which implies that

$$\int_{t_0}^{\infty} \left(\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\,ds \right)^{1/(p(t)-1)} dt = \infty\,.$$

This is a contradiction. □

We finally propose the following open problem: Does equation (1.1) have a nonoscillatory solution which is decreasing to 0 as $t \to \infty$ in the case when (1.5) is false? If the nonexistence of such a solution is proved, then the condition (1.5) can be removed from Theorem 1.1. Otherwise, it will be a discrepancy between equations (1.1) and (1.3).

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Faculty of Liberal Arts, Sciences and Global Education,
Osaka Metropolitan University,
1-1 Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531, Japan
E-mail: `kodai@omu.ac.jp`

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MAXWELL'S EQUATIONS REVISITED - MENTAL IMAGERY AND MATHEMATICAL SYMBOLS

Matthias Geyer, Jan Hausmann, Konrad Kitzing, Madlyn Senkyr, and Stefan Siegmund

Abstract. Using Maxwell's mental imagery of a tube of fluid motion of an imaginary fluid, we derive his equations $\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, $\operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$, $\operatorname{div}\mathbf{D} = \varrho$, $\operatorname{div}\mathbf{B} = 0$, which together with the constituting relations $\mathbf{D} = \varepsilon_0\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$, form what we call today Maxwell's equations. Main tools are the divergence, curl and gradient integration theorems and a version of Poincare's lemma formulated in vector calculus notation. Remarks on the history of the development of electrodynamic theory, quotations and references to original and secondary literature complement the paper.

1. Introduction

Maxwell developed his famous equations

$$\operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \quad \operatorname{div}\mathbf{D} = \varrho\,,$$
$$\operatorname{curl}\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}\,, \quad \operatorname{div}\mathbf{B} = 0\,,$$
$$\mathbf{D} = \varepsilon_0\mathbf{E}\,, \quad \mathbf{B} = \mu_0\mathbf{H}\,,$$

in what Hon and Goldstein [12] call an odyssey in electromagnetics consisting of four stations:

– Station 1 (1856-1858): on Faraday's lines of force [18]
– Station 2 (1861-1862): on physical lines of force [19]
– Station 3 (1865): A dynamical theory of the electromagnetic field [20]
– Station 4 (1873): A treatise on electricity and magnetism [21]

Maxwell's original work is a rich source for methodological inspiration [22, 21]. Many excellent books [4, 12, 24] and papers [1, 2, 27] have been written on Maxwell's construction of an electrodynamic theory, the transcription of Maxwell's equations to vector analysis notation [13, 28] and their contemporary presentation with differential forms [8, 16, 23, 30]. When it comes to teaching Maxwell's equations many aspects of the didactic transposition process from Maxwell's original work, to

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textbooks and then to classroom are lost. As pointed out in [14] this is, in particular, the case for the displacement current term $\frac{\partial \mathbf{D}}{\partial t}$ insertion. In this article we present a method for introducing Maxwell's displacement current concept, within the framework of an integral formulation of Maxwell's equations (for other didactic derivations of Maxwell's equations, see e.g. [5, 9, 14]). Our method is deliberately not formulated in the elegant and more advanced differential form calculus, but the readily available vector calculus notation. We stay very close to Maxwell's original work in the following specific aspects:

- We start out with Maxwell's first paper on Faraday's lines of force [18].
- We apply his analogy of an imaginary fluid to electricity and magnetism.
- We use his mental imagery of a tube of fluid motion.

Historical Remarks are added for the reader's convenience and as references to Maxwell's work and secondary literature. Mathematical preliminaries are the well-known integration theorems

$$f_1 - f_2 = \int_{\text{curve}} \operatorname{grad} f \cdot d\mathbf{s}, \int_{\text{curve}} \mathbf{a} \cdot d\mathbf{s} = \int_{\text{surface}} \operatorname{curl} \mathbf{a} \cdot d\mathbf{S}, \int_{\text{surface}} \mathbf{b} \cdot d\mathbf{S} = \int_{\text{volume}} \operatorname{div} \mathbf{b} \, dV$$

which are recalled in Theorem 2.3, together with a version of Poincare's Lemma.

Historical Remark 1.1 (Mental Imagery & Mathematical Symbols)**.** On September 15th, 1870, Maxwell gave an *Address to the Mathematical and Physical Sections of the British Association* (see e.g. the reprint in [22, vol. 2, pp. 218-219]) in which he also explains the role of mental imagery and mathematical symbols:

> The human mind is seldom satisfied, and is certainly never exercising its highest functions, when it is doing the work of a calculating machine. What the man of science, whether he is a mathematician or a physical inquirer, aims at is, to acquire and develop clear ideas of the things he deals with. [...]
>
> But if he finds that clear ideas are not to be obtained by means of processes the steps of which he is sure to forget before he has reached the conclusion, it is much better that he should turn to another method, and try to understand the subject by means of well-chosen illustrations derived from subjects with which he is more familiar.
>
> [...] [A] truly scientific illustration is a method to enable the mind to grasp some conception or law in one branch of science, by placing before it a conception or a law in a different branch of science, and directing the mind to lay hold of that mathematical form which is common to the corresponding ideas in the two sciences [...].
>
> The correctness of such an illustration depends on whether the two systems of ideas which are compared together are really analogous in form, or whether, in other words, the corresponding physical quantities really belong to the same mathematical class. When this condition is fulfilled, the illustration is not only convenient for teaching science in a pleasant and easy manner, but the recognition of the formal analogy between the two systems of ideas leads to a knowledge of both, more profound than could be obtained by studying each system separately.
>
> [...] [S]cientific truth should be presented in different forms, and should be regarded as equally scientific, whether it appears in the robust form and the vivid colouring of a physical illustration, or in the tenuity and paleness of a symbolical expression.

> Time would fail me if I were to attempt to illustrate by examples the scientific value of the classification of quantities. I shall only mention the name of that important class of magnitudes having direction in space which Hamilton has called vectors, and which form the subject-matter of the Calculus of Quaternions, a branch of mathematics which, when it shall have been thoroughly understood by men of the illustrative type, and clothed by them with physical imagery, will become, perhaps under some new name, a most powerful method of communicating truly scientific knowledge to persons apparently devoid of the calculating spirit.

$\diamond$

In this paper we adopt Maxwell's empathetic perception that scientific truth should be presented in different forms, using mental imagery as well as mathematical symbols. We follow his analogy between a model of an imaginary fluid and his equations of electrodynamics which are indeed analogous in form and belong to the same mathematical class. A truly scientific illustration taking the role of mental imagery will suggest itself by visualizing the concepts corresponding to the mathematical symbols used to describe the analogy, in our case this will be the divergence, curl and gradient integration theorems and a geometric interpretation of Poincaré's Lemma in terms of the integration theorems.

Historical Remark 1.2 (Quaternions & Vector Analysis)**.** Maxwell's prevision that the Calculus of Quaternions under some new name would become a most powerful method of communicating scientific knowledge (see Remark 1.1) came true with the early development of vector analysis [3] from quaternions

$$q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad (q_0, q_1, q_2, q_3 \in \mathbb{R}) \quad \text{with} \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\,.$$

Hamilton who discovered the quaternions in 1843 noted already the special role of what he called a *vector*

$$q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

compared to what he called the *scalar part* q_0 of $q \coloneqq q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. His notation $Sq = q_0$ and $Vq = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ applied to the product of two vectors

$$S(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) = -(xx' + yy' + zz')\,,$$
$$V(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) = (yz' - zy')\mathbf{i} + (zx' - xz')\mathbf{j} + (xy' - yx')\mathbf{k}\,,$$

founded the vector calculus notation of the scalar and cross product, respectively, which was then most prominently developed and promoted by Heaviside (see e.g. [13] and the references therein) and by Gibbs [10, 11] who coined most of the notation

$$\nabla f \coloneqq \operatorname{grad} f \coloneqq \big(\partial_1 f, \partial_2 f, \partial_3 f\big) \quad \text{for} \quad f \in C^1(\mathbb{R}^3, \mathbb{R}),$$
$$\nabla \times \mathbf{a} \coloneqq \operatorname{curl} \mathbf{a} \coloneqq \big(\partial_2 \mathbf{a}_3 - \partial_3 \mathbf{a}_2, \partial_3 \mathbf{a}_1 - \partial_1 \mathbf{a}_3, \partial_1 \mathbf{a}_2 - \partial_2 \mathbf{a}_1\big)\,,$$
$$\nabla \cdot \mathbf{a} \coloneqq \operatorname{div} \mathbf{a} \coloneqq \partial_1 \mathbf{a}_1 + \partial_2 \mathbf{a}_2 + \partial_3 \mathbf{a}_3 \quad \text{for} \quad \mathbf{a} \in C^1(\mathbb{R}^3, \mathbb{R}^3)\,,$$

which is still used today in vector calculus. $\diamond$

Heaviside developed vector analysis independently of Gibbs and it was him who expressed Maxwell's equations in vector calculus notation (see e.g. [1, 28] and the many references therein).

$$\left.\begin{aligned} a &= \tfrac{dH}{dy} - \tfrac{dG}{dz} \\ b &= \tfrac{dF}{dz} - \tfrac{dH}{dx} \\ c &= \tfrac{dG}{dx} - \tfrac{dF}{dy} \end{aligned}\right\} \quad \text{(A)} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\left.\begin{aligned} P &= c\tfrac{dy}{dt} - b\tfrac{dz}{dt} - \tfrac{dF}{dt} - \tfrac{d\Psi}{dx} \\ Q &= a\tfrac{dz}{dt} - c\tfrac{dx}{dt} - \tfrac{dG}{dt} - \tfrac{d\Psi}{dy} \\ R &= b\tfrac{dx}{dt} - a\tfrac{dy}{dt} - \tfrac{dH}{dt} - \tfrac{d\Psi}{dz} \end{aligned}\right\} \quad \text{(B)} \qquad \mathbf{E} = \mathbf{v} \times \mathbf{B} - \tfrac{\partial \mathbf{A}}{\partial t} - \nabla\varphi$$

$$\left.\begin{aligned} X &= vc - wb \\ Y &= wa - uc \\ Z &= ub - va \end{aligned}\right\} \quad \text{(C)} \qquad \mathbf{F} = \mathbf{J} \times \mathbf{B}$$

$$\left.\begin{aligned} a &= \alpha + 4\pi A \\ b &= \beta + 4\pi B \\ c &= \gamma + 4\pi C \end{aligned}\right\} \quad \text{(D)} \qquad \mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$$

$$\left.\begin{aligned} 4\pi u &= \tfrac{d\gamma}{dy} - \tfrac{d\beta}{dz} \\ 4\pi v &= \tfrac{d\alpha}{dz} - \tfrac{d\gamma}{dx} \\ 4\pi w &= \tfrac{d\beta}{dx} - \tfrac{d\alpha}{dy} \end{aligned}\right\} \quad \text{(E)} \qquad \mathbf{J} = \nabla \times \mathbf{H}$$

$$\mathfrak{D} = \tfrac{1}{4\pi}\mathfrak{E} \quad \text{(F)} \qquad \mathbf{D} = \epsilon\mathbf{E}$$

$$\mathfrak{K} = C\mathfrak{E} \quad \text{(G)} \qquad \mathbf{J}_c = \sigma\mathbf{E}$$

$$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}} \quad \text{(H)} \qquad \mathbf{J} = \mathbf{J}_c + \tfrac{\partial \mathbf{D}}{\partial t}$$

$$\left.\begin{aligned} u &= p + \tfrac{df}{dt} \\ v &= q + \tfrac{dg}{dt} \\ w &= r + \tfrac{dh}{dt} \end{aligned}\right\} \quad \text{(H*)} \qquad \mathbf{J} = \mathbf{J}_c + \tfrac{\partial \mathbf{D}}{\partial t}$$

$$\mathfrak{C} = (C + \tfrac{1}{4\pi}K\tfrac{d}{dt})\mathfrak{E} \quad \text{(I)} \qquad \mathbf{J} = \sigma\mathbf{E} + \epsilon\tfrac{\partial \mathbf{E}}{\partial t}$$

$$\left.\begin{aligned} u &= CP + \tfrac{1}{4\pi}K\tfrac{dP}{dt} \\ v &= CQ + \tfrac{1}{4\pi}K\tfrac{dQ}{dt} \\ w &= CR + \tfrac{1}{4\pi}K\tfrac{dR}{dt} \end{aligned}\right\} \quad \text{(I*)} \qquad \mathbf{J} = \sigma\mathbf{E} + \epsilon\tfrac{\partial \mathbf{E}}{\partial t}$$

$$\rho = \tfrac{df}{dx} + \tfrac{dg}{dy} + \tfrac{dh}{dz} \quad \text{(J)} \qquad \varrho = \nabla \cdot \mathbf{D}$$

$$\sigma = lf + mg + nh + l'f' + m'g' + n'h' \quad \text{(K)} \qquad \varrho_s = \mathbf{n} \cdot (\mathbf{D_1} - \mathbf{D_2})$$

$$\mathfrak{B} = \mu\mathfrak{H} \quad \text{(L)} \qquad \mathbf{B} = \mu\mathbf{H}$$

FIG. 1: Maxwell's original equations.

Figure 1 displays Maxwell's original set of equations [21, vol. 2, pp. 215-=233] which were almost completely in coordinate-wise "longhand" notation, together with their interpretation in modern Gibbsean vector calculus notation (cp. also [16, p. 3]).

2. Vector analysis, integration theorems and Poincaré's Lemma

One of the cornerstones of vector analysis is the fact that the sequence

$$0 \to \mathbb{R} \to C^\infty(\Omega,\mathbb{R}) \stackrel{\text{grad}}{\to} C^\infty(\Omega,\mathbb{R}^3) \stackrel{\text{curl}}{\to} C^\infty(\Omega,\mathbb{R}^3) \stackrel{\text{div}}{\to} C^\infty(\Omega,\mathbb{R}) \to 0$$

is exact. We work with the following more detailed description of this statement.

Theorem 2.1 (Poincaré's Lemma)**.** *Let $\Omega \subset \mathbb{R}^3$ be open and star shaped with star center $x_0 \in \Omega$.*

(a) *For each $\mathbf{a} \in C^1(\Omega,\mathbb{R}^3)$*

$$\operatorname{curl}\mathbf{a} = 0 \qquad \Leftrightarrow \qquad \exists f \in C^1(\Omega,\mathbb{R})\colon \mathbf{a} = \operatorname{grad} f\,,$$

e.g. $f(\mathbf{x}) := \int_0^1 \langle \mathbf{a}(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)), \mathbf{x}-\mathbf{x}_0\rangle\, dt$ for $\mathbf{x} \in \Omega$.

(b) *For each $\mathbf{b} \in C^1(\Omega,\mathbb{R}^3)$*

$$\operatorname{div}\mathbf{b} = 0 \qquad \Leftrightarrow \qquad \exists \mathbf{a} \in C^1(\Omega,\mathbb{R}^3)\colon \mathbf{b} = \operatorname{curl}\mathbf{a}\,,$$

e.g. $\mathbf{a}(\mathbf{x}) := \int_0^1 t\mathbf{b}(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))dt \times (\mathbf{x}-\mathbf{x}_0)$ for $\mathbf{x} \in \Omega$.

(c) *For each $g \in C^1(\Omega,\mathbb{R})$*

$$\exists \mathbf{b} \in C^1(\Omega,\mathbb{R}^3)\colon g = \operatorname{div}\mathbf{b}\,,$$

e.g. $\mathbf{b}(\mathbf{x}) := \int_0^1 t^2 g(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))(\mathbf{x}-\mathbf{x}_0)dt$ for $\mathbf{x} \in \Omega$.

Proof. Theorem 2.1 is a special case of Poincaré's Lemma for differential forms (see e.g. [15, Theorems 11.49 & 17.14 and Lemma 17.27]). We prove only (a). The proofs of (b) and (c) have the same structure due to the fact that the exterior derivative of a differential form specializes to div, curl and grad, respectively.

(a) The implication ($\Leftarrow$) follows easily by computing $\operatorname{curl}\operatorname{grad} f = 0$. To show the implication ($\Rightarrow$) define $\mathbf{w}(\mathbf{x}) := \mathbf{x} - \mathbf{x}_0$, $(w_1,w_2,w_3) := \mathbf{w}$, $(a_1,a_2,a_3) := \mathbf{a}$ and $f(\mathbf{x}) := \int_0^1 \sum_{k=1}^3 a_k(\mathbf{x}_0 + t\mathbf{w}(\mathbf{x}))w_k(\mathbf{x})dt$ for $\mathbf{x} \in \Omega$. Let $j \in \{1,2,3\}$. We use

$\partial_j a_k = \partial_k a_j$ for $k \in \{1, 2, 3\}$ to compute

$$\begin{aligned}
\partial_j f(\cdot) &= \int_0^1 \sum_{k=1}^{3} \partial_j \big(a_k(\mathbf{x}_0 + t\mathbf{w}(\cdot)) w_k(\cdot) \big)\, dt \\
&= \int_0^1 \Big(\sum_{k=1}^{3} \big\langle \operatorname{grad} a_k(\mathbf{x}_0 + t\mathbf{w}(\cdot)), t\mathbf{e}_j \big\rangle w_k(\cdot) + a_j(\mathbf{x}_0 + t\mathbf{w}(\cdot)) \Big)\, dt \\
&= \int_0^1 t \sum_{k=1}^{3} \partial_j a_k(\mathbf{x}_0 + t\mathbf{w}(\cdot)) w_k(\cdot)\, dt + \int_0^1 a_j(\mathbf{x}_0 + t\mathbf{w}(\cdot))\, dt \\
&= \int_0^1 t \sum_{k=1}^{3} \partial_k a_j(\mathbf{x}_0 + t\mathbf{w}(\cdot)) w_k(\cdot)\, dt + \int_0^1 a_j(\mathbf{x}_0 + t\mathbf{w}(\cdot))\, dt \\
&= \int_0^1 t \big\langle \operatorname{grad} a_j(\mathbf{x}_0 + t\mathbf{w}(\cdot)), \mathbf{w}(\cdot) \big\rangle\, dt + \int_0^1 a_j(\mathbf{x}_0 + t\mathbf{w}(\cdot))\, dt\,.
\end{aligned}$$

However, for $\mathbf{x} \in \Omega$

$$t \mapsto \big\langle \operatorname{grad} a_j(\mathbf{x}_0 + t\mathbf{w}(\mathbf{x})), \mathbf{w}(\mathbf{x}) \big\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \big(t \mapsto a_j(\mathbf{x}_0 + t\mathbf{w}(\mathbf{x})) \big)\,.$$

We conclude by partial integration that for $\mathbf{x} \in \Omega$

$$\begin{aligned}
\partial_j f(\mathbf{x}) = t a_j(\mathbf{x}_0 + t\mathbf{w}(\mathbf{x})) \Big|_{t=0}^{1} &- \int_0^1 a_j(\mathbf{x}_0 + t\mathbf{w}(\mathbf{x}))\, dt \\
&+ \int_0^1 a_j(\mathbf{x}_0 + t\mathbf{w}(\mathbf{x})) dt = a_j(\mathbf{x})\,. \qquad \square
\end{aligned}$$

Historical Remark 2.2 (Early Days of Electrodynamics)**.** The early days of *electrodynamics* - a term coined by Ampère [6] - read like an exciting detective novel and are excellently presented e.g. in [4, 12, 24] and the references therein. The following timeline is adapted from [27].

1785	Coulomb's Law is published
1813	Gauss's Divergence Theorem is discovered
1820	Ørsted discovers that an electric current creates a magnetic field
1820	Ampère's work founds electrodynamics; Biot-Savart Law is discovered
1826	Ampère's Formula is published
1831	Faraday's Law is published
1856	Maxwell publishes "On Faraday's lines of force"
1861	Maxwell publishes "On physical lines of force"
1865	Maxwell publishes "A dynamical theory of the electromagnetic field"
1873	Maxwell publishes "A Treatise on Electricity and Magnetism"
1888	Hertz discovers radio waves
1940	Einstein popularizes the name "Maxwell's Equations"

In 1785 Coulomb published his law, which states that the force between two electrical charges is proportional to the product of the charges and inversely proportional to the square of the distance between them (see e.g. [31] for a documentary).

In 1813 Gauss discovered a special case of the divergence theorem. Todays version, that the surface integral of a vector field over a closed surface is equal to the volume integral of the divergence over the region inside the surface, was formulated and proved by Ostrogradsky (not by Gauss) 13 years later. Maxwell credits the divergence theorem to Ostrogradsky. He cited a paper by Ostrogradsky from the correct year but with a wrong title. This citation was later removed but can still be found in the first edition of Maxwell's Treatise on Electricity and Magnetism.

In 1820 Ørsted discovered that a compass needle was deflected from the magnetic north direction by a nearby electric current. He thus confirmed a direct relationship between electricity and magnetism. Already on September 30, 1820, Biot and Savart announced their results for the distance dependence of the magnetic force exerted by a long, straight current-carrying wire on a magnetic needle.

In 1826 Ampère's formula on the force between two infinitesimal current elements was published. Maxwell respected Ampère's work and his genius but he questioned the action-at-a-distance concept which was commonly accepted then as the state-of-the-art also by Ampère e.g. in his formula. When Maxwell later developed his field theory of electromagnetism Ampère's formula fell into disuse. Sometime around the 1930s Ampère's name came to be associated with "the first law of circulation", then also called Ampère's Law which, however, is not due to Ampère. One should therefore speak of Ampère's formula which was published in 1826.

Faraday was one of the most influential scientists in history and definitely during his time. He was an excellent experimentalist who conveyed his ideas in clear and simple language (see e.g. [32] for a documentary). He published from 1831 until 1854 in the "Philosophical Transactions of the Royal Society" a series of articles with the title "Experimental Researches in Electricity". He chose this also as the title of a three volume book in which he summarised his research results on electricity in 30 series of articles consisting of 3430 numbered paragraphs. Maxwell acknowledged and accepted the phenomena discovered by Ørsted, Ampère, and others but, principally, he depended on the conceptual framework and experimental achievements of Faraday [12, p. 4] (see [7] for a reprint of Faraday's Experimental Researches in Electricity). In the preface to his Treatise on Electricity and Magnetism of 1873 [16, p. viii], Maxwell recalled:

> [B]efore I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's Experimental Researches in Electricity.

Maxwell's initial publication [17] of 1856 (an abstract) on electromagnetism placed his methodology at the forefront [12, Sec. 1.3, p. 9]. For details we recommend the excellent discussion of the importance of methodology for Maxwell in Hon and Goldstein [12, Secs. 1.3 & 1.4] from which we quote freely. At the beginning of the abstract [17] Maxwell indicated that the methodology he adopted was a modified version of the formal analogy invoked by Thomson [26, 'On the uniform motion of heat in homogeneous solid bodies, and its connexion with the mathematical theory of electricity', pp. 1–14]:

> The method pursued in this paper is a modification of that mode of viewing electrical phenomena in relation to the theory of the uniform conduction of heat, which was first pointed out by Professor W. Thomson, [...] Instead of using the analogy of heat, a fluid, the properties of which are entirely at our disposal [i.e., purely imaginary], is assumed as the vehicle of mathematical reasoning.

Maxwell actually stated that he would modify Thomson's methodology of analogy that relates two distinct physical domains via the same mathematical structure. He used Faraday's concept of lines of force as processes of reasoning and introduced an imaginary imponderable and incompressible fluid that permeates a medium whose resistance is directly proportional to the velocity of the fluid. $\diamond$

We now recall the classical integration theorems, called gradient theorem, curl theorem (theorem of Stokes or Kelvin-Stokes theorem) and divergence theorem (theorem of Gauß or Ostrogradsky-Gauß theorem), and fix the notation for the integration of a function $f\colon M \to \mathbb{R}$ on a k-dimensional submanifold $M \subset \mathbb{R}^n$. For simplicity assume that $\Phi\colon U \to V$, with $U \subset \mathbb{R}^k$ open, is a chart of M such that $\operatorname{supp} f \subset V$. Then with the Gramian $g(\mathbf{u}) := \det(D\Phi(\mathbf{u})^\top D\Phi(\mathbf{u}))$ the integral is defined by

$$\int_M f(\mathbf{x})\, dS(\mathbf{x}) := \int_U f(\Phi(\mathbf{u}))\sqrt{g(\mathbf{u})}\, d\mathbf{u}\,.$$

We follow common physics notation, omit the argument $\mathbf{x}$, i.e. we write $\int_M f\, dS$, and for $n = 3$ notationally distinguish between the cases $k = 1, 2, 3$ by writing ds, dS and dV instead of dS, respectively. Moreover, to further bridge the notational gap, we typeset vectorial line and surface elements in bold face as displayed and defined in the following theorem.

Theorem 2.3 (Classical Integration Theorems)**.** *Let $\Omega \subset \mathbb{R}^3$ be open.*

Gradient Theorem. *Let $\gamma\colon [0,1] \to \Omega$ be piecewise continuously differentiable and $f \in C^1(\Omega, \mathbb{R})$. Then*

$$\int_0^1 \langle \operatorname{grad} f(\gamma(t)), \gamma'(t)\rangle\, dt = f(\gamma(1)) - f(\gamma(0))\,,$$

and with $\mathbf{p} = \gamma(0)$, $\mathbf{q} = \gamma(1)$, even shorter and more common in physics literature

$$\int_\gamma \operatorname{grad} f \cdot d\mathbf{s} = f(\mathbf{q}) - f(\mathbf{p})\,.$$

Curl Theorem. *Let $M \subset \Omega$ be a bounded piecewise smooth oriented two-dimensional manifold with unit normal field $\nu\colon M \to \mathbb{R}^3$ and piecewise smooth boundary curve ∂M with induced unit tangent field $\tau\colon \partial M \to \mathbb{R}^3$, and $\mathbf{a} \in C^1(\Omega, \mathbb{R}^3)$. Then*

$$\int_M \langle \operatorname{curl} \mathbf{a}, \nu\rangle\, dS = \int_{\partial M} \langle \mathbf{a}, \tau\rangle\, ds\,,$$

and even shorter and more common in physics literature

$$\int_M \operatorname{curl}\mathbf{a}\cdot d\mathbf{S} = \int_{\partial M} \mathbf{a}\cdot d\mathbf{s}\,.$$

Divergence Theorem. *Let $M \subset \Omega$ be bounded open with piecewise smooth boundary ∂M, $\nu\colon \partial M \to \mathbb{R}^3$ the outer unit normal field, and $\mathbf{b} \in C^1(\Omega,\mathbb{R}^3)$. Then*

$$\int_M \operatorname{div}\mathbf{b}\, dV = \int_{\partial M} \langle \mathbf{b},\nu\rangle\, dS\,,$$

and even shorter and more common in physics literature

$$\int_M \operatorname{div}\mathbf{b}\, dV = \int_{\partial M} \mathbf{b}\cdot d\mathbf{S}\,.$$

Proof. The proofs of these classical integration theorems can be found in many textbooks. We refer to [25, Chapter 8, p. 261, Stokes' Theorem] for a unified proof of all three statements using differential forms. □

Whereas the classical integration theorems 2.3 allow for an integral interpretation of the differential operators div, curl and grad, the following theorem additionally also provides the converse implication.

Theorem 2.4 (Geometric interpretation of gradient, curl and divergence)**.** *Let $\Omega \subset \mathbb{R}^3$ be non-empty and open.*

(a) *Let $\Gamma := \{\gamma\colon [0,1] \to \Omega \mid \gamma$ is piecewise continuously differentiable$\}$. Then for each $f \in C^1(\Omega,\mathbb{R})$ and $\mathbf{a} \in C(\Omega,\mathbb{R}^3)$ the following statements are equivalent:*
  (i) $\forall \gamma \in \Gamma\colon \int_\gamma \mathbf{a}\cdot d\mathbf{s} = f(\gamma(1)) - f(\gamma(0))$.
  (ii) $\mathbf{a} = \operatorname{grad} f$.

(b) *Let $\mathcal{M}$ be the set of bounded oriented piecewise smooth two-dimensional submanifolds M of Ω with piecewise smooth boundary ∂M. Then for each $\mathbf{a} \in C^1(\Omega,\mathbb{R}^3)$ and $\mathbf{b} \in C(\Omega,\mathbb{R}^3)$ the following statements are equivalent:*
  (i) $\forall M \in \mathcal{M}\colon \int_M \mathbf{b}\cdot d\mathbf{S} = \int_{\partial M} \mathbf{a}\cdot d\mathbf{s}$.
  (ii) $\mathbf{b} = \operatorname{curl}\mathbf{a}$.

(c) *Let $\mathcal{M}$ be the set of bounded open subsets M of Ω with piecewise smooth boundary. Then for each $\mathbf{b} \in C^1(\Omega,\mathbb{R}^3)$ and $g \in C(\Omega,\mathbb{R})$ the following statements are equivalent:*
  (i) $\forall M \in \mathcal{M}\colon \int_M g\, dV = \int_{\partial M} \mathbf{b}\cdot d\mathbf{S}$.
  (ii) $g = \operatorname{div}\mathbf{b}$.

For the proof of Theorem 2.4 we use the following lemma.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^3$ be non-empty and open. Let $\mathbf{v} \in C(\Omega,\mathbb{R}^3)$ and $g \in C(\Omega,\mathbb{R})$. The following statements hold:*

(a) *If for all paths $\gamma\colon [0,1] \to \Omega$, $t \mapsto (1-t)\mathbf{x}_0 + t\mathbf{x}_1$ where $\mathbf{x}_0,\mathbf{x}_1 \in \Omega$, $\int_\gamma \mathbf{v}\cdot d\mathbf{s} = 0$, then $\mathbf{v} = 0$.*

(b) *If for all discs $D(\mathbf{x},\mathbf{n},r) \subset \Omega$ with center $\mathbf{x} \in \Omega$, radius $r > 0$, unit normal $\mathbf{n} \in \mathbb{R}^3$ and the orientation induced by $\mathbf{n}$, $\int_{D(\mathbf{x},\mathbf{n},r)} \mathbf{v}\cdot d\mathbf{S} = 0$, then $\mathbf{v} = 0$.*

(c) *If for all balls* $B(\mathbf{x}, r) \subset \Omega$ *with center* $\mathbf{x} \in \Omega$ *and radius* $r > 0$*,* $\int_{B(\mathbf{x},r)} g \, dV = 0$*, then* $g = 0$*.*

Proof. We only prove (b). Suppose to the contrary, that $\mathbf{v} \neq 0$. Then there is $\mathbf{x} \in \Omega$ with $\mathbf{v}(\mathbf{x}) \neq 0$. We define $\mathbf{n} := \mathbf{v}(\mathbf{x})/\|\mathbf{v}(\mathbf{x})\|$. Then the function $g(\mathbf{y}) := \langle \mathbf{v}(\mathbf{y}), \mathbf{n} \rangle$ is continuous in Ω and satisfies $g(\mathbf{x}) = \|\mathbf{v}(\mathbf{x})\| > 0$. Hence there is $r > 0$, s.t. $B(\mathbf{x}, r) \subseteq \Omega$ and for all $\mathbf{y} \in B(\mathbf{x}, r)$ it holds that $g(\mathbf{y}) > 0$. Then the disk $D(\mathbf{x}, \mathbf{n}, r) := \{\mathbf{y} \in B(\mathbf{x}, r) \mid \langle \mathbf{y} - \mathbf{x}, \mathbf{n} \rangle = 0\}$ has unit normal field $D(\mathbf{x}, \mathbf{n}, r) \ni \mathbf{y} \mapsto \mathbf{n}$ and can be parameterized almost globally by a parameterization $\gamma : \operatorname{dom} \gamma \subset \mathbb{R}^2 \to D(\mathbf{x}, \mathbf{n}, r)$, where $\operatorname{dom} \gamma$ has non-zero measure. We compute

$$0 = \int_{D(\mathbf{x},\mathbf{n},r)} \mathbf{v} \cdot d\mathbf{S} = \int_{\operatorname{dom} \gamma} \langle \mathbf{v}(\gamma(s,t)), \mathbf{n} \rangle \, d(s,t) = \int_{\operatorname{dom} \gamma} g(\gamma(s,t)) \, d(s,t) > 0,$$

which is a contradiction. □

Proof of Theorem 2.4. The proposition (a)(ii) ⇒ (i) follows by an application of the Fundamental Theorem of Calculus and (b)(ii) ⇒ (i) respectively (c)(ii) ⇒ (i) follows from the Curl respectively the Divergence Theorem. The converse implications (a)(i) ⇒ (ii), (b)(i) ⇒ (ii) and (c)(i) ⇒ (ii) can be deduced again from the Fundamental Theorem of Calculus, the Curl and the Divergence Theorem and Lemma 2.5. We provide the proof only for (b)(i) ⇒ (ii). By an application of the Curl Theorem, we compute for $M \in \mathcal{M}$,

$$\int_M \big(\mathbf{b} - \operatorname{curl} \mathbf{a}\big) \cdot d\mathbf{S} = 0 \, .$$

Since all discs $D(\mathbf{x}, \mathbf{n}, r) \subseteq \Omega$, with center $\mathbf{x} \in \mathbb{R}^3$, unit normal $\mathbf{n} \in \mathbb{R}^3$ and radius $r > 0$ are elements of $\mathcal{M}$, we deduce from Lemma 2.5 that $\mathbf{b} - \operatorname{curl} \mathbf{a} = 0$. □

Combining Poincaré's Lemma 2.1 with the geometric interpretation of gradient, curl and divergence in Theorem 2.4, we obtain our main tool for the construction of Maxwell's equations.

Lemma 2.6 (Geometric interpretation of Poincaré's Lemma)**.** *Let* $\Omega \subset \mathbb{R}^3$ *be open and star shaped.*

(a) *For each* $\mathbf{a} \in C^1(\Omega, \mathbb{R}^3)$

$$\operatorname{curl} \mathbf{a} = 0 \quad \Leftrightarrow \quad \exists f \in C^1(\Omega, \mathbb{R}) \colon \text{Theorem 2.4(a)(i) and (a)(ii) hold.}$$

(b) *For each* $\mathbf{b} \in C^1(\Omega, \mathbb{R}^3)$

$$\operatorname{div} \mathbf{b} = 0 \quad \Leftrightarrow \quad \exists \mathbf{a} \in C^1(\Omega, \mathbb{R}^3) \colon \text{Theorem 2.4(b)(i) and (b)(ii) hold.}$$

(c) *For each* $g \in C^1(\Omega, \mathbb{R})$

$$\exists \mathbf{b} \in C^1(\Omega, \mathbb{R}^3) \colon \text{Theorem 2.4(c)(i) and (c)(ii) hold.}$$

3. Maxwell's imaginary fluid

In this section we model Maxwell's imaginary fluid based on his original work and applications of the geometric interpretation of Poincaré's Lemma 2.6. All occurring quantities are assumed to be smooth enough for their integrals and derivatives to exist as noted.

Historical Remark 3.1 (Maxwell's Imaginary Fluid and Tube of Fluid Motion)**.** In [18] Maxwell introduces a hypothetical fluid as a purely imaginary substance with a collection of imaginary properties. We quote from [22, pp. 160–162].

> (1) The substance here treated of must not be assumed to possess any of the properties of ordinary fluids except those of freedom of motion and resistance to compression. [...]
>
> *The portion of fluid which at any instant occupied a given volume, will at any succeeding instant occupy an equal volume.*
>
> This law expresses the incompressibility of the fluid, and furnishes us with a convenient measure of its quantity, namely its volume. The unit of quantity of the fluid will therefore be the unit of volume.
>
> (2) The direction of motion of the fluid will in general be different at different points of the space which it occupies, but since the direction is determinate for every such point, we may conceive a line to begin at any point and to be continued so that every element of the line indicates by its direction the direction of motion at that point of space. Lines drawn in such a manner that their direction always indicates the direction of fluid motion are called lines of fluid motion. [...]

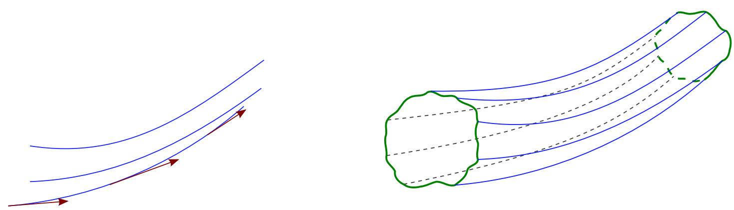

Fig. 2: Lines of fluid motion and tube of fluid motion.

> (3) If upon any surface which cuts the lines of fluid motion we draw a closed curve, and if from every point of this curve we draw a line of motion, these lines of motion will generate a tubular surface which we may call a tube of fluid motion. Since this surface is generated by lines in the direction of fluid motion no part of the fluid can flow across it, so that this imaginary surface is as impermeable to the fluid as a real tube.
>
> (7) [...] if the origin of the tube or its termination be within the space under consideration, then we must conceive the fluid to be supplied by a source within that space, capable of creating and emitting unity of fluid in unity of time, and to be afterwards swallowed up by a sink capable of receiving and destroying the same amount continually. [...] ◇

The imaginary fluid which Maxwell introduces in property (1) of the Historical Remark 3.1 occupies volume in a domain $\Omega \subset \mathbb{R}^3$ and has, in particular, a velocity depending on time t and described by a velocity field

$$\mathbf{v} \colon \mathbb{R} \times \Omega \to \mathbb{R}^3 , \quad (t, \mathbf{x}) \mapsto \mathbf{v}(t, \mathbf{x}) .$$

Let S be an oriented surface, more precisely, a bounded piecewise smooth oriented two-dimensional manifold in Ω. Then

$$\int_S \mathbf{v}(t, \cdot) \cdot d\mathbf{S}$$

describes the flow at time t through the surface S, also called the *flux* through S. With this interpretation we rename $\mathbf{v}$ and call it from now on (*fluid*) *flux density*, see Figure 3.

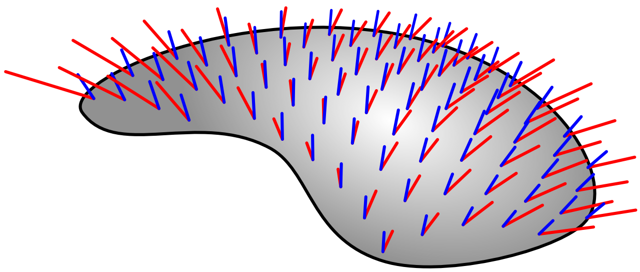

FIG. 3: Flux density $\mathbf{v}(t, \cdot)$ and unit normal field ν on surface S.

Maxwell's tube of fluid motion $T \subset \Omega$ as introduced in property (3) of the Historical Remark 3.1 has a bounding surface $S = \partial T$ consisting of a lateral area and two base areas. The flux through the lateral area is zero, since it consists of lines of fluid motion and hence the total flux through the surface S of T is the sum of the fluxes through the two base areas. Maxwell introduces in property (7) of the Historical Remark 3.1 sources and sinks, creating and swallowing up the imaginary fluid, respectively, on the base areas of the tube of fluid motion. Maxwell's imaginary fluid has therefore, in particular, a *source density*

$$\varrho : \mathbb{R} \times \Omega \to \mathbb{R} , \quad (t, \mathbf{x}) \mapsto \varrho(t, \mathbf{x}) ,$$

which specifies by its integral over the tube of fluid motion

$$\int_T \varrho(t, \cdot) \, dV$$

the sources and sinks in T which create or swallow up fluid at time t. Maxwell's imaginary fluid is incompressible and at any time t the fluid produced by the sources and sinks in T has to equal the flux of the fluid through S, i.e. the balance law

$$\int_T \varrho(t, \cdot) \, dV = \int_S \mathbf{v}(t, \cdot) \cdot d\mathbf{S} \tag{3.1}$$

holds. Maxwell uses in [18] the mental imagery of a tube of fluid motion to describe and develop properties of the imaginary fluid. This and various other aspects of the development of electrodynamic theory are discussed in detail in many good papers and books, see e.g. [24] and the references therein. We leave now Maxwell's original considerations from Remark 3.1 but continue in this section to describe his imaginary fluid as an analogy for the development of Maxwell's equations in the next section.

We start with a further simplification of notation by omitting the t-dependence of vector fields, i.e. we write e.g. $\int \mathbf{v} \cdot d\mathbf{S}$ instead of $\int \mathbf{v}(t, \cdot) \cdot d\mathbf{S}$. It is understood that integrands and therefore integral expressions with abbreviated notation do depend on time although t is not explicitly written. Moreover, if an operator div, curl or grad is applied to a function which is defined on $\mathbb{R} \times \Omega$, then the partial derivatives involved are only with respect to the space variables in Ω, e.g. $\operatorname{div} \mathbf{v}$ is short for $(t, \mathbf{x}) \mapsto \operatorname{div} \mathbf{v}(t, \cdot)|_{\mathbf{x}}$.

The considerations which lead to the balance law (3.1) of Maxwell's imaginary incompressible fluid suggest that it should not only hold for integration domains which are tubes of fluid motion but for each bounded open subset $M \subset \Omega$ with piecewise smooth boundary ∂M

$$\int_M \varrho \, dV = \int_{\partial M} \mathbf{v} \cdot d\mathbf{S} \tag{3.2}$$

ensuring that the fluid produced in M equals the flux through ∂M. By the Divergence Theorem 2.3(c)

$$\int_M \operatorname{div} \mathbf{v} \, dV = \int_{\partial M} \mathbf{v} \cdot d\mathbf{S}$$

and hence the integral $\int_M (\varrho - \operatorname{div} \mathbf{v}) \, dV$ vanishes for each bounded open subset $M \subset \Omega$ with piecewise smooth boundary. As a consequence of Lemma 2.5(c)

$$\operatorname{div} \mathbf{v} = \varrho.$$

So far we have established for Maxwell's imaginary fluid (which has the unit m^3 of volume) the properties *(fluid) flux* described by

$$\boxed{\begin{aligned} \mathbf{v} : \mathbb{R} \times \Omega &\to \mathbb{R}^3, \\ (t, x) &\mapsto \mathbf{v}(t, x) \end{aligned} \qquad (\textit{flux density} \text{ with unit } [\mathbf{v}] = \tfrac{\mathrm{m}}{\mathrm{s}})}$$

and *sinks and sources* described by

$$\boxed{\begin{aligned} \varrho : \mathbb{R} \times \Omega &\to \mathbb{R}\,, \\ (t, x) &\mapsto \varrho(t, x) \end{aligned} \qquad (\textit{source density} \text{ with unit } [\varrho] = \tfrac{\mathrm{m}^3}{\mathrm{m}^3 \mathrm{s}} = \tfrac{1}{\mathrm{s}})}$$

which are linked by the balance law (3.2), also called *Continuity Equation for the Fluid Flux*, that equates for each bounded open $M \subset \Omega$ with piecewise smooth boundary the sources in M with the flux through ∂M

Continuity Equation for the Fluid Flux (integral form)

$$\int_M \varrho \, dV = \int_{\partial M} \mathbf{v} \cdot d\mathbf{S} \qquad \text{(with unit } \tfrac{\mathrm{m}^3}{s}\text{)}$$

(*sources in* M = *flux through* ∂M),

or equivalently in differential form

Continuity Equation for the Fluid Flux (differential form)

$$\varrho = \operatorname{div} \mathbf{v} \qquad \text{(with unit } \tfrac{1}{s}\text{)}$$

(*source density* = *divergence of flux density*).

The source density ϱ and hence also the sources $\int_M \varrho \, dV$ in a bounded open $M \subset \Omega$ with piecewise smooth boundary depend on time t. The time derivative $\partial_t \int_M \varrho \, dV = \int_M \partial_t \varrho \, dV$ of the sources in M describes the production rate of sources in M. The geometric interpretation of Poincaré's Lemma 2.6(c), applied to $-\partial_t \varrho$, yields a vector field $\mathbf{j} : \mathbb{R} \times \Omega \to \mathbb{R}^3$ with $\int_M \partial_t \varrho \, dV = -\int_{\partial M} \mathbf{j} \cdot d\mathbf{S}$. A positive sign of $\int_M \partial_t \varrho \, dV$ means that over time there are more sources in M, consequently $\int_{\partial M} \mathbf{j} \cdot d\mathbf{S}$ describes a current of sources from inside of M across the boundary ∂M. We therefore have established a *current of sources* described by

$$\begin{aligned} \mathbf{j} : \mathbb{R} \times \Omega &\to \mathbb{R}^3, \\ (t, x) &\mapsto \mathbf{j}(t, x) \end{aligned} \qquad (\textit{source current density} \text{ with unit } [\mathbf{j}] = \tfrac{\mathrm{m}}{\mathrm{s}^2})$$

which is linked to the *source production rate density* $\partial_t \varrho$ by a balance law that we call *Continuity Equation for the Source Current* and that equates for each bounded open $M \subset \Omega$ with piecewise smooth boundary the source production rate in M with the source current through ∂M

Continuity Equation for the Source Current (integral form)

$$\int_M \partial_t \varrho \, dV = -\int_{\partial M} \mathbf{j} \cdot d\mathbf{S} \qquad \text{(with unit } \tfrac{\mathrm{m}^3}{s^2}\text{)}$$

(*source production rate in* M = $-$*source current through* ∂M),

or equivalently in differential form

Continuity Equation for the Source Current (differential form)

$$\partial_t \varrho = -\operatorname{div} \mathbf{j} \qquad \text{(with unit } \tfrac{1}{s^2}\text{)}$$

(*source production rate density* = $-$*divergence of source current density*).

Note that at this stage of our discussion of the properties of Maxwell's imaginary fluid only the divergence of the source current density $\mathbf{j}$ is uniquely determined and $\mathbf{j}$ plus an arbitrary divergence-free vector field would also be a possible source current density of the imaginary fluid.

We aim now for an application of the geometric interpretation of Poincaré's Lemma 2.6(b) to $\partial_t \mathbf{v} + \mathbf{j}$, which is possible because it is divergence-free

$$\operatorname{div}(\partial_t \mathbf{v} + \mathbf{j}) = \partial_t \operatorname{div} \mathbf{v} + \operatorname{div} \mathbf{j} = \partial_t \varrho - \partial_t \varrho = 0\,.$$

The geometric interpretation of Poincaré's Lemma 2.6(b) yields therefore a vector field $\mathbf{h}\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$ with $\partial_t \mathbf{v}+\mathbf{j} = \operatorname{curl} \mathbf{h}$ and the equivalent geometric interpretation that for each bounded piecewise smooth oriented two-dimensional submanifold M of Ω with piecewise smooth boundary ∂M the balance law $\int_M (\partial_t \mathbf{v}+\mathbf{j})\cdot d\mathbf{S} = \int_{\partial M} \mathbf{h}\cdot d\mathbf{s}$ holds. It is insightful to explore this balance law for the two special cases (i) $\partial_t \mathbf{v} = 0$ and (ii) $\mathbf{j} = 0$, and M being a surface bounded by a closed curve (visualize M as a disk bounded by a circle ∂M). In case (i) of a stationary fluid flux density $\mathbf{v}$ the source current $\int_M \mathbf{j} \cdot d\mathbf{S}$ through M equals the line integral $\int_{\partial M} \mathbf{h} \cdot d\mathbf{s}$ of the vector field $\mathbf{h}$ along the closed boundary curve ∂M. In case (ii) of a vanishing source current it is the time change $\partial_t \int_M \mathbf{v} \cdot d\mathbf{S}$ of the flux of the imaginary fluid through M which equals $\int_{\partial M} \mathbf{h} \cdot d\mathbf{s}$. From that perspective, the sum of the source current and the flux change rate of the imaginary fluid through the surface M are described by the path integral $\int_{\partial M} \mathbf{h} \cdot d\mathbf{s}$ of $\mathbf{h}$ along the boundary curve ∂M. We call $\mathbf{h}$ the *fluid field* and $\int_{\partial M} \mathbf{h} \cdot d\mathbf{s}$ the *circulation of the fluid field* along ∂M, see Figure 4.

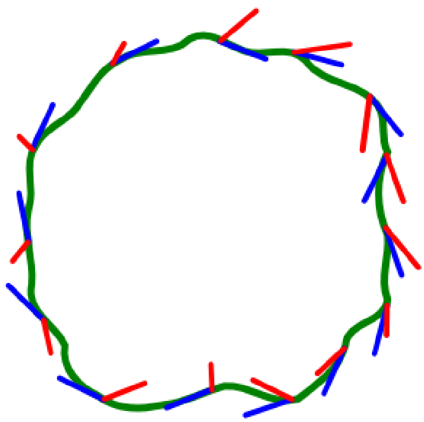

FIG. 4: Fluid field $\mathbf{h}(t,\cdot)$ and unit tangent field on curve ∂M.

We have now established

$$\begin{aligned} \mathbf{h} : \mathbb{R} \times \Omega &\to \mathbb{R}^3\,, \\ (t,x) &\mapsto \mathbf{h}(t,x) \end{aligned} \qquad (\textit{fluid field} \text{ with unit } [\mathbf{h}] = \tfrac{\mathrm{m}^2}{\mathrm{s}^2})$$

and the change rate $\partial_t \int_M \mathbf{v} \cdot d\mathbf{S}$ of the flux of the imaginary fluid through a surface M as an additive contribution to the source current $\int_M \mathbf{j} \cdot d\mathbf{S}$ through M. We call $\int_M \partial_t \mathbf{v} \cdot d\mathbf{S}$ the *flux change current* and also for later reference when the imaginary fluid analogy is applied to electricity and magnetism *displacement current*. It is linked to the fluid field $\mathbf{h}$ by a balance law, that we call *Circulation Law for the Fluid Field*, or short *Circulation Law*, and that equates for each bounded piecewise smooth oriented two-dimensional submanifold M of Ω with piecewise smooth boundary ∂M the sum of the flux change current and the source current through M with the circulation of the fluid field along ∂M

Circulation Law (integral form)

$$\int_M (\partial_t \mathbf{v} + \mathbf{j}) \cdot d\mathbf{S} = \int_{\partial M} \mathbf{h} \cdot d\mathbf{s} \qquad \text{(with unit } \tfrac{\mathrm{m}^3}{\mathrm{s}^2}\text{)}$$

(*flux change current + source current through M = circulation of fluid field along ∂M*),

or equivalently in differential form

Circulation Law (differential form)

$$\partial_t \mathbf{v} + \mathbf{j} = \operatorname{curl} \mathbf{h} \qquad \text{(with unit } \tfrac{\mathrm{m}}{\mathrm{s}^2}\text{)}$$

(*flux change current + source current densities = curl of fluid field*).

Note that only the curl of the fluid field $\mathbf{h}$ is uniquely determined and $\mathbf{h}$ plus an arbitrary vector field with vanishing curl would also be a possible fluid field of the imaginary fluid.

Remark 3.2 (Interpretation of Circulation Law for Tube of Fluid Motion)**.** For a bounded piecewise smooth oriented two-dimensional submanifold M of Ω with piecewise smooth boundary ∂M both terms $\int_M \partial_t \mathbf{v} \cdot d\mathbf{S}$ and $\int_M \mathbf{j} \cdot d\mathbf{S}$ with positive sign contribute to an increase of fluid "on the other side of M", an increase on the side to which the unit normal field of M is pointing. To state this more precisely, consider two cases (i) M is the closed surface of a tube T of fluid motion with outer unit normal field and empty boundary ∂M, and case (ii) where M is a base area of a tube T of fluid motion with induced orientation, cp. also Figure 2. In case (i), using the fact that an integral over the empty set ∂M equals zero, the Circulation Law becomes $\partial_t \int_M \mathbf{v} \cdot d\mathbf{S} = -\int_M \mathbf{j} \cdot d\mathbf{S}$ with the interpretation that an instantaneous increase of the flux out of T is balanced by a current of sources into T. In case (ii) a positive source current $\int_M \mathbf{j} \cdot d\mathbf{S}$ through M out of T contributes to a decrease of fluid production in T by a decrease of sources in T. A positive term $\int_M \partial_t \mathbf{v} \cdot d\mathbf{S} = \partial_t \int_M \mathbf{v} \cdot d\mathbf{S}$ can be interpreted as an increase of the flux from inside of T through M out of T, because M has the induced orientation of T with an outer unit normal field. The circulation $\int_{\partial M} \mathbf{h} \cdot d\mathbf{s}$ of the fluid field $\mathbf{h}$ along ∂M equals the sum of both terms. ◇

4. Electricity and magnetism

Historical Remark 4.1 (Maxwell's Imaginary Fluid Analogy applied to Electricity and Magnetism)**.** In [18] Maxwell prepared the application of the imaginary fluid analogy to electricity and magnetism. We quote from [22, pp. 175, 178].

> *Application of the Idea of Lines of Force.*
>
> I have now to shew how the idea of lines of fluid motion as described above may be modified so as to be applicable to the sciences of statical electricity, permanent magnetism, magnetism of induction, and uniform galvanic currents, reserving the laws of electro-magnetism for special consideration. [...]
>
> Now we found in (18) that the velocity of our imaginary fluid due to a source S at a distance r varies inversely as r^2. Let us see what will be the effect of substituting such a source for every particle of positive electricity. [...]
>
> *Theory of Permanent Magnets.*
>
> A magnet is conceived to be made up of elementary magnetized particles, each of which has its own north and south poles, the action of which upon other north and south poles is governed by laws mathematically identical with those of electricity. Hence the same application of the idea of lines of force can be made to this subject, and the same analogy of fluid motion can be employed to illustrate it.

Imaginary Fluid	Electric and Magnetic Concepts	
fluid source density	*electric charge density*	*magnetic charge density*
$\varrho\colon \mathbb{R}\times\Omega\to\mathbb{R},$ $(t,x)\mapsto\varrho(t,x)$ with unit $\frac{1}{\mathrm{s}}$	$\varrho\colon \mathbb{R}\times\Omega\to\mathbb{R},$ $(t,x)\mapsto\varrho(t,x)$ with unit $\frac{\mathrm{C}}{\mathrm{m}^3}$	$\varrho^{\mathrm{m}}\colon \mathbb{R}\times\Omega\to\mathbb{R},$ $(t,x)\mapsto 0$ with unit $\frac{\mathrm{T}}{\mathrm{m}}$
fluid flux density	*electric flux density*	*magnetic flux density*
$\mathbf{v}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{v}(t,x)$ with unit $\frac{\mathrm{m}}{\mathrm{s}}$	$\mathbf{D}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{D}(t,x)$ with unit $\frac{\mathrm{C}}{\mathrm{m}^2}$	$\mathbf{B}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{B}(t,x)$ with unit T
source current density	*electric current density*	*magnetic current density*
$\mathbf{j}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{j}(t,x)$ with unit $\frac{\mathrm{m}}{\mathrm{s}^2}$	$\mathbf{j}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{j}(t,x)$ with unit $\frac{\mathrm{A}}{\mathrm{m}^2}$	$\mathbf{j}^{\mathrm{m}}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto 0$ with unit $\frac{\mathrm{T}}{\mathrm{s}}$
fluid field	*magnetic field*	*electric field*
$\mathbf{h}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{H}(t,x)$ with unit $\frac{\mathrm{m}^2}{\mathrm{s}^2}$	$\mathbf{H}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{H}(t,x)$ with unit $\frac{\mathrm{A}}{\mathrm{m}}$	$\mathbf{E}\colon \mathbb{R}\times\Omega\to\mathbb{R}^3,$ $(t,x)\mapsto\mathbf{E}(t,x)$ with unit $\frac{\mathrm{V}}{\mathrm{m}}$

We follow Maxwell and apply the analogy of the imaginary fluid to electricity and magnetism. The experimentally observed absence of magnetic monopoles is reflected by setting the *magnetic charge density* ϱ^{m} and the *magnetic current density* $\mathbf{j}^{\mathrm{m}}$ equal to zero. The above table shows the correspondence between concepts for the imaginary fluid and their counterparts for electricity and magnetism.

Note that by convention the electric field $\mathbf{E}$ does not correspond to the fluid field $\mathbf{h}$ but to its negative $-\mathbf{h}$, as can be seen below in the analog of the Circulation Law for the Fluid Field (Faraday's Law). The names for $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{E}$ emphasize the correspondence to the imaginary fluid analogy. Whereas $\mathbf{E}$ is consistently named *electric field* in the literature, also other names are used for $\mathbf{H}$, $\mathbf{D}$ and $\mathbf{B}$ as listed in the following table, see also [8].

H	Alternative denominations for ***magnetic field***
	magnetic field intensity, *magnetic field strength*, *magnetizing force*
B	Alternative denominations for ***magnetic flux density***
	magnetic induction field, *magnetic field*
D	Alternative denominations for ***electric flux density***
	electric displacement field, *electric induction*

The Continuity Equation for the Fluid Flux of the imaginary fluid now becomes *Gauss' Law* for the electric flux and the *Magnetic Flux Continuity*, respectively. In integral form they hold for each bounded open $M \subset \Omega$ with piecewise smooth boundary ∂M.

Gauss' Law (integral form)

$$\int_M \varrho \, dV = \int_{\partial M} \mathbf{D} \cdot d\mathbf{S} \qquad \text{(with unit C)}$$

$$(\textit{electric charge in } M = \textit{electric flux through } \partial M),$$

Magnetic Flux Continuity (integral form)

$$0 = \int_{\partial M} \mathbf{B} \cdot d\mathbf{S} \qquad \text{(with unit Tm}^2\text{)}$$

$$(\textit{magnetic charge in } M = \textit{magnetic flux through } \partial M),$$

or equivalently in differential form

Gauss' Law (differential form)

$$\varrho = \operatorname{div}\mathbf{D} \qquad (\text{with unit } \tfrac{\mathrm{C}}{\mathrm{m}^3})$$

(*electrical charge density* = *divergence of electric flux density*),

Magnetic Flux Continuity (differential form)

$$0 = \operatorname{div}\mathbf{B} \qquad (\text{with unit } \tfrac{\mathrm{T}}{\mathrm{m}})$$

(*magnetic charge density* = *divergence of magnetic flux density*).

The Continuity Equation for the Source Current of the imaginary fluid becomes the *Continuity Equation for the Electric Current*. The magnetic current is assumed to vanish. In integral form it holds for each bounded open $M \subset \Omega$ with piecewise smooth boundary ∂M.

Continuity Equation for the Electric Current (integral form)

$$\int_M \partial_t \varrho \, dV = -\int_{\partial M} \mathbf{j} \cdot d\mathbf{S} \qquad (\text{with unit A})$$

(*charge production rate in* M = $-$*electric current through* ∂M),

or equivalently in differential form

Continuity Equation for the Electric Current (differential form)

$$\partial_t \varrho = -\operatorname{div}\mathbf{j} \qquad (\text{with unit } \tfrac{\mathrm{A}}{\mathrm{m}^3})$$

(*electric charge production rate density* = $-$ *divergence of electric current density*).

The change rate of the electric flux $\partial_t \int_M \mathbf{D} \cdot d\mathbf{S}$ through a surface M is called *displacement current* through M. The Circulation Law for the Fluid Field applied to the magnetic field $\mathbf{H}$ becomes *Ampère's Law*, also called *Ampère's Circuital Law with Maxwell's Correction*, and it equates for each bounded piecewise smooth oriented two-dimensional submanifold M of Ω with piecewise smooth boundary ∂M the sum of the displacement current and the electric current through M with the circulation of the magnetic field along ∂M.

Ampère's Law (integral form)

$$\int_M (\partial_t \mathbf{D} + \mathbf{j}) \cdot d\mathbf{S} = \int_{\partial M} \mathbf{H} \cdot d\mathbf{s} \qquad \text{(with unit A)}$$

(*displacement current + electric current through M = circulation of magnetic field along ∂M*),

or equivalently in differential form

Ampère's Law (differential form)

$$\partial_t \mathbf{D} + \mathbf{j} = \operatorname{curl} \mathbf{H} \qquad \text{(with unit } \tfrac{\mathrm{A}}{\mathrm{m}^2}\text{)}$$

(*displacement current + electric current densities = curl of magnetic field*).

Remark 4.2 (Interpretation of Displacement Current)**.** As described in Remark 3.2, the displacement current $\int_M \partial_t \mathbf{D} \cdot d\mathbf{S}$, as well as the electric current $\int_M \mathbf{j} \cdot d\mathbf{S}$, contribute to an electricity increase on the "other side of M". Whereas the electric current $\int_M \mathbf{j} \cdot d\mathbf{S}$ describes the transport of charge, the displacement current $\partial_t \int_M \mathbf{D} \cdot d\mathbf{S}$ describes an increased flux of electricity. The sum of both yield an electricity current with unit Ampère that equals the circulation $\int_{\partial M} \mathbf{H} \cdot d\mathbf{s}$ of the magnetic field along ∂M. For an interpretation of the displacement current which is similar in spirit and also uses the integral representation of Ampère's Law, see [9]. ◇

The Circulation Law for the Fluid Field applied to the electric field $\mathbf{E}$ (which corresponds to $-\mathbf{h}$) becomes *Faraday's Law*, also called *Maxwell-Faraday Equation*. In integral form it holds for each bounded piecewise smooth oriented two-dimensional submanifold M of Ω with piecewise smooth boundary ∂M.

Faraday's Law (integral form)

$$-\int_M \partial_t \mathbf{B} \cdot d\mathbf{S} = \int_{\partial M} \mathbf{E} \cdot d\mathbf{s} \qquad \text{(with unit V)}$$

(*negative of rate of change of magnetic flux through M = circulation of electric field along ∂M*),

or equivalently in differential form

Faraday's Law (differential form)

$$-\partial_t \mathbf{B} = \operatorname{curl} \mathbf{E} \qquad \text{(with unit } \tfrac{\mathrm{V}}{\mathrm{m}^2}\text{)}$$

(*negative of rate of change of magnetic flux density = curl of electric field*).

We thus have arrived at Maxwell's equations

$$\operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \operatorname{div} \mathbf{D} = \varrho,$$
$$\operatorname{curl} \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}, \quad \operatorname{div} \mathbf{B} = 0,$$

which are completed by material dependent constitutive relations, e.g.

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H},$$

in free space with permittivity ε_0 and permeability μ_0. For theoretical and experimental approaches to derive constitutive relations in complex media see e.g. [29] and the references therein.

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Institute for Materials Science and Max Bergmann Center of Biomaterials
& Dresden Center for Computational Materials Science (DCMS),
TU Dresden, Germany
E-mail: `matthias.geyer@tu-dresden.de`

Faculty of Mathematics, TU Dresden, Germany
E-mail: `jan.hausmann@tu-dresden.de`

Faculty of Mathematics, Institute of Analysis,
TU Dresden, Germany
E-mail: `konrad.kitzing@tu-dresden.de`

Faculty of Mathematics, TU Dresden, Germany
E-mail: `madlyn_dorothee.senkyr@mailbox.tu-dresden.de`

Faculty of Mathematics, Center for Dynamics and Institute of Analysis,
TU Dresden, Germany
E-mail: `stefan.siegmund@tu-dresden.de`

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STABLE PERIODIC SOLUTIONS IN SCALAR PERIODIC DIFFERENTIAL DELAY EQUATIONS

Anatoli Ivanov and Sergiy Shelyag

Abstract. A class of nonlinear simple form differential delay equations with a T-periodic coefficient and a constant delay $\tau > 0$ is considered. It is shown that for an arbitrary value of the period $T > 4\tau - d_0$, for some $d_0 > 0$, there is an equation in the class such that it possesses an asymptotically stable T-period solution. The periodic solutions are constructed explicitly for the piecewise constant nonlinearities and the periodic coefficients involved, by reduction of the problem to one-dimensional maps. The periodic solutions and their stability properties are shown to persist when the nonlinearities are "smoothed" at the discontinuity points.

1. Introduction

Differential delay equations serve as mathematical models of various phenomena in numerous applications where delays are intrinsic features of their functioning. An extensive list of applications can be found in e.g. monographs [4, 8, 10] with further references therein. The theoretical basics of the equations are given in monographs [3, 6].

The scalar differential delay equation

$$x'(t) = -\mu x(t) + g\big(x(t-\tau)\big), \quad g \in C(\mathbb{R},\mathbb{R}), \quad \mu \geq 0 \tag{1.1}$$

is one of the simplest by its form and still exhibiting a variety of quite complex dynamics and a broad range of applications. Depending on the particular form of the nonlinearity g it is well-known under particular names such as Mackey-Glass model [2, 9], Lasota-Wazewska equation [12], Nicholson's blowflies model [1], some other named models [10].

Equations of form (1.1) were studied in numerous publications primarily with respect to the property of global asymptotic stability of the equilibrium and the existence of nontrivial periodic solutions. In the presence of the negative feedback property for the nonlinearity g and the instability of the linearized equation about

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the unique equilibrium it is shown that the equation typically possesses a nontrivial periodic solution slowly oscillating about the equilibrium. The standard techniques used to prove the existence of periodic solutions are the ejective fixed point theory with its modifications [3, 6]. The exact value of the periods for such periodic solutions is generally not known; it can be arbitrary and varies continuously under continuous changes of g and μ.

A natural extension of model (1.1) is the following equation with the periodic coefficient $a(t)$:

$$x'(t) = -\mu x(t) + a(t) f\big(x(t-\tau)\big)\,, \qquad t \geq 0\,, \tag{1.2}$$

where f, $a \in C(\mathbb{R},\mathbb{R})$ and $a(t+T) \equiv a(t)$ for some positive period $T \geq \tau > 0$. The presence of the non-autonomous periodic input $a(t)$ can be justified in corresponding biological models by various factors, for example, by seasonal changes in the negative feedback [4, 8, 10].

Given equation (1.2) with the T-periodic coefficient $a(t)$ one can ask a natural question whether it admits periodic solutions with the same period. The primary objective of this note is to show that such periodic solutions exist for a wide continuous range of periods $T \geq 4\tau - \delta_0$ for some small $\delta_0 > 0$. The periodic solutions are constructed explicitly in terms of piece-wise constant functions f and a, and their continuous approximations. The special case of $\mu = 0$ is considered in this paper. The extension to the case of $\mu > 0$ is straightforward, however, it requires substantial additional space for adequate exposition and will be treated in a separate paper.

2. Preliminaries

Consider the scalar differential delay equation

$$x'(t) = a(t) f\big(x(t-\tau)\big)\,, \qquad t \geq 0\,, \tag{2.1}$$

with a T-periodic coefficient $a(t) \geq 0$, $a(t+T) \equiv a(t)$, $T \geq \tau$, and a nonlinearity $f(x)$ satisfying the negative feedback assumption $x \cdot f(x) < 0\ \forall x \in \mathbb{R}, x \neq 0$.

For the continuous functions f and a the standard choice of the initial set for equation (2.1) is the Banach space of continuous functions on the initial interval $[-\tau, 0]$: $\mathbb{X} = C([-\tau,0],\mathbb{R})$. For an arbitrary initial function $\phi \in \mathbb{X}$ there exists a unique solution $x(t) = x(t;\phi)$ to equation (2.1) defined for all $t \geq 0$. It is obtained by forward integration. At every $t \geq 0$ the solution $x(t)$ can be viewed as an element of space $\mathbb{X}$ by the following representation: $\mathbb{X} \ni x_t(s) := x(t+s), s \in [-\tau, 0]$.

For the initial basic construction of periodic solutions in Section 3 the functions f and a are piecewise constant. For arbitrary $\phi \in \mathbb{X}$ the corresponding solution $x(t;\phi)$ is explicitly built for all $t \geq 0$ by direct integration. It turns out to be a piecewise affine function differentiable everywhere except at discrete isolated set of points (in fact, it is a finite set of points on every finite interval $[0, t_0], t_0 > 0$).

We are interested in oscillation of solutions about the equilibrium $x(t) \equiv 0$. Sufficient conditions for the oscillation can be easily found in relevant available publications on the issue (see e.g. the monograph [5] and further references therein).

In particular, when $\sup_{t\in[0,T]}\int_t^{t+\tau} a(s)\,ds > -1/f'(0)$ all solutions of equation (2.1) oscillate. We shall assume this condition to hold throughout the paper.

Due to the negative feedback assumption the important role in the dynamics is played by the slowly oscillating solutions. A solution is called slowly oscillating if the distance between its any two zeros is greater than the delay $\tau > 0$. Any initial function $\phi \in \mathbb{X}$ such that $\phi(s) > 0\ \forall s \in [-\tau, 0]$ gives rise to a slowly oscillating solution, under the assumption of oscillation of all solutions.

Define two sets

$$K_+ := \{\phi \in \mathbb{X} \mid \phi(s) \geq 0\ \forall s \in [-\tau, 0],\ \phi \not\equiv 0\}$$

and

$$K_- := \{\phi \in \mathbb{X} \mid \phi(s) \leq 0\ \forall s \in [-\tau, 0],\ \phi \not\equiv 0\}\,.$$

It is a straightforward calculation to verify that for arbitrary $\phi \in K_+, \phi(0) \neq 0$, the corresponding solution $x(t;\phi)$ has an increasing sequence of zeros $0 < z_1 < z_2 < z_3 < \cdots$ such that $z_{k+1} - z_k > \tau, k \in \mathbb{N}$, and

$$x(t) < 0\ \ \forall t \in (z_{2k-1}, z_{2k}) \quad \text{and} \quad x(t) > 0\ \ \forall t \in (z_{2k}, z_{2k+1}).$$

Therefore, the solution $x(t;\phi)$ is slowly oscillating [3, 6]. Similar property is valid for any solution $x(t;\psi), \psi \in K_-, \psi(0) \neq 0$.

3. Piecewise constant nonlinearities

In this section we consider the particular case of equation (2.1) when $\tau = 1$:

$$x'(t) = a(t)f\big(x(t-1)\big)\,, \qquad t \geq 0\,. \tag{3.1}$$

Note that the case of general delay $\tau > 0$ can always be normalized to $\tau = 1$ by time rescaling $t = \tau \cdot s$. We start with the case when the nonlinearity f is the negative sign function

$$f(x) = f_0(x) = -\mathrm{sign}(x) = \begin{cases} +1 & \text{if } \ x < 0 \\ \ \ 0 & \text{if } \ x = 0 \\ -1 & \text{if } \ x > 0\,, \end{cases}$$

and the T-periodic coefficient $a(t)$ is a piecewise constant function defined by two positive constants a_1, a_2 as

$$a(t) = A_0(t) = \begin{cases} a_1 & \text{if } \ t \in [0, p_1) \\ a_2 & \text{if } \ t \in [p_1, p_1 + p_2) \\ \text{periodic extension on } \mathbb{R} \text{ outside interval } [0,T), T = p_1 + p_2\,, \end{cases}$$

where a_1, a_2, p_1, p_2 are all positive constants.

Due to the piecewise constant values of both $f(x)$ and $a(t)$ the forward solutions to (3.1) can be calculated explicitly; they are piecewise affine continuous functions for $t \geq 0$ differentiable everywhere except at a countable set of isolated points (where the solution changes slope). We will consider initial functions $\phi(s) \in C([-1, 0], \mathbb{R})$ which give rise to slowly oscillating solutions. Without loss of generality one can assume that $\phi \in K_+$ and $\phi(s) > 0\ \forall s \in [-1, 0]$. The corresponding solution

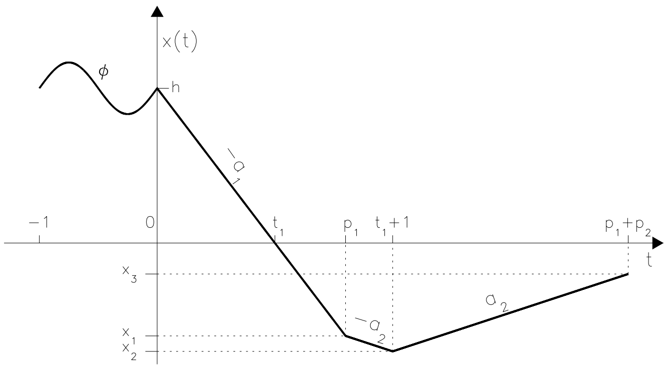

FIG. 1: Solution of Eq. (3.1) for $t \in [0, T]$.

$x(t) = x(t; \phi), t \geq 0$, depends only on the value $\phi(0) := h > 0$ and does not depend on the remaining values $\phi(s) > 0, s \in [-1, 0)$ on the initial interval.

Given $\phi \in K_+$ and its corresponding solution $x(t; \phi)$ we would like to find conditions such that its segment $x_{p_1+p_2}(s)$ belongs to the set K_-, in other words, such that the translation operator by the period $T = p_1 + p_2$ along the solutions maps the set K_+ into K_-.

The following is the explicit calculation of the solution $x(t; h)$, $t \geq 0$, for $h > 0$. See Figure 1 for the geometric representation of the solution.

On the interval $[0, p_1]$ the solution is given by $x(t) = h - a_1 t$. We assume that $x_1 := x(p_1) = h - a_1 p_1 < 0$. Therefore, there exists the unique value $t_1 = h/a_1 < p_1$ such that $x(t_1) = 0$.

We also assume that $p_1 - t_1 < 1$, implying $t_1 + 1 > p_1$. Then on the interval $[p_1, t_1 + 1]$ the solution x is given by $x(t) = x_1 - a_2(t - p_1)$. Set $x_2 := x(t_1 + 1) = (1 - a_2/a_1)h - a_2 + p_1(a_2 - a_1)$. Since $x_1 < 0$ and $a_2 > 0$ we also have that $x_2 < 0$ is valid. Therefore, the segment of the solution $x(t), t \in [t_1, t_1 + 1]$, belongs to the set K_-.

We next suppose that $t_1 + 1 < p_1 + p_2$. On the interval $[t_1 + 1, p_1 + p_2]$ the solution x is found as $x(t) = x_2 + a_2[t - (t_1 + 1)]$. Set $x_3 := x(p_1 + p_2)$ and additionally assume that $x_3 < 0$. The value of x_3 is easily calculated as

$$x_3 = x(p_1 + p_2) = \left(1 - 2\frac{a_2}{a_1}\right) h - a_1 p_1 - a_2(2 - 2p_1 - p_2) := F_1(h) = mh - b\,.$$

We note that the piecewise affine solution $x(t; h)$ is continuous on $[0, p_1 + p_2]$ and differentiable everywhere except at points $t = p_1$ and $t = t_1 + 1$.

Likewise, when $\psi \in K_-$ and $\psi(0) = h < 0$ analogous calculations yield

$$x_3 = x(p_1 + p_2; \psi) = \left(1 - 2\frac{a_2}{a_1}\right) h + a_1 p_1 + a_2(2 - 2p_1 - p_2) := F_2(h) = mh + b\,.$$

We would like to guarantee that the slope $m = 1 - 2a_2/a_1$ of the affine maps F_1, F_2 satisfies $|m| < 1$, and the y-intercept $b = a_1 p_1 + a_2(2 - 2p_1 - p_2) > 0$ is positive. An easy calculation leads to the following conclusion:

Proposition 3.1. *Suppose that* $a_2 < a_1$. *Then* $|m| = |1 - 2a_2/a_1| < 1$. *If in addition* $(a_1/a_2 - 2)p_1 > p_2 - 2$ *then* $b = a_1p_1 + a_2(2 - 2p_1 - p_2) > 0$ *is positive.*

Define the piecewise affine map F by

$$F(h) = \begin{cases} F_1(h) & \text{if } \ h > 0 \\ F_2(h) & \text{if } \ h < 0\,. \end{cases} \tag{3.2}$$

Under the assumptions of Proposition 3.1 map F has a unique attracting two-cycle $\{h_1^*, h_2^*\} = \{-b/(1+m), b/(1+m)\}$ which attracts all initial values $h \in \mathbb{R}$ when $-1 < m < 0\,(a_2 < a_1 < 2a_2)$ and all $h \in (-b/m, b/m)$ when $0 < m < 1\,(2a_2 < a_1)$. This two-cycle of F corresponds to asymptotically stable slowly oscillating periodic solution of equation (3.1) with the period $T = 2(p_1+p_2)$. It is also easy to see that the periodic solution has the following symmetry property $x(t+p_1+p_2) = -x(t) \ \ \forall t \in \mathbb{R}$. This is due to the fact that $F_2(-h) = -F_1(h), h > 0$.

The above consideration immediately implies the following

Corollary 3.2. *Suppose that* a_1, a_2, p_1, p_2 *satisfy the assumptions of Proposition 3.1. Then the corresponding equation* (3.1) *has a unique asymptotically stable slowly oscillating symmetric periodic solution* $x_*(t)$ *with the period* $T = 2(p_1 + p_2)$.

Consider next the case of equation (3.1) when $f(x) = f_0(x)$ and the piecewise constant coefficient $a(t)$ is defined by four constants as

$$a(t) = A_1(t) = \begin{cases} a_1 & \text{if } \ t \in [0, p_1) \\ a_2 & \text{if } \ t \in [p_1, p_1 + p_2) \\ a_3 & \text{if } \ t \in [p_1 + p_2, p_1 + p_2 + p_3) \\ a_4 & \text{if } \ t \in [p_1 + p_2 + p_3, p_1 + p_2 + p_3 + p_4) \\ \text{periodic extension on } \mathbb{R} \text{ outside interval } [0, T), \\ T := p_1 + p_2 + p_3 + p_4\,, \end{cases} \tag{3.3}$$

where a_1, a_2, a_3, a_4, p_1, p_2, p_3, p_4 are all positive constants.

Proposition 3.3. *Assume that the two quadruples* a_1, a_2, p_1, p_2 *and* a_3, a_4, p_3, p_4 *each satisfy the conditions of Proposition 3.1 and that in addition the inequalities* $m_1 = 1 - 2a_2/a_1 < b_1/b_2$, $m_2 = 1 - 2a_4/a_3 < b_2/b_1$ *are valid. Then the differential delay equation* (3.1) *possesses a unique asymptotically stable slowly oscillating periodic solution with the period* $T = p_1 + p_2 + p_3 + p_4$.

The validity of Proposition 3.3 is seen from the analogous construction of the piecewise affine map F similar to that given my (3.2), where F_1 is built of the quadruple a_1, a_2, p_1, p_2 while F_2 is derived from the quadruple a_3, a_4, p_3, p_4. The periodic solution is defined by the unique attracting 2-cycle of the interval map F, which is given explicitly by $(h_1^*, h_2^*) = ((m_1b_2 - b_1)/(1 - m_1m_2), (b_2 - m_2b_1)/(1 - m_1m_2))$, $h_1^* > 0$, $h_2^* < 0$. Such 2-cycle exists if the additional assumptions of the proposition on values of m_1, m_2, b_1, b_2 are met. Note that one of the inequalities for m_1, m_2 is satisfied by default.

Next we would like to see what values of the period T can be achieved for the stable periodic solutions, based on values $a_i, p_i, 1 \le i \le 4$, defining the piecewise constant coefficient $a(t)$. According to Proposition 3.1, one must first have that $a_1 > a_2$ and $a_3 > a_4$ are satisfied so that both $|m_1| < 1$, $|m_2| < 1$ are valid. Since we also require that $b_1 = p_1(a_1-2a_2)+a_2(2-p_2) > 0$, $b_2 = p_3(a_3-2a_4)+a_4(2-p_4) > 0$, both can be achieved if $a_1 > 2a_2$, $a_3 > 2a_4$ and $p_2 < 2$, $p_4 < 2$. To get arbitrarily large values of the period T one can proceed in several ways. One is to keep values $p_2 < 2$, $p_4 < 2$ fixed and increase either or both values of a_1, a_3 indefinitely. Another way is to define the coefficient $a(t)$ by modifying $A_1(t)$ in (3.3) beyond the initial period $T_1 = p_1 + p_2 + p_3 + p_4$ by $A_2(t) := A_1(t)$ for $t \in [0, T_1)$ and $A_2(t) \equiv 0$ for $t \in [T_1, T_1 + p_5)$, for some $p_5 > 0$ (which can be any). By increasing the value of p_5 the new period $T = p_1 + p_2 + p_3 + p_4 + p_5$ of the periodic solution can be made continuously arbitrarily large.

In view of the construction and consideration above we arrive at the following statement

Theorem 3.4. *There is a constant $T_0 > 2$ such that for arbitrary period T within the range $T_0 \le T < \infty$ there are choices of values a_i, p_i, $1 \le i \le 4$, such that equation* (3.1) *with nonlinearity $f = f_0$ and the respective coefficient $a = A_2$ has an asymptotically stable slowly oscillating periodic solution with the period T.*

Since by the very construction the periodic solutions are slowly oscillating each semi-cycle is of the length greater than 1. Therefore the period of any such periodic solution is always greater than 2. Period $T = 4$ is achieved when each of the semi-periods $T_1 = p_1 + p_2$ and $T_2 = p_3 + p_4$ is 2. We obtain the period 4 solution for this particular choice of the constants' values $a_1 = 7$, $a_2 = 3$, $a_3 = 6$, $a_4 = 2.5$ and $p_1 = p_2 = 1$, $p_3 = 1.2$, $p_4 = 0.8$. Due to the continuous dependence of the period T on the constants' values the smaller period can be achieved by their perturbation. We numerically observed stable periodic solutions by changing the above values proportionally up to when $T_1 = T_2 = 1.8$, thus making the period $T = 3.6$ (one can choose this value as T_0 in the statement). It would be of interest to derive a sharper estimate for T_0.

4. Smoothed nonlinearities

In this section we consider equation (2.1) where the piecewise constant functions $f(x)$ and $a(t)$ of Section 3 are replaced by close to them continuous nonlinearities. The basic idea is to make functions f and a continuous (or even smooth) in a small neighborhood of every discontinuity point by connecting the respective two constant values by a line segment.

We start first with the case $f(x) = f_0(x)$ and $a(t) = A_0(t)$. Let $\delta_0 > 0$ be small, and for every $\delta \in (0, \delta_0]$ introduce the continuous functions $f_\delta(x)$ and $A_0^\delta(t)$ by:

$$f(x) = f_\delta(x) = \begin{cases} +1 & \text{if } x \le -\delta \\ -1 & \text{if } x \ge \delta \\ -(1/\delta)x & \text{if } x \in [-\delta, \delta], \end{cases} \tag{4.1}$$

and
(4.2)

$$a(t) = A_0^\delta(t) = \begin{cases} a_2 + \frac{a_1 - a_2}{2\delta}(t+\delta) \quad \text{if } \ t \in [-\delta, \delta] \\ a_1 \quad \text{if } \ t \in [\delta, p_1 - \delta) \\ a_1 + \frac{a_2 - a_1}{2\delta}[t - (p_1 - \delta)] \quad \text{if } \ t \in [p_1 - \delta, p_1 + \delta] \\ a_2 \quad \text{if } \ t \in [p_1 + \delta, p_1 + p_2 - \delta) \\ a_2 + \frac{a_1 - a_2}{2\delta}[t - (p_2 - \delta)] \quad \text{if } \ t \in [p_1 + p_2 - \delta, p_1 + p_2 + \delta] \\ \text{periodic extension on } \mathbb{R} \text{ outside interval } [0, T), T = p_1 + p_2 \,. \end{cases}$$

Note that in the above definition of $A_0^\delta(t)$ there is an intentional overlap in values of the function on the intervals $[-\delta, \delta]$ and $[p_2 - \delta, p_2 + \delta]$ (where they are the same due to the intended periodicity). Likewise to (4.2), we define the continuous functions $A_1^\delta(t)$ and $A_2^\delta(t)$ based on the earlier defined piecewise constant coefficients $A_1(t)$ and $A_2(t)$ and with the same respective periods.

It is a well known fact that such small δ-perturbation of the nonlinearity f and the coefficient a lead to small smooth perturbations of the map F (away from its discontinuity point $h = 0$) (see e.g. [7, 11] for more relevant details). Below we outline the justification of this fact by showing the continuous dependence on δ and smoothness of the corresponding map for the value $x_1(\delta)$.

For $\delta \geq 0$ the value $x_1(\delta) = x(p_1; h)$ is explicitly calculated by direct integration as

$$\begin{aligned} x_1(\delta) &= h - \int_0^\delta a_\delta(t)\,dt - a_1(p_1 - 2\delta) - \int_{p_1-\delta}^{p_1} a_\delta(t)\,dt \\ &= h - a_1 p_1 + 2a_1\delta - \int_0^\delta a_\delta(t)\,dt - \int_{p_1-\delta}^{p_1} a_\delta(t)\,dt \\ &= x_1(0) + \tilde{x}_1(\delta)\,, \end{aligned}$$

where $\tilde{x}_1(\delta)$ is continuous in δ with $\tilde{x}_1(0) = 0$.

Similar calculations for the next two values $x_2(\delta)$ and $x_3(\delta)$ lead to the expression $x_3(\delta) = F_1(h, \delta) = F_1(h) + \tilde{F}_1(h, \delta)$ where $F_1(h)$ is as in (3.2) and $\tilde{F}_1(h, \delta)$ is continuous in h, δ and continuously differentiable in h with $\tilde{F}_1(h, 0) = 0$ and $\partial(\tilde{F}_1(h, \delta))/\partial h \leq M$, where positive constant M is independent of $\delta \geq 0$. Analogous calculations are valid for $F_2(h, \delta)$ (we omit those calculations and particular details of the expressions). Therefore, by the continuity for small $\delta > 0$ map $F(h, \delta)$ as in (3.2) has an attracting two-cycle close to that when $\delta = 0$.

The above considerations give us the following statement:

Theorem 4.1. *There exist $T_0 > 2$ and $\delta_0 > 0$ such that for arbitrary T with $T_0 < T < \infty$ and any $0 < \delta < \delta_0$ differential delay equation* (3.1) *with $f(x) = f_\delta(x)$ and T-periodic $a(t) = A_1^\delta(t)$ (or $A_2^\delta(t)$) has an asymptotically stable slowly oscillating solution with the period T.*

5. Discussion and conclusions

The results of Sections 3 and 4 derived for differential delay equation (3.1) can be extended to the more general equation (2.1) with $\mu > 0$ and piecewise constant or smoothed functions $f(x)$ and $a(t)$. The calculations become more involved and complex, however, as the solutions are now piecewise exponential of the form $x(t) = A\exp\{-\mu t\} + B, A, B$ - constant. The resulting dynamics can become more complicated as well: besides the stability and periodicity they can exhibit the chaotic behaviors. The basic idea of the analysis is the same as for equation (3.1): a reduction of the dynamics to that of interval maps. A review paper [7] provides examples of such analyses as well as references to other publications.

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Department of Mathematics, Pennsylvania State University,
44 University Drive, Dallas, PA 18612, USA
E-mail: afi1@psu.edu

School of Information Technology, Deakin University,
Melbourne, Victoria, 3125, Australia
E-mail: sergiy.shelyag@deakin.edu.au

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DELAY-DEPENDENT STABILITY CONDITIONS FOR FUNDAMENTAL CHARACTERISTIC FUNCTIONS

Hideaki Matsunaga

Abstract. This paper is devoted to the investigation on the stability for two characteristic functions $f_1(z) = z^2 + pe^{-z\tau} + q$ and $f_2(z) = z^2 + pze^{-z\tau} + q$, where p and q are real numbers and $\tau > 0$. The obtained theorems describe the explicit stability dependence on the changing delay τ. Our results are applied to some special cases of a linear differential system with delay in the diagonal terms and delay-dependent stability conditions are obtained.

1. Introduction

We consider two characteristic functions

$$f_1(z) = z^2 + pe^{-z\tau} + q$$

and

$$f_2(z) = z^2 + pze^{-z\tau} + q,$$

where p and q are real numbers and $\tau > 0$. Equations $f_1(z) = 0$ and $f_2(z) = 0$ are the characteristic equations of linear differential equations

$$x''(t) + px(t-\tau) + qx(t) = 0 \tag{1.1}$$

and

$$x''(t) + px'(t-\tau) + qx(t) = 0 \tag{1.2}$$

with the delay τ, respectively.

A quasi-polynomial $f(z)$ is said to be *stable* if all zeros of $f(z)$ have negative real parts. In studying the stability of a characteristic function, main concern is on the stability region, the maximal region in the space of parameters for which the characteristic function is stable. Clarifying the dependence of all parameters on stability is important; however, it is not easy for a more general quasi-polynomial that contains $f_1(z)$ and $f_2(z)$. In this case, the quasi-polynomial or the zero solution of the corresponding delay differential equation may switch finite times from stability

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to instability and vice versa as a parameter increases. Such phenomena for changing parameter are often referred to as *stability switches*; see, e.g., [2, 3].

In 1966, Hsu and Bhatt [5] first presented the following stability results for $f_1(z)$ and $f_2(z)$; see also Stépán [7, Corollary 3.4 and Theorem 3.8].

Theorem A. *Let $\tau = 1$. Then function $f_1(z)$ is stable if and only if there exists a nonnegative integer m such that either*

$$p > 0\,, \quad p < q - (2m+1)^2\pi^2 \quad \textit{and} \quad p < -q + (2m+2)^2\pi^2$$

or

$$p < 0\,, \quad p > -q + 4m^2\pi^2 \quad \textit{and} \quad p > q - (2m+1)^2\pi^2\,.$$

Theorem B. *Let $\tau = 1$. Then function $f_2(z)$ is stable if and only if*

$$p > 0, \quad q > 0 \quad \textit{and} \quad p < -\frac{2}{\pi}q + \frac{\pi}{2}$$

or there exists a nonnegative integer m such that either

$$p > 0\,, \quad p < \frac{2}{(4m+3)\pi}q - \frac{(4m+3)\pi}{2} \quad \textit{and} \quad p < -\frac{2}{(4m+5)\pi}q + \frac{(4m+5)\pi}{2}$$

or

$$p < 0\,, \quad p > -\frac{2}{(4m+1)\pi}q + \frac{(4m+1)\pi}{2} \quad \textit{and} \quad p > \frac{2}{(4m+3)\pi}q - \frac{(4m+3)\pi}{2}.$$

Notice that Theorems A and B provide the stability conditions for (1.1) and (1.2), respectively, and depend on the parameters p and q with $\tau = 1$. A natural question then arises: how do the stability conditions for $f_1(z)$ and $f_2(z)$ depend on the delay τ with fixed p and q? The purpose of this paper is to answer the question. As an application, we can obtain delay-dependent stability conditions for some special cases of a linear differential system with delay in the diagonal terms.

2. Main results

Our main results are stated as follows:

Theorem 2.1. *Function $f_1(z)$ is stable if and only if either*

$$0 < 5p < 3q \quad \textit{and} \quad \tau \in (\tau_{2,0}, \tau_{1,1}) \cup (\tau_{2,1}, \tau_{1,2}) \cup \cdots \cup (\tau_{2,k_1-1}, \tau_{1,k_1}) \tag{2.1}$$

or

$$0 < -p < q \quad \textit{and} \quad \tau \in (\tau_{1,0}, \tau_{2,0}) \cup (\tau_{1,1}, \tau_{2,1}) \cup \cdots \cup (\tau_{1,k_2-1}, \tau_{2,k_2-1})\,. \tag{2.2}$$

Here $\tau_{1,n}$, $\tau_{2,n}$, k_1, and k_2 are defined as

$$\tau_{1,n} = \frac{2n\pi}{\sqrt{p+q}}, \quad \tau_{2,n} = \frac{(2n+1)\pi}{\sqrt{-p+q}}, \quad n = 0, 1, 2, \ldots,$$

$$k_1 = \left\lceil \frac{2\sqrt{-p+q} - \sqrt{p+q}}{2(\sqrt{p+q} - \sqrt{-p+q}\,)} \right\rceil, \quad k_2 = \left\lceil \frac{\sqrt{p+q}}{2(\sqrt{-p+q} - \sqrt{p+q}\,)} \right\rceil, \tag{2.3}$$

where $\lceil\cdot\rceil$ denotes the ceiling function, namely, $\lceil x\rceil = \min\{s \in \mathbb{Z} \mid x \le s\}$.

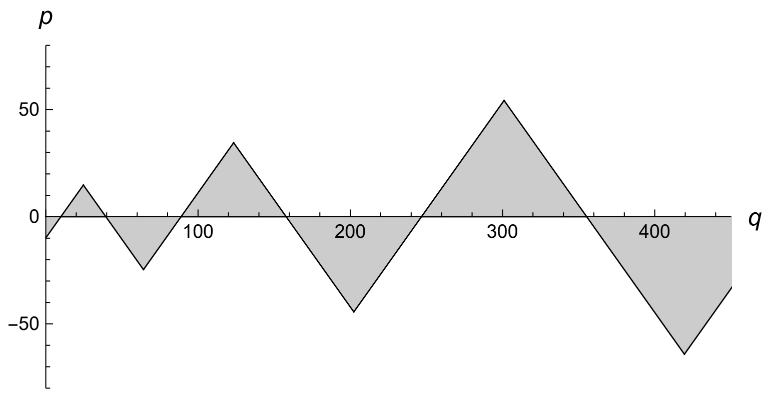

FIG. 1: Stability region of $f_1(z)$ with $\tau = 1$ (Theorem A).

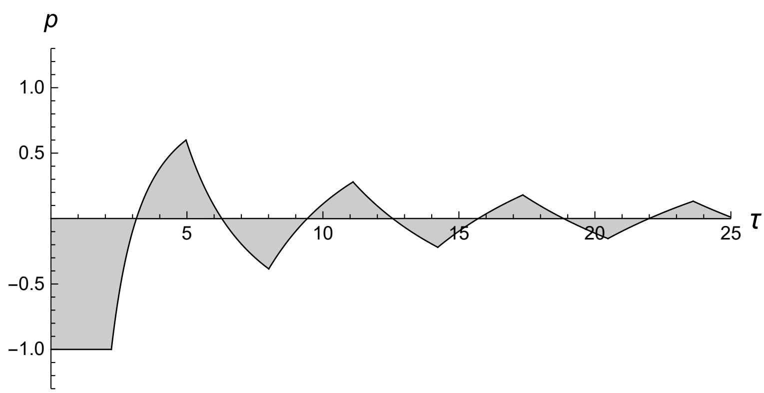

FIG. 2: Stability region of $f_1(z)$ with $q = 1$ (Theorem 2.1).

Theorem 2.2. *Function $f_2(z)$ is stable if and only if any one of the following three conditions holds:*

$$p > 0, \quad 15p^2 \geq 4q \quad \text{and} \quad 0 < \tau < \tau_{3,0}, \tag{2.4}$$

$$p > 0, \quad 15p^2 < 4q \quad \text{and} \quad \tau \in (0, \tau_{3,0}) \cup (\tau_{4,0}, \tau_{3,1}) \cup \cdots \cup (\tau_{4,k_3-1}, \tau_{3,k_3}), \tag{2.5}$$

$$p < 0, \quad 3p^2 < 4q \quad \text{and} \quad \tau \in (\tau_{3,0}, \tau_{4,0}) \cup (\tau_{3,1}, \tau_{4,1}) \cup \cdots \cup (\tau_{3,k_4-1}, \tau_{4,k_4-1}). \tag{2.6}$$

Here $\tau_{3,n}$, $\tau_{4,n}$, k_3, and k_4 are defined as

$$\tau_{3,n} = \frac{(4n+1)\pi}{p + \sqrt{p^2 + 4q}}, \quad \tau_{4,n} = \frac{(4n+3)\pi}{-p + \sqrt{p^2 + 4q}}, \quad n = 0, 1, 2, \ldots,$$

$$k_3 = \left\lceil \frac{\sqrt{p^2 + 4q}}{4p} - 1 \right\rceil, \quad k_4 = \left\lceil -\frac{\sqrt{p^2 + 4q}}{4p} - \frac{1}{2} \right\rceil. \tag{2.7}$$

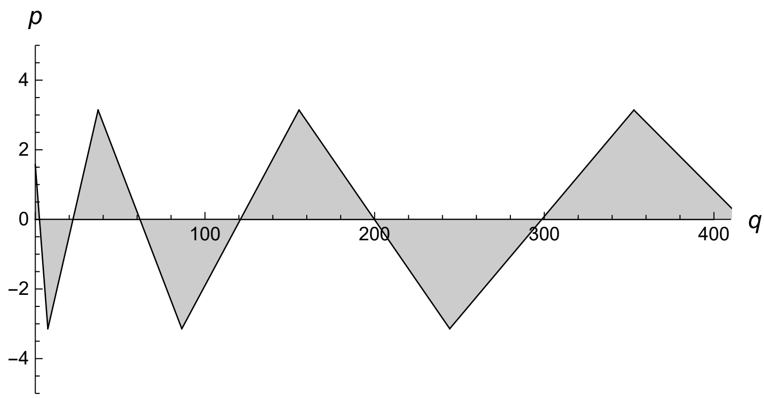

FIG. 3: Stability region of $f_2(z)$ with $\tau = 1$ (Theorem B).

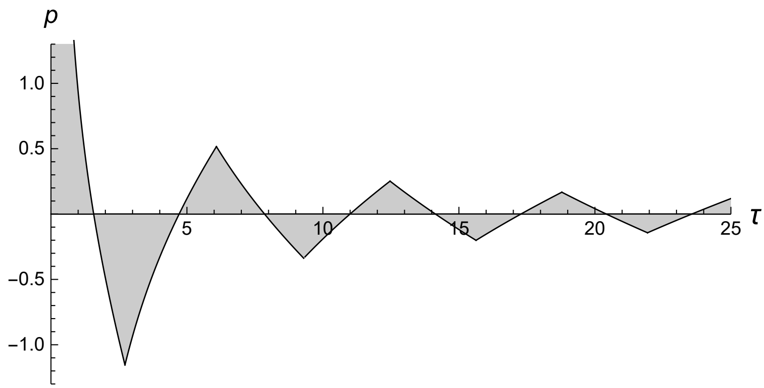

FIG. 4: Stability region of $f_2(z)$ with $q = 1$ (Theorem 2.2).

Remark 2.3. Theorems 2.1 and 2.2 show that as τ increases from 0, both $f_1(z)$ and $f_2(z)$ switch finite times from stability to instability and vice versa under suitable conditions, and they become unstable eventually; see Figs. 2 and 4.

Proof of Theorem 2.1. Let $\lambda = z\tau$ and $g_1(\lambda) = \tau^2 f_1(z)$. Then

$$g_1(\lambda) = \lambda^2 + p\tau^2 e^{-\lambda} + q\tau^2 \,.$$

Clearly, the stability of $f_1(z)$ is equivalent to that of $g_1(\lambda)$. Thus, we will prove that function $g_1(\lambda)$ is stable if and only if either (2.1) or (2.2) holds.

By Theorem A, function $g_1(\lambda)$ is stable if and only if there exists a nonnegative integer m such that either

$$p > 0\,, \quad p\tau^2 < q\tau^2 - (2m+1)^2\pi^2 \quad \text{and} \quad p\tau^2 < -q\tau^2 + (2m+2)^2\pi^2 \tag{2.8}$$

or

$$(2.9)\qquad p<0\,,\quad p\tau^2 > -q\tau^2 + 4m^2\pi^2 \quad\text{and}\quad p\tau^2 > q\tau^2 - (2m+1)^2\pi^2\,.$$

It follows that

$$\begin{aligned}(2.8) \iff&\; p>0\,,\quad (-p+q)\tau^2 > (2m+1)^2\pi^2\,,\quad (p+q)\tau^2 < (2m+2)^2\pi^2\\ \iff&\; q>p>0\,,\quad \tau > \frac{(2m+1)\pi}{\sqrt{-p+q}}\,,\quad \tau < \frac{(2m+2)\pi}{\sqrt{p+q}}\\ \iff&\; q>p>0\,,\quad \tau_{2,m} < \tau < \tau_{1,m+1}.\end{aligned}$$

Notice that k_1 defined by (2.3) is the smallest nonnengative integer that satisfies $\tau_{2,k_1} > \tau_{1,k_1+1}$ because $\tau_{2,k} > \tau_{1,k+1}$ is equivalent to

$$k > \frac{2\sqrt{-p+q} - \sqrt{p+q}}{2(\sqrt{p+q} - \sqrt{-p+q}\,)}\ (> -1).$$

Suppose that

$$\frac{2\sqrt{-p+q} - \sqrt{p+q}}{2(\sqrt{p+q} - \sqrt{-p+q}\,)} \le 0\,,$$

namely, $3q \le 5p$. Then we obtain $k_1 = 0$ and $\tau_{2,k} > \tau_{1,k+1}$ for $k = 0, 1, 2, \ldots$. In this case, no nonnegative integer m that satisfies (2.8) exist. Hence, if $0 < 5p < 3q$, then $k_1 \ge 1$ and

$$0 < \tau_{2,0} < \tau_{1,1} < \tau_{2,1} < \tau_{1,2} < \cdots < \tau_{2,k_1-1} < \tau_{1,k_1} < \tau_{1,k_1+1} < \tau_{2,k_1}\,,$$

which indicates that (2.8) holds if and only if (2.1) holds.

Similarly, we observe that

$$\begin{aligned}(2.9) \iff&\; p<0\,,\quad (p+q)\tau^2 > 4m^2\pi^2\,,\quad (-p+q)\tau^2 < (2m+1)^2\pi^2\\ \iff&\; -q<p<0\,,\quad \tau > \frac{2m\pi}{\sqrt{p+q}}\,,\quad \tau < \frac{(2m+1)\pi}{\sqrt{-p+q}}\\ \iff&\; -q<p<0,\quad \tau_{1,m} < \tau < \tau_{2,m}.\end{aligned}$$

Notice that k_2 defined by (2.3) is the smallest positive integer that satisfies $\tau_{1,k_2} > \tau_{2,k_2}$ because $\tau_{1,k} > \tau_{2,k}$ is equivalent to

$$k > \frac{\sqrt{p+q}}{2(\sqrt{-p+q} - \sqrt{p+q}\,)}\ (> 0)\,.$$

Therefore, we obtain

$$0 = \tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \tau_{2,1} < \cdots < \tau_{1,k_2-1} < \tau_{2,k_2-1} < \tau_{2,k_2} < \tau_{1,k_2}\,,$$

which implies that (2.9) holds if and only if (2.2) holds. This completes the proof. □

Proof of Theorem 2.2. Let $\lambda = z\tau$ and $g_2(\lambda) = \tau^2 f_2(z)$. Then

$$g_2(\lambda) = \lambda^2 + p\tau\lambda e^{-\lambda} + q\tau^2.$$

Clearly, the stability of $f_2(z)$ is equivalent to that of $g_2(\lambda)$. Thus, we will prove that function $g_2(\lambda)$ is stable if and only if (2.4), (2.5), or (2.6) holds.

From Theorem B, function $g_2(\lambda)$ is stable if and only if

$$p > 0, \quad q > 0\,, \quad p\tau < -\frac{2q\tau^2}{\pi} + \frac{\pi}{2} \tag{2.10}$$

or there exists a nonnegative integer m such that either

$$p > 0, \quad p\tau < \frac{2q\tau^2}{(4m+3)\pi} - \frac{(4m+3)\pi}{2}\,, \quad p\tau < -\frac{2q\tau^2}{(4m+5)\pi} + \frac{(4m+5)\pi}{2} \tag{2.11}$$

or

$$p < 0\,, \quad p\tau > -\frac{2q\tau^2}{(4m+1)\pi} + \frac{(4m+1)\pi}{2}, \quad p\tau > \frac{2q\tau^2}{(4m+3)\pi} - \frac{(4m+3)\pi}{2}\,. \tag{2.12}$$

It is easy to see that

$$(2.10) \iff p > 0\,, \quad q > 0\,, \quad 0 < \tau < \frac{-p + \sqrt{p^2 + 4q}}{4q}\pi = \tau_{3,0},$$

which coincides with (2.4). We observe that

$$(2.11) \iff p > 0, \quad \begin{cases} 4q\tau^2 - 2(4m+3)p\pi\tau - (4m+3)^2\pi^2 > 0 \\ 4q\tau^2 + 2(4m+5)p\pi\tau - (4m+5)^2\pi^2 < 0 \end{cases}$$

$$\iff p > 0\,, \quad q > 0\,, \quad \tau_{4,m} < \tau < \tau_{3,m+1}.$$

Notice that k_3 defined by (2.7) is the smallest nonnegative integer that satisfies $\tau_{4,k_3} > \tau_{3,k_3+1}$ because $\tau_{4,k} > \tau_{3,k+1}$ is equivalent to

$$k > \frac{\sqrt{p^2 + 4q}}{4p} - 1 \ (> -1).$$

Suppose that $\sqrt{p^2 + 4q}/(4p) - 1 \le 0$, namely, $15p^2 \ge 4q$. Then we obtain $k_3 = 0$ and $\tau_{4,k} > \tau_{3,k+1}$ for $k = 0, 1, 2, \ldots$. In this case, no nonnegative integer m that satisfies (2.11) exist. Hence, if $p > 0$ and $15p^2 < 4q$, then $k_3 \ge 1$ and

$$0 < \tau_{3,0} < \tau_{4,0} < \tau_{3,1} < \cdots < \tau_{4,k_3-1} < \tau_{3,k_3} < \tau_{3,k_3+1} < \tau_{4,k_3}.$$

These facts indicate that (2.10) or (2.11) holds if and only if (2.4) or (2.5) holds.

Similarly, we observe that

$$(2.12) \iff p < 0, \quad \begin{cases} 4q\tau^2 + 2(4m+1)p\pi\tau - (4m+1)^2\pi^2 > 0 \\ 4q\tau^2 - 2(4m+3)p\pi\tau - (4m+3)^2\pi^2 < 0 \end{cases}$$

$$\iff p < 0, \quad q > 0, \quad \tau_{3,m} < \tau < \tau_{4,m}.$$

Notice that k_4 defined by (2.7) is the smallest nonnegative integer that satisfies $\tau_{3,k_4} > \tau_{4,k_4}$ because $\tau_{3,k} > \tau_{4,k}$ is equivalent to

$$k > -\frac{\sqrt{p^2 + 4q}}{4p} - \frac{1}{2} \ (> -1).$$

Suppose that $-\sqrt{p^2 + 4q}/(4p) - 1/2 \le 0$, namely, $3p^2 \ge 4q$. Then we obtain $k_4 = 0$ and $\tau_{3,k} > \tau_{4,k}$ for $k = 0, 1, 2, \ldots$. In this case, no nonnegative integer m that satisfies (2.12) exist. Therefore, if $p < 0$ and $3p^2 < 4q$, then $k_4 \ge 1$ and

$$0 < \tau_{3,0} < \tau_{4,0} < \tau_{3,1} < \tau_{4,1} < \cdots < \tau_{3,k_4-1} < \tau_{4,k_4-1} < \tau_{4,k_4} < \tau_{3,k_4},$$

which implies that (2.12) holds if and only if (2.6) holds. This completes the proof. □

3. Application

In this section, we investigate the asymptotic stability of the zero solution of a linear delay differential system

$$\begin{cases} x'(t) = -ax(t-r) - by(t), \\ y'(t) = -cx(t) - dy(t-r), \end{cases} \tag{3.1}$$

where a, b, c, d are real numbers and $r > 0$. The characteristic equation of (3.1) is given by

$$\det \begin{pmatrix} z + ae^{-zr} & b \\ c & z + de^{-zr} \end{pmatrix} = 0\,,$$

that is,

$$z^2 + (a+d)e^{-zr} + ade^{-2zr} - bc = 0. \tag{3.2}$$

When $a = d$, equation (3.2) is reduced to

$$\begin{cases} \big(z + ae^{-zr} + \sqrt{bc}\,\big)\big(z + ae^{-zr} - \sqrt{bc}\,\big) = 0 & (bc \geq 0)\,, \\ \big(z + ae^{-zr} + i\sqrt{-bc}\,\big)\big(z + ae^{-zr} - i\sqrt{-bc}\,\big) = 0 & (bc < 0)\,. \end{cases} \tag{3.3}$$

In 2009, the author [6] presented delay-dependent stability conditions for (3.1) with $a = d$ by using root analysis of (3.3). Consequently, let us treat other two special cases $a + d = 0$ and $ad = 0$.

Consider first the case $a + d = 0$. Then equation (3.2) becomes

$$z^2 - a^2e^{-2zr} - bc = 0. \tag{3.4}$$

By applying Theorem 2.1 to equation (3.4) with $p = -a^2$, $q = -bc$, $\tau = 2r$ and $\tau_{j,n} = 2r_{j,n}$ $(j = 1, 2)$, we obtain the following corollary.

Corollary 3.1. *Let $d = -a$. Then the zero solution of (3.1) is asymptotically stable if and only if*

$$a^2 + bc < 0 \quad \text{and} \quad r \in (r_{1,0}, r_{2,0}) \cup (r_{1,1}, r_{2,1}) \cup \cdots \cup (r_{1,\ell_2-1}, r_{2,\ell_2-1})\,.$$

Here $r_{1,n}$, $r_{2,n}$, and ℓ_2 are defined as

$$r_{1,n} = \frac{n\pi}{\sqrt{-a^2 - bc}}\,, \quad r_{2,n} = \frac{(n + 1/2)\pi}{\sqrt{a^2 - bc}}\,, \quad n = 0, 1, 2, \ldots,$$

$$\ell_2 = \left\lceil \frac{\sqrt{-a^2 - bc}}{2(\sqrt{a^2 - bc} - \sqrt{-a^2 - bc})} \right\rceil.$$

Next, consider the case $ad = 0$. Without loss of generality, we may assume $d = 0$. Then equation (3.2) becomes

$$z^2 + aze^{-zr} - bc = 0\,. \tag{3.5}$$

By applying Theorem 2.2 to equation (3.5) with $p = a$, $q = -bc$, $\tau = r$ and $\tau_{j,n} = r_{j,n}$ $(j = 3, 4)$, we obtain the following corollary.

Corollary 3.2. *Let $d = 0$. Then the zero solution of* (3.1) *is asymptotically stable if and only if any one of the following three conditions holds:*

(i) $a > 0, \quad 15a^2 \geq -4bc > 0 \quad$ *and* $\quad 0 < r < r_{3,0}$,

(ii) $a > 0, \quad 15a^2 < -4bc \quad$ *and* $\quad r \in (0, r_{3,0}) \cup (r_{4,0}, r_{3,1}) \cup \cdots \cup (r_{4,\ell_3-1}, r_{3,\ell_3})$,

(iii) $a < 0, \quad 3a^2 < -4bc \quad$ *and* $\quad r \in (r_{3,0}, r_{4,0}) \cup (r_{3,1}, r_{4,1}) \cup \cdots \cup (r_{3,\ell_4-1}, r_{4,\ell_4-1})$.

Here $r_{3,n}$, $r_{4,n}$, ℓ_3, and ℓ_4 are defined as

$$r_{3,n} = \frac{(4n+1)\pi}{a + \sqrt{a^2 - 4bc}}, \quad r_{4,n} = \frac{(4n+3)\pi}{-a + \sqrt{a^2 - 4bc}}, \quad n = 0, 1, 2, \ldots,$$

$$\ell_3 = \left\lceil \frac{\sqrt{a^2 - 4bc}}{4a} - 1 \right\rceil, \quad \ell_4 = \left\lceil -\frac{\sqrt{a^2 - 4bc}}{4a} - \frac{1}{2} \right\rceil.$$

In the remainder case $ad \neq 0$ and $a \neq \pm d$, although two delays r and $2r$ make the distribution of roots of (3.2) much more complicated, some new explicit stability conditions for (3.1) have been obtained; see [4] for details.

Finally, it turns out that Čermák and Kisela [1] have extended some parts of Theorem 2.1 to a stability criterion of fractional delay differential equations.

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Department of Mathematics,
Osaka Metropolitan University,
Sakai 599-8531, Japan
E-mail: hideaki.matsunaga@omu.ac.jp

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APPROXIMATION OF LIMIT CYCLE OF DIFFERENTIAL SYSTEMS WITH VARIABLE COEFFICIENTS

Masakazu Onitsuka

Dedicated to Professor Tetsuo Furumochi on the occasion of his 75th birthday

Abstract. The behavior of the approximate solutions of two-dimensional nonlinear differential systems with variable coefficients is considered. Using a property of the approximate solution, so called conditional Ulam stability of a generalized logistic equation, the behavior of the approximate solution of the system is investigated. The obtained result explicitly presents the error between the limit cycle and its approximation. Some examples are presented with numerical simulations.

1. Introduction

We consider the two-dimensional nonlinear differential system

$$
\begin{aligned}
x' &= f(t)x + g(t)y - \frac{f(t)}{\kappa} x \left(x^2+y^2\right)^{\frac{\alpha}{2}}\,,\\
y' &= -g(t)x + f(t)y - \frac{f(t)}{\kappa} y \left(x^2+y^2\right)^{\frac{\alpha}{2}}\,,
\end{aligned}
\tag{1.1}
$$

and its perturbed system

$$
\begin{aligned}
x' &= f(t)x + g(t)y - \frac{f(t)}{\kappa} x \left(x^2+y^2\right)^{\frac{\alpha}{2}} + p_1(t)\,,\\
y' &= -g(t)x + f(t)y - \frac{f(t)}{\kappa} y \left(x^2+y^2\right)^{\frac{\alpha}{2}} + p_2(t)\,,
\end{aligned}
\tag{1.2}
$$

where f, g, p_1 and p_2 are real-valued continuous functions for $t \geq 0$, and α and κ are positive constants. If $f = g \equiv 1$, $\alpha = 2$ and $\kappa = 1$, then (1.1) is reduces to the

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differential system

$$x' = x + y - x\left(x^2 + y^2\right)\,, \qquad y' = -x + y - y\left(x^2 + y^2\right)\,. \tag{1.3}$$

This system is well-known to have exactly one stable limit cycle $x^2 + y^2 = 1$ (see, [13]); that is, on the phase plane, there is the orbit of the unique periodic solution of (1.3) that rotates infinitely on the unit circle, and any orbit of the solution except the zero solution $(x(t), y(t)) \equiv (0, 0)$ and the periodic solution approaches while rotating to the unit circle.

On the other hand, if $p_1 \not\equiv 0 \not\equiv p_2$, then the differential system

$$x' = x + y - x\left(x^2 + y^2\right) + p_1(t)\,, \qquad y' = -x + y - y\left(x^2 + y^2\right) + p_2(t)\,. \tag{1.4}$$

does not have the zero solution and it is unknown whether it has a periodic solution. Needless to say, it will be very difficult to derive the conditions for the system to have a limit cycle because (1.4) is a nonautonomous differential system. If it is an autonomous system, many tools can be used, for example, the well-known Poincaré-Bendixon theorem, but a different approach will be needed for nonautonomous systems. See [2, 3, 7, 9, 10, 11, 14, 20] for recent results related to limit cycles. Here, instead of looking for the periodic orbit or limit cycle of (1.4), it can be regarded as a perturbed system of (1.3). If we impose some constraints on p_1 and p_2, we would expect the solution of (1.4) to be an approximation of the solution of (1.3). A well-known tool is the linear approximation method, but unfortunately (1.3) dealt with here is a nonlinear system. In this study, we will introduce a new tool for approximating nonlinear systems. It provides an approximation of the limit cycle by using a property called conditional Ulam stability for a scalar nonlinear equation. The definition of conditional Ulam stability will be given in the next section.

Define $\|(x, y)\| := \sqrt{x^2 + y^2}$. The main result of this study is as follows.

Theorem 1.1. *Suppose that there exists $\underline{f} > 0$ such that*

$$f(t) \geq \underline{f} \quad \text{for} \quad t \geq 0\,. \tag{1.5}$$

Let $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, $\|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, let $(x(t), y(t))$ and $(\xi(t), \eta(t))$ be the solutions of (1.1) *and* (1.2) *with*

$$\big(x(0), y(0)\big) = \big(\xi(0), \eta(0)\big) = (x_0, y_0)\,, \tag{1.6}$$

respectively. If

$$\|\big(p_1(t), p_2(t)\big)\| \leq \varepsilon \quad \text{for} \quad t \geq 0\,, \tag{1.7}$$

then $(x(t), y(t))$ and $(\xi(t), \eta(t))$ exist on $[0, \infty)$. Furthermore,

$$\min\left\{\|\big(\xi(t), \eta(t)\big)\|, \|\big(x(t), y(t)\big)\|\right\} \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}\,,$$

and

$$\big|\|(\xi(t),\eta(t))\| - \|(x(t),y(t))\|\big| \le \max\Big\{\frac{\alpha+1}{\alpha\underline{f}}, \frac{\alpha+1}{\alpha^2\underline{f}}\Big\}\varepsilon$$

for $t \in [0,\infty)$.

If $f = g \equiv 1$, $\alpha = 2$ and $\kappa = 1$, then we immediately obtain the following result.

Corollary 1.2. *Let* $\varepsilon \in \Big(0, \frac{2}{3\sqrt{3}}\Big]$, $\|(x_0,y_0)\| \in \Big[\frac{1}{\sqrt{3}},\infty\Big)$, *let* $(x(t),y(t))$ *and* $(\xi(t),\eta(t))$ *be the solutions of* (1.3) *and* (1.4) *with* (1.6), *respectively. If* (1.7) *holds, then* $(x(t),y(t))$ *and* $(\xi(t),\eta(t))$ *exist on* $[0,\infty)$. *Furthermore,*

$$\min\big\{\|(\xi(t),\eta(t))\|, \|(x(t),y(t))\|\big\} \ge \frac{1}{\sqrt{3}},$$

and

$$\big|\|(\xi(t),\eta(t))\| - \|(x(t),y(t))\|\big| \le \frac{3}{2}\varepsilon$$

for $t \in [0,\infty)$.

We denote a circle with radius $R > 0$ centered at the origin by C_R. Let (x_0,y_0) on the circle $C_{\frac{1}{\sqrt{3}}}$ or, be outside the circle $C_{\frac{1}{\sqrt{3}}}$. Now we consider the solutions $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6) and

$$p_1(t) = \frac{2}{3\sqrt{3}}\big(1 - 2\max\big\{\cos\sqrt{t}, 0\big\}\big) \quad \text{and} \quad p_2(t) = 0\,, \tag{1.8}$$

respectively. From $\|(p_1(t),p_2(t))\| \le \frac{2}{3\sqrt{3}}$ for $t \ge 0$, we can choose $\varepsilon = \frac{2}{3\sqrt{3}}$, and using Corollary 1.2, we see that $(x(t),y(t))$ and $(\xi(t),\eta(t))$ on the circle $C_{\frac{1}{\sqrt{3}}}$ or, are outside the circle $C_{\frac{1}{\sqrt{3}}}$, and

$$\big|\|(\xi(t),\eta(t))\| - \|(x(t),y(t))\|\big| \le \frac{3}{2}\varepsilon = \frac{1}{\sqrt{3}} \quad \text{for} \quad t \in [0,\infty)\,. \tag{1.9}$$

Figure 1 shows the orbits corresponding to $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6), (1.8) and

$$(x_0,y_0) = \Big(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\Big)\,. \tag{1.10}$$

Figure 2 shows the orbits corresponding to $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6), (1.8) and

$$(x_0,y_0) = (1.2, 1.2)\,. \tag{1.11}$$

The circle $C_{\frac{1}{\sqrt{3}}}$ is drawn with broken line. Figure 3 shows the orbits corresponding to $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6), (1.8) and

$$(x_0,y_0) = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)\,; \tag{1.12}$$

that is, (x_0,y_0) on the unit circle. This means that the orbit of $(x(t),y(t))$ represent the limit cycle. From this, we can conclude that the orbit of $(\xi(t),\eta(t))$ represent an approximation of the limit cycle. Note here that all orbits in Figures 1–3 are drawn for $0 \le t \le 150$. If we draw more time than 150, the red curve will fill the inside of the lip-like area. From (1.9), we see that the orbit corresponding to $(\xi(t), \eta(t))$ of (1.4) with (1.6), (1.8) and (1.12) is inside the circle $C_{1+\frac{1}{\sqrt{3}}}$ for $t \in [0, \infty)$. The purpose of this study is to explicitly present the error between the limit cycle of (1.1) and its approximation.

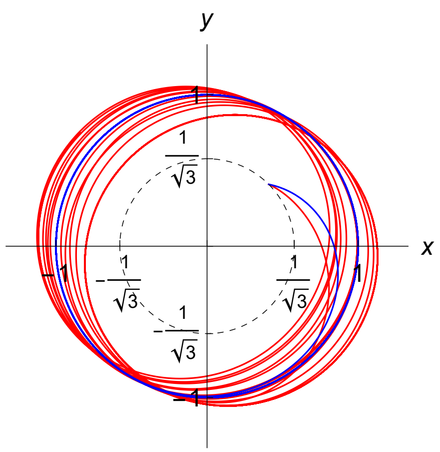

FIG. 1: Orbits of (1.3) (blue) and (1.4) (red) with (1.8) and (1.10).

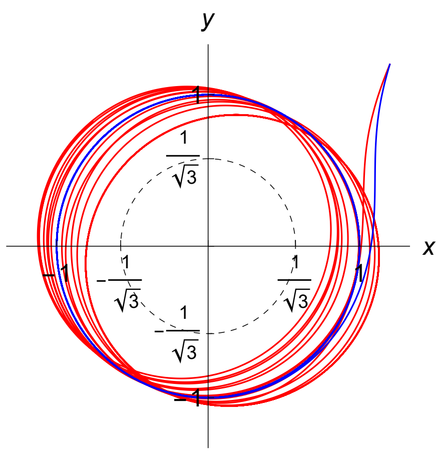

FIG. 2: Orbits of (1.3) (blue) and (1.4) (red) with (1.8) and (1.11).

In Section 2, we introduce the concept of conditional Ulam stability and present previous results that play an important role in this study. In Section 3, we prove Theorem 1.1 by using a previous result. In Section 4, we present the second main result and prove it. In Section 5, we give two examples of variable coefficients and present numerical simulations.

2. Conditional Ulam stability

In this section, we consider the nonautonomous generalized logistic equation

$$z' = h(t)z\left(1 - \frac{z^{\alpha}}{K}\right), \tag{2.1}$$

where h is a positive continuous function for $t \geq 0$, and α and K are positive constants. Especially when $h(t)$ is a constant, (2.1) is called the Richards model, which is one of the models that describe infectious diseases. Here, z, h, and K represent the cumulative number of cases/deaths, growth rate, and final epidemic size, respectively. In [16], the present author studied conditional Ulam stability of (2.1). Conditional Ulam stability is a property that guarantees the difference between the approximate solution and the exact solution to be finite. The exact definition is as follows.

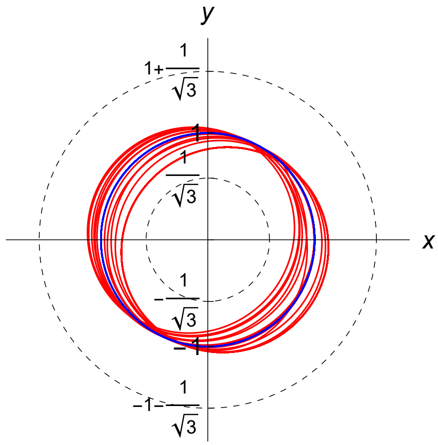

FIG. 3: Orbits of (1.3) (blue) and (1.4) (red) with (1.8) and (1.12).

Definition 2.1. Let $A \subseteq (0,\infty)$ and $B \subseteq \mathbb{R}$ be nonempty sets. Define the class
$\mathcal{C}_B := \left\{ x \in C^1[0,T_x) : x(0) \in B, T_x > 0 \text{ with } T_x = \infty \text{ or } |x(t)| \to \infty \text{ as } t \nearrow T_x \right\}$.
Note that $[0,T_x)$ refers to the maximal existence interval of $x(t)$. The nonlinear differential equation

$$z' = F(t,z) \tag{2.2}$$

is *conditionally Ulam stable* on $[0, \min\{T_z, T_\zeta\})$ with A in the class $\mathcal{C}_B$ if there exists $L > 0$ such that for any $\varepsilon \in A$ and any approximate solution $\zeta \in \mathcal{C}_B$ that satisfy

$$|\zeta' - F(t,\zeta)| \le \varepsilon \quad \text{for} \quad t \in [0, T_\zeta) ,$$

there exists a solution $z \in \mathcal{C}_B$ of (2.2) such that $|\zeta(t) - z(t)| \le L\varepsilon$ for $t \in [0, \min\{T_z, T_\zeta\})$. We call such an L an *Ulam constant* for (2.2) on $[0, \min\{T_z, T_\zeta\})$.

If $A = (0,\infty)$ and $B = \mathbb{R}$, then this definition is exactly the same as that for the standard Ulam stability. See [1,4,5,6,8,12,15,17,18,19] for previous studies on standard and conditional Ulam stabilities. In [16], the present author obtained the following results.

Theorem 2.2 ([16, Theorem 5.1])**.** *Suppose that there exists* $\underline{h} > 0$ *such that*

$$h(t) \ge \underline{h} \quad \textit{for} \quad t \ge 0 .$$

Let $\varepsilon \in \left(0, \frac{\alpha \underline{h} K^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, $z_0 \in \left[\left(\frac{K}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, *let* $z \in C^1[0,T_z)$ *be the solution of* (2.1) *with* $z(0) = z_0$, *and let* $\zeta \in C^1[0,T_\zeta)$ *be the solution of the perturbed* *nonautonomous Richards model*

$$\zeta' = h(t)\zeta\Big(1 - \frac{\zeta^\alpha}{K}\Big) + p(t)\,, \quad |p(t)| \le \varepsilon \tag{2.3}$$

with $\zeta(0) = z_0$, where $p(t)$ is a real-valued continuous function. Then the global existence of the solutions of (2.1) and (2.3) with $z(0) = \zeta(0) = z_0$ is guaranteed. That is, $T_z = T_\zeta = \infty$ holds. Furthermore,

$$|\zeta(t) - z(t)| \le \max\Big\{\frac{\alpha+1}{\alpha\underline{h}}, \frac{\alpha+1}{\alpha^2\underline{h}}\Big\}\varepsilon \quad \textit{for} \quad t \in [0,\infty)\,.$$

Theorem 2.3 ([16, Theorem 5.2]). *Suppose that there exists $\underline{h} > 0$ such that*

$$h(t) \ge \underline{h} \quad \textit{for } t \ge 0.$$

Let $A = \Bigg(0, \frac{\alpha\underline{h}K^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\Bigg]$ and $B = \Big[\Big(\frac{K}{\alpha+1}\Big)^{\frac{1}{\alpha}}, \infty\Big)$. Then (2.1) is conditionally Ulam stable on $[0,\infty)$ with A in the class $\mathcal{C}_B$. Furthermore, $L = \max\Big\{\frac{\alpha+1}{\alpha\underline{h}}, \frac{\alpha+1}{\alpha^2\underline{h}}\Big\}$ is an Ulam constant on $[0,\infty)$.

In this study, we will especially use Theorem 2.2, which is clearly given the initial values, to help analyze the approximate solutions of (1.1).

3. Proof of main result

Using the polar transformation $x = r\cos\theta$ and $y = r\sin\theta$ to (1.1) and (1.2), we obtain the systems

$$\begin{aligned} r' &= f(t)r\Big(1 - \frac{r^\alpha}{\kappa}\Big)\,,\\ r\theta' &= -g(t)r\,, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} r' &= f(t)r\Big(1 - \frac{r^\alpha}{\kappa}\Big) + p_1(t)\cos\theta + p_2(t)\sin\theta,\\ r\theta' &= -g(t)r - p_1(t)\sin\theta + p_2(t)\cos\theta \end{aligned} \tag{3.2}$$

for $t \ge 0$, respectively. In this section, first we present the proof of main result by using (3.1), (3.2) and Theorem 2.2.

Proof of Theorem 1.1. Suppose that there exists $\underline{f} > 0$ such that (1.5) holds. Given an arbitrary $\varepsilon \in \Bigg(0, \frac{\alpha\underline{f}\kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\Bigg]$, suppose that (1.7) holds. Let $(x(t), y(t))$ and $(\xi(t), \eta(t))$ be the solutions of (1.1) and (1.2) with (1.6) and

$$\|(x_0, y_0)\| \in \Big[\Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}}, \infty\Big)\,,$$

respectively. Now we consider the solution $(\rho(t),\phi(t))$ of (3.2) corresponding to $(\xi(t),\eta(t))$. Then, by (1.7), we see that

$$\begin{aligned}\left\|\begin{pmatrix}\rho'-f(t)\rho\left(1-\frac{\rho^\alpha}{\kappa}\right)\\ \rho\big(\phi'+g(t)\big)\end{pmatrix}\right\| &= \left\|\begin{pmatrix}p_1(t)\cos\phi+p_2(t)\sin\phi\\ -p_1(t)\sin\phi+p_2(t)\cos\phi\end{pmatrix}\right\|\\ &= \sqrt{(p_1(t)\cos\phi+p_2(t)\sin\phi)^2+(-p_1(t)\sin\phi+p_2(t)\cos\phi)^2}\\ &= \sqrt{p_1^2(t)+p_2^2(t)}=\|\big(p_1(t),p_2(t)\big)\|\le\varepsilon \quad \text{for} \quad t\ge 0\,.\end{aligned}$$

Hence we obtain

$$\left|\rho'-f(t)\rho\left(1-\frac{\rho^\alpha}{\kappa}\right)\right|\le\left\|\begin{pmatrix}\rho'-f(t)\rho\left(1-\frac{\rho^\alpha}{\kappa}\right)\\ \rho\big(\phi'+g(t)\big)\end{pmatrix}\right\|\le\varepsilon \quad \text{for} \quad t\ge 0\,. \tag{3.3}$$

Moreover, by (1.6) we know that

$$\rho(0)=\|\big(\xi(0),\eta(0)\big)\|=\|(x_0,y_0)\|\in\left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},\infty\right).$$

Next we consider the solution $(r(t),\theta(t))$ of (3.1) corresponding to $(x(t),y(t))$. Then, from (1.6) it follows that

$$r(0)=\|(x(0),x(0))\|=\rho(0).$$

Using Theorem 2.2 with $h=f$, $K=\kappa$, $z=r$, $\zeta=\rho$, we conclude that $r(t)$ and $\rho(t)$ exist on $[0,\infty)$ and

$$|\rho(t)-r(t)|\le\max\left\{\frac{\alpha+1}{\alpha\underline{f}},\frac{\alpha+1}{\alpha^2\underline{f}}\right\}\varepsilon \quad \text{for} \quad t\in[0,\infty)\,;$$

that is,

$$\big|\|\big(\xi(t),\eta(t)\big)\|-\|\big(x(t),y(t)\big)\|\big|\le\max\left\{\frac{\alpha+1}{\alpha\underline{f}},\frac{\alpha+1}{\alpha^2\underline{f}}\right\}\varepsilon \quad \text{for} \quad t\in[0,\infty)\,.$$

Note here that $r(t)$ and $\rho(t)$ are non-negative on $[0,\infty)$ because $r(t)=\|(x(t),y(t))\|$ and $\rho(t)=\|(\xi(t),\eta(t))\|$.

Next, we will show that

$$\|(\xi(t),\eta(t))\|=\rho(t)\ge\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} \quad \text{for} \quad t\in[0,\infty)\,.$$

To prove this fact, we assume that there exists $t_1>0$ such that

$$\rho(t_1)<\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}.$$

From the continuity of $\rho(t)$, $\rho(t)$ is negative near $t=t_1$. Using this with $\rho(0)\in\left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},\infty\right)$, we see that there exists $0\le t_2<t_1$ such that

$$\rho(t_2)=\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},$$

and

$$\rho(t) < \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} \quad \text{for} \quad t \in (t_2, t_1]\,. \tag{3.4}$$

Now we consider the case (i) $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right)$. From (1.5) and (3.3), we have

$$\begin{aligned}\rho'(t_2) &\geq f(t_2)\rho(t_2)\Big(1 - \frac{\rho^\alpha(t_2)}{\kappa}\Big) - \varepsilon = f(t_2)\frac{\alpha\kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} - \varepsilon \\ &\geq \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} - \varepsilon > 0\,.\end{aligned}$$

From this with the continuity of $\rho'(t)$, $\rho'(t)$ is positive near $t = t_2$. Thus, we see that there exists $0 < \delta \leq t_1 - t_2$ such that

$$\rho'(t) > 0 \quad \text{for} \quad t \in [t_2, t_2 + \delta]\,.$$

Therefore,

$$\rho(t) \geq \rho(t_2) = \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} \quad \text{for} \quad t \in [t_2, t_2 + \delta]\,.$$

This contradicts (3.4).

Next we consider the case (ii) $\varepsilon = \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}$. From $\rho \geq 0$, (1.5) and (3.4), we have

$$f(t)\rho(t)\Big(1 - \frac{\rho^\alpha(t)}{\kappa}\Big) \geq f(t)\rho(t)\Big(1 - \frac{1}{\alpha+1}\Big) > \frac{\alpha \underline{f}}{\alpha+1}\rho(t) \geq 0 \quad \text{for} \quad t \in (t_2, t_1]\,.$$

Hence, by (3.3), we have

$$\begin{aligned}\left(\rho(t)e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}\right)' &= \left(\rho'(t) - \frac{\alpha \underline{f}}{\alpha+1}\rho(t)\right)e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} \\ &> \left[\rho'(t) - f(t)\rho(t)\left(1 - \frac{\rho^\alpha(t)}{\kappa}\right)\right]e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} \\ &\geq -\varepsilon e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}\end{aligned}$$

for $t \in (t_2, t_1]$. Integrating this inequality and using

$$\rho(t_2) = \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad \varepsilon = \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}},$$

we obtain

$$\begin{aligned}\rho(t)e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} &> \rho(t_2) + \frac{\alpha+1}{\alpha \underline{f}}\varepsilon\left(e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} - 1\right) \\ &= \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} + \frac{\alpha+1}{\alpha \underline{f}}\frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\left(e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} - 1\right) \\ &= \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}\end{aligned}$$

for $t \in (t_2, t_1]$. This contradicts (3.4). Hence we can conclude that

$$\rho(t) \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} \quad \text{for } \ t \in [0,\infty).$$

If $p_1 = p_2 \equiv 0$, then $\rho \equiv r$. Therefore,

$$r(t) \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} \quad \text{for } \ t \in [0,\infty).$$

This completes the proof of Theorem 1.1. □

4. Second main result

By assuming stronger condition to $\|(p_1(t), p_2(t))\|$, we can also obtain a relationship between θ and ϕ. The following theorem is the second main result in this paper.

Theorem 4.1. *Suppose that there exists $\underline{f} > 0$ such that (1.5) holds. Let $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, $\|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, let $(x(t), y(t))$ and $(\xi(t), \eta(t))$ be the solutions of (1.1) and (1.2) with (1.6), respectively. If there exists $\beta > 1$ such that*

$$\|(p_1(t), p_2(t))\| \leq \frac{\varepsilon}{(t+1)^{\beta}} \quad \textit{for} \quad t \geq 0, \tag{4.1}$$

then the global existence of $(x(t), y(t))$ and $(\xi(t), \eta(t))$ is guaranteed. Furthermore, the following holds: Let $(r(t), \theta(t))$ and $(\rho(t), \phi(t))$ be the solutions of (3.1) and (3.2) corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$, respectively. Then $(r(t), \theta(t))$ and $(\rho(t), \phi(t))$ exist on $[0, \infty)$, and

$$\min\{\rho(t), r(t)\} \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},$$

$$|\rho(t) - r(t)| \leq \max\left\{\frac{\alpha+1}{\alpha \underline{f}}, \frac{\alpha+1}{\alpha^2 \underline{f}}\right\}\varepsilon \quad \textit{and} \quad |\phi(t) - \theta(t)| \leq \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta-1}$$

for $t \in [0, \infty)$.

Proof. Suppose that there exists $\underline{f} > 0$ such that (1.5) holds. Given an arbitrary $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, suppose that (1.7) holds. Let $(x(t), y(t))$ and $(\xi(t), \eta(t))$ be the solutions of (1.1) and (1.2) with (1.6) and $\|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, respectively. In addition, let $(r(t), \theta(t))$ and $(\rho(t), \phi(t))$ be the solutions of (3.1) and (3.2) corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$, respectively. Note that (4.1) implies (1.7). Then, using the same method as the proof of Theorem 1.1, we see that

$$r(0) = \rho(0) = \|(\xi(0), \eta(0))\| = \|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right);$$

$$|\rho(\phi' + g(t))| \le \left\| \begin{pmatrix} \rho' - f(t)\rho\left(1 - \frac{\rho^\alpha}{\kappa}\right) \\ \rho(\phi' + g(t)) \end{pmatrix} \right\| \le \frac{\varepsilon}{(t+1)^\beta} \quad \text{for} \quad t \ge 0; \tag{4.2}$$

and $r(t)$ and $\rho(t)$ exist on $[0, \infty)$; and

$$\min\{\rho(t), r(t)\} \ge \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \tag{4.3}$$

and

$$|\rho(t) - r(t)| \le \max\left\{\frac{\alpha+1}{\alpha \underline{f}}, \frac{\alpha+1}{\alpha^2 \underline{f}}\right\} \varepsilon \quad \text{for} \quad t \in [0, \infty).$$

Next we will prove that $\theta(t)$ and $\phi(t)$ exist on $[0, \infty)$ and

$$|\phi(t) - \theta(t)| \le \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta - 1} \quad \text{for } t \in [0, \infty).$$

Define

$$q(t) := \rho(t)(\phi'(t) + g(t))$$

for $t \in [0, \infty)$. Then, by (4.2), we have

$$|q(t)| \le \frac{\varepsilon}{(t+1)^\beta} \quad \text{for} \quad t \ge 0.$$

Since $\rho(t)$ is positive on $[0, \infty)$, we can solve the above differential equation. Then we obtain

$$\phi(t) = \phi(0) + \int_0^t \left(g(s) + \frac{q(s)}{\rho(s)}\right) ds \quad \text{for } t \in [0, \infty).$$

Because $\rho(t)$ exists on $[0, \infty)$, $\phi(t)$ exists on $[0, \infty)$. Obviously, $\theta'(t) + g(t) = 0$ is also solved and we obtain

$$\theta(t) = \theta(0) + \int_0^t g(s) ds \quad \text{for } t \in [0, \infty).$$

By (1.6), we have $\phi(0) = \theta(0)$, and so that

$$|\phi(t) - \theta(t)| \le \int_0^t \frac{|q(s)|}{\rho(s)} ds \le \int_0^t \frac{\varepsilon}{\rho(s)(s+1)^\beta} ds \quad \text{for } t \in [0, \infty).$$

From this with (4.3) it follows that

$$\begin{aligned} |\phi(t) - \theta(t)| &\le \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \varepsilon \int_0^t \frac{1}{(s+1)^\beta} ds \\ &\le \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta-1} \left[1 - \frac{1}{(t+1)^{\beta-1}}\right] \\ &< \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta-1} \end{aligned}$$

for $t \in [0, \infty)$. This completes the proof of Theorem 4.1. $\square$

5. Examples

In this section, we will present two examples of variable coefficients. Let (x_0, y_0) on the circle $C_{\frac{1}{\sqrt{3}}}$ or, be outside the circle $C_{\frac{1}{\sqrt{3}}}$. Consider the the solutions $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.8) and

$$\alpha = 2, \quad \kappa = 1, \quad f(t) = \frac{|\sin t|}{t+1} + 1, \quad g(t) = \cos t + 0.5, \tag{5.1}$$

respectively. From $\underline{f} = 1$ and $\|(p_1(t), p_2(t))\| \le \frac{2}{3\sqrt{3}}$ for $t \ge 0$, we can choose $\varepsilon = \frac{2}{3\sqrt{3}}$, and using Theorem 1.1, we conclude that $(x(t), y(t))$ and $(\xi(t), \eta(t))$ on the circle $C_{\frac{1}{\sqrt{3}}}$ or, are outside of $C_{\frac{1}{\sqrt{3}}}$, and (1.9) holds. Figure 4 shows the orbits corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.8), (1.12) and (5.1), for $0 \le t \le 150$. From (3.1), we see that (1.1) has a limit cycle as the unit circle. Thus, the orbit of $(x(t), y(t))$ represent the limit cycle, and the orbit of $(\xi(t), \eta(t))$ represent an approximation of the limit cycle. Note that $(x(t), y(t))$ rotates in the opposite direction each time the sign of g changes on the unit circle. However, because $\lim_{t\to\infty} \theta(t) = -\infty$ holds, we will call the unit circle the limit cycle here.

Next we consider the the solutions $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.12), (5.1) and

$$p_1(t) = \frac{2}{3\sqrt{3}(t+1)^2}\left(1 - 2\max\left\{\cos\sqrt{t}, 0\right\}\right) \quad \text{and} \quad p_2(t) = 0, \tag{5.2}$$

respectively. From $\underline{f} = 1$ and

$$\|(p_1(t), p_2(t))\| \le \frac{2}{3\sqrt{3}(t+1)^2} \le \frac{2}{3\sqrt{3}} \quad \text{for} \quad t \ge 0,$$

we can choose $\varepsilon = \frac{2}{3\sqrt{3}}$ and $\beta = 2$. Let $(r(t), \theta(t))$ and $(\rho(t), \phi(t))$ be the solutions of (3.1) and (3.2) corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$, respectively. Using Theorem 4.1, we have

$$|\rho(t) - r(t)| \le \frac{3}{2}\varepsilon = \frac{1}{\sqrt{3}} \quad \text{and} \quad |\phi(t) - \theta(t)| \le \sqrt{3}\varepsilon = \frac{2}{3}$$

for $t \in [0, \infty)$. Figure 5 shows the orbits corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.12), (5.1) and (5.2), for $0 \le t \le 50$.

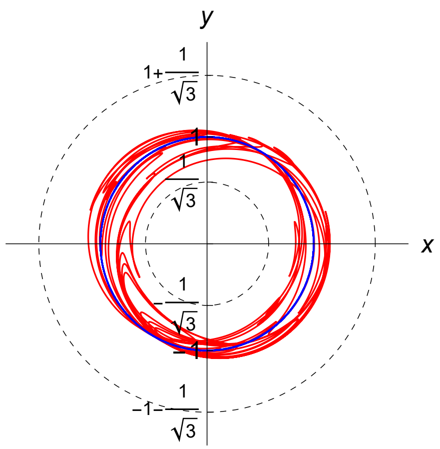

FIG. 4: Orbits of (1.1) (blue) and (1.2) (red) with (1.6), (1.8), (1.12) and (5.1).

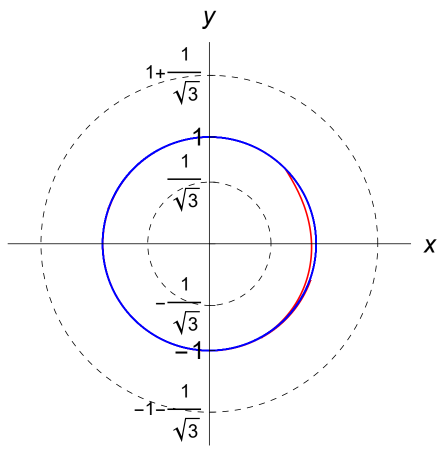

FIG. 5: Orbits of (1.1) (blue) and (1.2) (red) with (1.6), (1.12), (5.1) and (5.2).

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Department of Applied Mathematics,
Okayama University of Science,
Okayama, 700-0005, Japan
E-mail: onitsuka@ous.ac.jp; onitsuka@xmath.ous.ac.jp

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NONLOCAL SEMILINEAR SECOND-ORDER DIFFERENTIAL INCLUSIONS IN ABSTRACT SPACES WITHOUT COMPACTNESS

Martina Pavlačková and Valentina Taddei

Abstract. We study the existence of a mild solution to the nonlocal initial value problem for semilinear second-order differential inclusions in abstract spaces. The result is obtained by combining the Kakutani fixed point theorem with the approximation solvability method and the weak topology. This combination enables getting the result without any requirements for compactness of the right-hand side or of the cosine family generated by the linear operator.

1. Introduction

The main goal of the paper is to investigate the existence of a solution to the following nonlocal initial value problem for semilinear second-order differential inclusion in a Banach space

$$\begin{cases} \ddot{x}(t) \in Ax(t) + F(t, x(t))\,, & \text{for a.a. } \ t \in [0,T]\,, \\ x(0) = g(x) \ \ \dot{x}(0) = h(x)\,. \end{cases} \tag{1.1}$$

Throughout the paper, we assume that

(i) E is a reflexive Banach space having a Schauder basis;

(ii) $A\colon D(A) \subset E \to E$ is a closed linear densely defined operator generating a cosine family $\{C(t)\}_{t\in\mathbb{R}}$;

(iii) $F\colon [0,T] \times E \multimap E$ is a multivalued mapping with nonempty, bounded, closed and convex values;

(iv) g, $h\colon C([0,T],E) \to E$.

Differential equations and inclusions in Banach spaces have been attracting quite big attention (see, e.g., [1,2,5,13,23,24]). In particular, as pointed out by Byszewski and Lakshmikantham in [11], the study of nonlocal conditions is of significance due to their applicability in many physical and engineering problems and also in other areas of applied mathematics. Since then several authors have been investigated

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problems with nonlocal initial conditions for different classes of abstract differential equations or inclusions (see, e.g., [4, 12, 14]).
One of the key tools that will be used in the paper is an approximation solvability method that was introduced in [6] to study fully nonlinear first-order problems in Hilbert spaces. Its application was afterwards extended to first-order semilinear problems in Banach spaces in [8] and to fully nonlinear second-order problems in Hilbert spaces in [7]. Recently, it was applied to Cauchy problems for semilinear second-order differential inclusions in [20].
Motivated by the above works, the main objective of this paper is proving the existence of a mild solution to the second-order semilinear differential inclusion in a Banach space satisfying nonlocal conditions without converting it into first-order problem. To obtain desired results, we will transfer the original problem into a sequence of problems in finite dimensional spaces using the approximation solvability method. Afterwards, the solvability of approximating problems will be shown by the Kakutani fixed point problem for multivalued mappings. Finally, the convergence of obtained solutions to the solution of the original problem will be proven. This procedure will enable to obtain the existence result under easily verifiable and not restrictive conditions on the cosine family generated by the linear operator or on the right-hand side and to avoid any requirement for compactness.

2. Preliminaries

In this section, the basic notions dealing with natural projections and cosine families will be mentioned.

A sequence $\{e_n\}_n$ of vectors in E is a *Schauder basis* for E if, for every $x \in E$, there exists a unique sequence of real numbers $\alpha_n = \alpha_n(x)$, $n \in \mathbb{N}$, such that $\|x - \sum_{i=1}^{n} \alpha_i e_i\| \to 0$, as $n \to \infty$.
Given a Schauder basis $\{e_n\}_n$ for E, let $E_n = \operatorname{span}\{e_1, \dots, e_n\}$ denote the n-dimensional Banach space generated by the first n vectors of the basis, and let $\mathbb{P}_n \colon E \to E_n$ be the *natural projection* of E onto E_n, i.e., $\mathbb{P}_n \left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k e_k$. It holds that $\mathbb{P}_n$ is linear and bounded, for every $n \in \mathbb{N}$, and that the sequence $\{\|\mathbb{P}_n\|\}_n$ is bounded, i.e. that there exists $K \geq 1$ such that

$$\|\mathbb{P}_n(x)\| \leq K\|x\| \quad \forall n \in \mathbb{N}\,,\ \forall\, x \in E\,.$$

For the main properties of the projection $\mathbb{P}_n$, we remind to [8], [9] and [19]. We recall, in particular, that if $x_n \rightharpoonup x$, then $\mathbb{P}_n(x_n) \rightharpoonup x$.
A one parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself is called a *strongly continuous cosine family* if

- $C(t + s) + C(s - t) = 2C(s)C(t)$, for all t, $s \in \mathbb{R}$;
- $C(0) = I$;
- the map $t \to C(t)x$ is continuous in $\mathbb{R}$, for each fixed $x \in E$.

If $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family, then there exist $M \geq 1$ and $\omega \geq 0$ such that, for all $t \in \mathbb{R}$,

$$\|C(t)\| \leq M e^{\omega|t|}\,. \tag{2.1}$$

We also recall that the map $c\colon [0,T]\times E\to E$ defined as $c(t,x)=C(t)x$ is continuous (see [20, Lemma 3]).
The one parameter family $\{S(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself defined, for all $t\in\mathbb{R}$ and $x\in E$, by

$$S(t)x=\int_0^t C(s)x\,ds$$

is called the *strongly continuous sine family* associate to the cosine family. It follows from the definition of $\{S(t)\}_{t\in\mathbb{R}}$ that, for every $t\in[0,T]$,

$$\|S(t)\|\le K_0\,,$$

where

$$K_0=\begin{cases} M\frac{e^{\omega T}-1}{\omega} & \text{if}\quad \omega\neq 0\\ MT & \text{if}\quad \omega=0\,.\end{cases}$$

For more information about sine and cosine families and their properties, see, e.g., [22].

The notion of a solution to (1.1) will be understood in a mild sense. Namely, by a *mild solution* of the problem (1.1) we mean a continuous function $x\colon[0,T]\to E$ such that, for all $t\in[0,T]$,

$$x(t)=C(t)\,g(x)+S(t)h(x)+\int_0^t S(t-s)f(s)\,ds\,,$$

where

$$f\in S^1_{F,\,x}=\{f\in L^1([0,T],E):f(t)\in F(t,x(t)),\ \text{for a.a. } t\in[0,T]\}\,.$$

3. Existence result

Theorem 3.1. *Consider the problem* (1.1) *and let $F:[0,T]\times E\multimap E$ satisfies the following assumptions:*

- $(F1)$ *$F(t,x)$ is nonempty, convex, closed, and bounded, for every $t\in[0,T]$ and $x\in E$,*
- $(F2)$ *for every $x\in E$, $F(\cdot,x)$ has a measurable selection,*
- $(F3)$ *for a.a. $t\in[0,T]$, $F(t,\cdot)\colon E^w\multimap E^w$, where E^ω denotes the space E endowed with the weak topology, is u.s.c.,*
- $(F4)$ *for every $n\in\mathbb{N}$, there exists $\varphi_n\in L^1([0,T],\mathbb{R})$, with*

$$\liminf_{n\to\infty}\frac{\|\varphi_n\|_{L^1}}{n}=0\,,$$

such that

$$\|z\|\le\varphi_n(t),$$

for a.a. $t\in[0,T]$, every $x\in E$ with $\|x\|\le n$ and every $z\in F(t,x)$.

Moreover let g and h satisfy:

- $(gh1)$ *$g,\ h\colon C([0,T],E)^w\to E^w$ are continuous;*

$(gh2)$

$$\lim_{n\to\infty}\frac{L_n}{n}=R\,,$$

where

$$L_n=\max\Big\{\sup_{\|x\|\le n}\|g(x)\|,\ \sup_{\|x\|\le n}\|h(x)\|\Big\},$$

with

$$R<\frac{1}{K(KMe^{\omega T}+K_0)}\,.$$

Then the problem (1.1) *has a solution.*

Proof. In order to get the existence of a solution to the problem (1.1), we will use the approximation solvability method. Thus, for each $m\in\mathbb{N}$, consider the multimap $G_m:[0,T]\times E\to E_m$ defined as $G_m=\mathbb{P}_m\circ F$ and the operator $\Sigma_m:C([0,T],E_m)\multimap C([0,T],E_m)$ defined as

$$\Sigma_m(q)(t)=\Big\{\mathbb{P}_mC(t)\,\mathbb{P}_mg(q)+\mathbb{P}_mS(t)h(q)+\int_0^t\mathbb{P}_mS(t-s)f(s)\,ds : f\in S^1_{G_m,\,q}\Big\}\,.$$

Let us note that the existence of a selection $f\in S^1_{G_m,\,q}$ is guaranteed, e.g., by [9, Proposition 2.2].

In order to show that Σ_m has a fixed point, we will prove that it satisfies all assumptions of the Kakutani fixed point theorem ([17, Theorem 1]). For this purpose, given $n\in\mathbb{N}$, we use the following notation

$$nB_m=\{q\in C([0,T];E_m):\|q(t)\|\le n,\quad \text{for every } t\in[0,T]\}\,.$$

Notice that $\Sigma_m(q)=\Sigma^1_m(q)+\Sigma^2_m(q)$, where Σ^1_m is a single valued map defined as

$$\Sigma^1_m(q)(t)=\mathbb{P}_mC(t)\,\mathbb{P}_mg(q)+\mathbb{P}_mS(t)h(q)\,,$$

while Σ^2_m is a multivalued map defined as

$$\Sigma^2_m(q)=\Big\{\int_0^t\mathbb{P}_mS(t-s)f(s)\,ds : f\in S^1_{G_m,\,q}\Big\}\,.$$

In [20, Theorem 1], we proved a result similar to the present one in the case when the non-linear term depends also on the first derivative, but the nonlocal conditions are replaced by the Cauchy conditions. In this proof, we shall outline only the differences with respect to the proof of the quoted result. In particular, it is possible to prove by using [18, Theorem 5.1.1] together with [20, Theorem 1] that Σ^2_m has convex values, a closed graph and that it maps bounded sets into relatively compact sets. Let us now prove that Σ^1_m satisfies the same properties. Clearly, Σ^1_m is convex valued, because it is single valued.

Assume that $\big(q_k,\,\Sigma^1_m(q_k)\big)\to(q,\,y)$ in $C([0,T],E_m)\times C([0,T],E_m)$, and let us prove that $y=\Sigma^1_m\,(q)$. According to $(gh1)$ and the boundedness of $C(t),S(t)$ and $\mathbb{P}_m$, since E_m is finite dimensional, it follows that

$$\Sigma^1_m(q_k)(t)\to\mathbb{P}_mC(t)\,\mathbb{P}_mg(q)+\mathbb{P}_mS(t)h(q)\,,$$

for every $t\in[0,T]$. Since the convergence in $C([0,T],E_m)$ implies the pointwise convergence, we get that $y=\Sigma^1_m\,(q)$, i.e. that Σ^1_m has a closed graph.

Take now $n \in \mathbb{N}$. Condition $(gh2)$ implies that $\{g(x) : x \in nB_m\}$ is bounded, thus $A = \{\mathbb{P}_m g(x) : x \in nB_m\}$ is relatively compact, because E_m is finite dimensional. Since $(t,x) \to C(t)x$ is continuous, it is uniformly continuous in the compact set $[0,T] \times \overline{A}$. We then get that, for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every t, $t_0 \in [0,T]$, $x \in nB_m$

$$\|C(t)\mathbb{P}_m g(x) - C(t_0)\mathbb{P}_m g(x)\| \leq \epsilon\,.$$

Moreover, for every $x \in nB_m$ there exists $\frac{d}{dt}S(t)h(x) = C(t)h(x)$. (2.1) and $(gh2)$ then imply that

$$\left\| \frac{d}{dt} S(t)h(x) \right\| \leq Me^{\omega T} L_n\,,$$

for every $t \in [0,T]$, $x \in nB_m$. Therefore, Σ_m^1 is equicontinuous in nB_m, for every $n \in \mathbb{N}$.

In order to show that Σ_m maps bounded sets into bounded sets and that there exists a bounded set $D \subset C^1([0,T]; E_m)$ such that $\Sigma_m(D) \subset D$, it is sufficient to notice that, according to $(F4)$ and $(gh2)$ for every n, $m \in \mathbb{N}$, $q \in nB_m$ and $h \in \Sigma_m(q)$, there exists $f \in S^1_{G_m,\,q}$ and $L_n \in \mathbb{R}$, $\varphi_n \in L^1([0,T],\mathbb{R})$ such that, for every $t \in [0,T]$, the following holds

$$\begin{aligned}
\|h(t)\| &\leq \|\mathbb{P}_m\|^2 \|C(t)\| \|g(q)\| + \|\mathbb{P}_m\| \|S(t)\| \|h(q)\| \\
&\quad + \int_0^T \|\mathbb{P}_m\| \|S(t-s)\| \|f(s)\|\, ds \\
&\leq K^2 M e^{\omega T} L_n + K K_0 L_n + K K_0 \|\varphi_n\|_{L^1}\,.
\end{aligned}$$

Therefore,

$$\|h\|_C \leq K^2 M e^{\omega T} L_n + K K_0 L_n + K K_0 \|\varphi_n\|_{L^1}$$

for every m, $n \in \mathbb{N}$, $q \in nB_m$, $h \in \Sigma_m(q)$. In particular, Σ_m^1 maps bounded sets into bounded and equicontinuous sets, i.e. relatively compact sets in the space $C([0,T],E_m)$.

Take $N > 0$ such that

$$\frac{L_N}{N} < \frac{1}{K(KMe^{\omega T} + K_0)} \quad \text{and} \quad \frac{\|\varphi_N\|_{L^1}}{N} < \frac{1}{KK_0}\left[1 - \frac{L_N}{N} K(KMe^{\omega T} + K_0)\right].$$

Such N exists because of $(F4)$ and $(gh2)$. Afterwards,

$$\frac{K^2 M e^{\omega T} L_N + K K_0 L_N + K K_0 ||\varphi_N||_{L^1}}{N} < 1\,,$$

which guarantees that $\Sigma_m(NB_m) \subset NB_m$, for all $m \in \mathbb{N}$.

Since Σ_m is closed and maps bounded sets into relatively compact sets, it has compact values; hence, it is u.s.c. Thus, $\Sigma_m : NB_m \multimap NB_m$ is a u.s.c. compact map with convex and closed values. Applying the Kakutani fixed point theorem, we obtain that, for all $m \in \mathbb{N}$, the operator Σ_m has a fixed point q_m. Because of the technique used, we are also able to localize the fixed point in the set

$$NB = \{q \in C([0,T],E) : \|q(t)\| \leq N,\ \text{for every } t \in [0,T]\}\,.$$

Let us now prove that the sequence $\{q_m\}_m$ found in previous step admits a subsequence pointwise weakly converging to a solution q of Problem (1.1).
The sequence $\{q_m\}_m$ satisfies, for all $m \in \mathbb{N}$ and $t \in [0,T]$,

$$q_m(t) = \mathbb{P}_m C(t)\, \mathbb{P}_m g(q_m) + \mathbb{P}_m S(t) h(q_m) + \int_0^t \mathbb{P}_m S(t-s) f_m(s)\, ds\,,$$

where $f_m \in S^1_{G_m, q_m}$, for every $m \in \mathbb{N}$.
Reasoning like in [20, Theorem 1], it is possible to prove that there exists a subsequence, still denoted as the sequence, and a function $f \in L^1([0,T], E)$ such that

$$\int_0^t \mathbb{P}_m S(t-s) f_m(s)\, ds \rightharpoonup \int_0^t S(t-s) f(s)\, ds\,,$$

for every $t \in [0,T]$.
Now, according to $(gh2)$, since $q_m \in NB$ for every $m \in \mathbb{N}$ and E is reflexive, there exists a subsequence, still denoted as the sequence, and $\overline{g}, \overline{h} \in E$ such that

$$g(q_m) \rightharpoonup \overline{g} \text{ and } h(q_m) \rightharpoonup \overline{h}\,,$$

which implies that

$$\mathbb{P}_m C(t)\, \mathbb{P}_m g(q_m) + \mathbb{P}_m S(t) h(q_m) \rightharpoonup C(t)\overline{g} + S(t)\overline{h}\,,$$

for every $t \in [0,T]$, i.e. that

$$q_m(t) \rightharpoonup q(t) = C(t)\overline{g} + S(t)\overline{h} + \int_0^t S(t-s) f(s)\, ds.$$

Thus, $q_m \rightharpoonup q$ in $C([0,T], E)$ (see [10, Theorem 4.3]). Hence, according to $(gh1)$, $\overline{g} = g(q)$ and $\overline{h} = h(q)$, while, reasoning like in the proof of [20, Theorem 1] we get that $f \in S^1_{F,\,q}$, and the proof is complete. $\square$

Remark 3.2. Let us note that assumption $(gh2)$ is satisfied, e.g., when (cf. assumption $(gh2)$ in [12]):

$(gh2^{'})$ there exists $Q > 0$ such that $\|g(q)\| \le Q$ and $\|h(q)\| \le Q$, for all $q \in C([0,T]; E)$.

In such a case, $R = 0$ and Theorem 1 can be proved also replacing condition $(F4)$ by the following one:

$(F4^{'})$ There exist $\alpha \in L^1([0,T], E)$ such that

$$\|z\| \le \alpha(t)(1 + \|x\|)\,,$$

for a.a. $t \in [0,T]$, every $x \in E$ and every $z \in F(t,x)$.

The only difference with respect to the proof of Theorem 1 concerns, in this case, the existence of a bounded set $H \subset C([0,T], E)$ such that, for every $m \in \mathbb{N}$, Σ_m maps $H \cap C([0,T], E_m)$ into itself. On this purpose, it is sufficient to reason like in [20, Theorem 2], observing that, denoted, for every fixed $j \in \mathbb{N}$,

$$q_j = \max_{t \in [0,T]} \int_0^T e^{-j(t-s)} \chi_{[0,t]}(s) \alpha(s)\, ds,$$

it is possible to prove that there exists a subsequence, still denoted as the sequence, such that $q_j \to 0$. Now take

$$H = \left\{x \in C([0,T],E) : \max_{t\in[0,T]} e^{-\bar{j}t}\|x(t)\| \le R\right\}$$

where $\bar{j} \in \mathbb{N}$ and $R \in \mathbb{R}$ are chosen such that

$$1 - KK_0 q_{\bar{j}} > 0\,,$$

and

$$R > \frac{K^2 M e^{\omega T} Q + KK_0 Q + KK_0\|\alpha\|_{L^1}}{1 - K_2 q_{\bar{j}}}\,.$$

Remark 3.3. We point out that our existence result is proved under quite weak assumptions. Indeed, similar results are obtained in literature for even more general equations and boundary conditions, but all with very strong assumptions.
In [3], an additional term $B\dot{x}$, with B linear and bounded, appears while $h(x) \equiv x_1 \in E$, but the authors assume that $C(t)$ is compact for every t. In [12], A generates a fundamental system and g and h are assumed bounded. Moreover, they have to satisfy, as well as F, a condition involving the Hausdorff measure of noncompactness. In [14,15,16], the left-hand side is of type $\frac{d}{dt}(\dot{x}(t) - p(t,x,\dot{x}))$ or the right-hand side is of type $F(t,x,x(a(t)),\dot{x},\dot{x}(b(t)))$ or $F(t,N(t)x)$ and g and h may depend also on $\dot{x}$. However, the existence results there are proved assuming that $S(t)$ is compact, for every t, or that F maps bounded sets into relatively compact ones, eventually that it satisfies a condition involving the Hausdorff measure of noncompactness. Moreover, g and h are assumed completely continuous and bounded or globally Lipschitz continuous. In [21], the nonlinear term depends also on the weighted average of the solution and g and h depends also on $\dot{x}$, but the nonlinear term, g and h are assumed globally Lipschitz continuous.

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Department of Applied Mathematics,
Moravian Business College Olomouc,
tř. Kosmonautů 1288/1, 779 00 Olomouc, Czech Republic
E-mail: pavlackovam@centrum.cz

Department of Sciences and Methods for Engineering,
University of Modena and Reggio Emilia,
Via G. Amendola, 2 - pad. Morselli, I-42122 Reggio Emilia, Italy
E-mail: valentina.taddei@unimore.it

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AROUND CERTAIN CRITICAL CASES IN STABILITY STUDIES IN HYDRAULIC ENGINEERING

Vladimir Rasvan

Abstract. It is considered the mathematical model of a benchmark hydroelectric power plant containing a water reservoir (lake), two water conduits (the tunnel and the turbine penstock), the surge tank and the hydraulic turbine; all distributed (Darcy-Weisbach) and local hydraulic losses are neglected,the only energy dissipator remains the throttling of the surge tank. Exponential stability would require asymptotic stability of the difference operator associated to the model. However in this case this stability is "fragile" i.e. it holds only for a rational ratio of the two delays, with odd numerator and denominator also. Otherwise this stability is critical (non-asymptotic and displaying an oscillatory mode).

1. Introduction. Problem statement

This paper has two starting points and the outcome is twofold. The first statement is explained below, the second one will be revealed towards the final part. Starting with the papers of A. D. Myshkis and his co-workers e.g. [1] and also with the papers of K. L. Cooke and his co-workers e.g. [3] the following methodology was established to deal with qualitative theory for non-standard BVP (Boundary Value Problems) for $1D$ hyperbolic PDEs (Partial Differential Equations). Integrating along the characteristics, a system of FDE (Functional Differential Equations) was associated to the BVP with initial conditions and the Cauchy problem (with initial conditions) for the FDEs. Consequently, any result obtained for one of the aforementioned mathematical objects is automatically projected back on the other one. Along almost half-century (starting from 1973-74) the author of this paper promoted this approach throughout his publications with reference to applications arising from Physics and Engineering, the most comprehensive presentation of the approach being given in [9], where the theorem of Cooke in [3] is proven completely.

Now we can turn to the qualitative problem of interest to us: (asymptotic) stability of the steady states (equilibria) for the BVPs mentioned above. This problem is reduced (equivalently) to the problem of the stability for the associated system of FDEs with deviated argument. Worth mentioning that in most applications the

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FDEs turn to be of neutral type. However the N(eutral)FDEs display a peculiar aspect of the stability problem. More precisely, if we consider even the simplest scalar NFDE, it is known - see e.g. [7], Corollary 1.7, p. 30 - that if the roots of its characteristic polynomial are located in $\mathbb{C}^-$ and its difference operator is stable, the stability is exponential. If the difference operator is unstable it is possible to have unbounded solutions while if the difference operator is in a critical case, the stability is at most non-exponential. Following the seminal papers of Hale and his co-workers (see [7] for complete references), the assumption on (strong) stability of the difference operator accompanied almost all development on NFDEs.

The present paper starts from the finding that, in spite of the aforementioned basic assumption on the difference operator, there exist important applications where it is not fulfilled. Various applications in Mechanical Engineering are modeled by NFDEs with the difference operator displaying critical stability [9]. Also Hydraulic Engineering (water hammer quenching, surge tank stability) is a source of such critically stable difference operators, but with even more interesting mathematical aspects. For this reason our choice went towards applications arising from Hydraulic Engineering.

2. Application description. The basic mathematical model

It is considered the standard hydroelectric plant composed of the water reservoir (lake), two water conduits (the tunnel and the penstock), the surge tank and the hydraulic turbine. The technological diagram can be seen in [10]. The mathematical model, considering distributed parameters of the water conduits, are as follows

$$
\begin{aligned}
&\partial_{\xi_i}\left(h_i + \tfrac{1}{2}\tfrac{T_{wi}}{T_i}q_i^2\right) + T_{wi}\partial_t q_i + \tfrac{1}{2}\tfrac{\lambda_i L_i}{D_i}\tfrac{T_{wi}}{T_i}q_i|q_i| = 0\,,\\
&\delta_i^2 T_{wi}\partial_t h_i + \partial_{\xi_i} q_i = 0 \ (\delta_i = T_{pi}/T_{wi}\ i = 1,2)\ ;\ h_1(0,t) \equiv 1\,,\\
&h_1(1,t) - \tfrac{1}{2}R_1\tfrac{T_{w1}}{T_1}q_1(1,t)|q_1(1,t)| = 1 + z(t) + R_s\tfrac{\mathrm{d}z}{\mathrm{d}t}\\
&\qquad = h_2(0,t) + \tfrac{1}{2}R_2\tfrac{T_{w2}}{T_2}q_2(0,t)|q_2(0,t)|\,,\\
&T_s\tfrac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = f_\theta\sqrt{h_2(1,t)}\,,\\
&T_a\tfrac{\mathrm{d}\varphi}{\mathrm{d}t} = q_2(1,t)h_2(1,t) - \nu_g\,.
\end{aligned}
\tag{2.1}
$$

The model contains rated state variables: the piezometric heads h_i $(i = 1,2)$ are rated to the lake head H_0; the water flows q_i are rated to $\bar{Q} = \alpha_q F_{\theta\,\max}\sqrt{H_0}$ - the maximal available flow at the turbine wicket gates; here $F_{\theta\,\max}$ is the maximal cross section area of the wicket gates and α_q - a flow coefficient; the rotating speed of the turbine is rated to the synchronous speed Ω_c and the available mechanical power to a resulting nominal power. The various time constants T_i, T_{wi}, T_{pi}, T_s, T_a are a result of the state variables rating (scaling) and they define (2.1) as a system with several time scales. The terms in λ_i define the so called Darcy-Weisbach hydraulic losses which are distributed along the water conduits. The terms in R_i define local hydraulic losses and the term in R_s defines the dynamic hydraulic losses due to the throttling of the surge tank. It has to be mentioned that the space coordinates along the water conduits are also rated to the lengths of the conduits $\xi_i = x_i/L_i$.

This basic model is rather complete and allows obtaining other models *via* various simplifying assumptions, which in most cases arise either from neglecting small terms or from engineering inferences suggested by the practical experience. In what follows we shall discuss the model adopted in [5], obtained from (2.1) by taking into account the following assumptions: i) the space variations of the dynamic heads $(1/2)(T_{wi}/T_i)q_i^2$ are negligible in comparison to the variations of the piezometric heads mainly during water hammer: according to [2], the variation of the piezometric head can reach several dozens of meters, while the variation of the dynamic head is at most 1 meter; this assertion is documented in [2] with exploitation data from hundreds of hydraulic power plants of the former USSR; ii) all distributed and local hydraulic losses are neglected, except the losses due to the throttling of the surge tank. Under these assumptions (2.1) become

$$\begin{aligned}
&\partial_{\xi_i} h_i + T_{wi}\partial_t q_i = 0\ ,\ \delta_i^2 T_{wi}\partial_t h_i + \partial_{\xi_i} q_i = 0\,,\\
&h_1(0,t) \equiv 1\ ;\ h_1(1,t) = 1 + z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = h_2(0,t)\,,\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = f_\theta\sqrt{h_2(1,t)}\,,\\
&T_a\frac{\mathrm{d}\varphi}{\mathrm{d}t} = q_2(1,t)h_2(1,t) - \nu_g\,.
\end{aligned} \tag{2.2}$$

This model is considered under water hammer: the water hammer is an abnormal regime generated by sudden load discharge at the hydraulic turbine level. Here, following [5], we shall consider the total turbine shutdown by complete instantaneous closing of the turbine wicket gates: $f_\theta \equiv 0$. Consequently the boundary condition of the penstock at $\xi_2 = 1$ becomes $q_2(1,t) \equiv 0$ and the turbine equation is "cut" (decoupled) from the rest of the model. The model is thus completely linear and represented by a non-standard BVP. We call it non-standard since the boundary conditions are coupled to an ODE and this ODE at its turn is controlled by the boundary conditions. It thus appears some kind of internal feedback which can either stabilize or destabilize the dynamic process of the water hammer.

3. The associated system of functional differential equations for stability analysis

We shall start from the model resulting from (2.2) and the condition $q_2(1,t) \equiv 0$

$$\begin{aligned}
&\partial_{\xi_i} h_i + T_{wi}\partial_t q_i = 0\ ,\ \delta_i^2 T_{wi}\partial_t h_i + \partial_{\xi_i} q_i = 0\,,\\
&h_1(0,t) \equiv 1\ ;\ h_1(1,t) = 1 + z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = h_2(0,t)\,,\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = 0\,,
\end{aligned} \tag{3.1}$$

and compute firstly its steady state by letting the time derivatives to 0

$$\begin{aligned}
&\bar{h}_i(\xi_i) \equiv \text{const}\ ;\ \bar{h}_1(0) = \bar{h}_1(1) = \bar{h}_2(0) = 1 + \bar{z} = 1\,,\\
&\bar{q}_i(\xi_1) \equiv \text{const}\ ;\ \bar{q}_1(1) = \bar{q}_2(0) = \bar{q}_2(1) = 0\,,
\end{aligned}$$

thus obtaining $\bar{h}_i = 1$, $\bar{z} = 0$, $\bar{q}_i = 0$; introduce the deviations $\chi_i(\xi_i, t) := h_i(\xi_i, t) - 1$; the variables $q_i(\xi_i, t)$, $z(t)$ obviously coincide with their deviations. The system in deviations reads

$$\begin{aligned} &\partial_{\xi_i}\chi_i + T_{wi}\partial_t q_i = 0\ ,\ \delta_i^2 T_{wi}\partial_t\chi_i + \partial_{\xi_i} q_i = 0\,, \\ &\chi_1(0,t) \equiv 0\ ;\ h_1(1,t) = z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = \chi_2(0,t)\,, \\ &T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = 0\,. \end{aligned} \tag{3.2}$$

To this system we associate the energy identities

$$\frac{1}{2}T_{wi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [q_i^2(\xi_i,t) + \delta_i^2\chi_i^2(\xi_i,t)]\mathrm{d}\xi_i + q_i(\xi_i,t)\chi_i(\xi_i,t)|_0^1 \equiv 0\,, \tag{3.3}$$

and the Riemann invariants (the forward and backward waves)

$$r_i^{\pm} = \frac{1}{2}(\delta_i\chi_i \pm q_i)\ \Leftrightarrow\ q_i = r_i^+ - r_i^-\ ,\ \chi_i = \frac{1}{\delta_i}(r_i^+ + r_i^-)\,. \tag{3.4}$$

Rewrite (3.2) in the Riemann invariants as follows

$$\begin{aligned} &\delta_i T_{wi}\partial_t r_i^{\pm} \pm \partial_{\xi_i} r_i^{\pm} = 0\ ;\ r_1^+(0,t) + r_1^-(0,t) \equiv 0\,, \\ &\frac{1}{\delta_1}(r_1^+(1,t) + r_1^-(1,t)) = z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{\delta_2}(r_2^+(0,t) + r_2^-(0,t))\,, \\ &T_s\frac{\mathrm{d}z}{\mathrm{d}t} = r_1^+(1,t) - r_1^-(1,t) - r_2^+(0,t) + r_2^-(0,t)\,, \\ &r_2^+(1,t) - r_2^-(1,t) \equiv 0\,. \end{aligned} \tag{3.5}$$

From now on we follow the methodology of [3,9]. Consider the two characteristic lines crossing some point (ξ_i, t) of the half plane $\{\xi_i, t | 0 \le \xi_i \le 1, t > 0\}$

$$\tau_i^{\pm}(\sigma;\xi_i,t) = t \pm \delta_i T_{wi}(\sigma - \xi_i)\ ,\ i = 1,2\,. \tag{3.6}$$

Since the Riemann invariants are constant along the characteristics (r_i^+ along τ_i^+ and r_i^- along τ_i^-), the following representation formulae are deduced

$$\begin{aligned} r_i^+(\xi_i,t) &= r_i^+(1, t + \delta_i T_{wi}(1-\xi_i))\,, \\ r_i^-(\xi_i,t) &= r_i^-(0, t + \delta_i T_{wi}\xi_i)\,. \end{aligned} \tag{3.7}$$

Let consider firstly those characteristics which can be extended on the entire interval $0 \le \sigma \le 1$. Defining $y_i^+(t) := r_i^+(1,t)$, $y_i^-(t) := r_i^-(0,t)$, we obtain

$$\begin{aligned} r_i^+(0,t) &= r_i^+(1, t + \delta_i T_{wi}) = y_i^+(t + \delta_i T_{wi})\,, \\ r_i^-(1,t) &= r_i^-(0, t + \delta_i T_{wi}) = y_i^-(t + \delta_i T_{wi})\,. \end{aligned} \tag{3.8}$$

These values are substituted in (3.5); we introduce further $w_i^{\pm}(t) := y_i^{\pm}(t + \delta_i T_{wi})$ in order to obtain the more "conventional" way of writing equations with deviated argument. Next, recall that stability studies are made for large $t > 0$; it is thus sufficient to take $t > \max\{\delta_1 T_{w1}, \delta_2 T_{w2}\}$ and eliminate the variables $w_1^+(t)$ and

$w_2^-(t)$. After an additional transformation requiring inversion of a 2×2 non-singular matrix, making also the following notations [5]

$$\begin{aligned}&\rho_1 := \frac{1+(\delta_2-\delta_1)R_s'}{1+(\delta_1+\delta_2)R_s'}\;,\;\rho_2 := \frac{1+(\delta_1-\delta_2)R_s'}{1+(\delta_1+\delta_2)R_s'}\;,\\ &R_s' := R_s/T_s\;;\;\vartheta := 2\delta_2 T_{w2}\;,\;\nu\vartheta := 2\delta_1 T_{w1}(\nu = (\delta_1 T_{w1})/(\delta_2 T_{w2}))\;,\end{aligned}$$

the following system of coupled delay differential and difference equations is obtained

$$\begin{aligned}&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = (\rho_1+\rho_2)[-\frac{\delta_1+\delta_2}{2}z(t) - w_1^-(t-\nu\vartheta) + w_2^+(t-\vartheta)]\,,\\ &w_1^-(t) = \frac{(\rho_1+\rho_2)\delta_1}{2}z(t) + \rho_1 w_1^-(t-\nu\vartheta) + (1-\rho_1)w_2^+(t-\vartheta)\,,\\ &w_2^+(t) = \frac{(\rho_1+\rho_2)\delta_2}{2}z(t) - (1-\rho_2)w_1^-(t-\nu\vartheta) - \rho_2 w_2^+(t-\vartheta)\,.\end{aligned} \tag{3.9}$$

The solutions of (3.9) can be constructed by steps provided initial conditions are given. For details the reader is sent to [10], Section 4. The one-to-one correspondence between the solutions of (3.9) and those of (3.5) is given by Theorem 4.1 of [10]; at its turn this correspondence strongly relies on (3.8) and on the representation formulae (3.7) re-written using the functions $w_i^\pm(t)$ as follows

$$\begin{aligned}&r_i^+(\xi_i,t) = y_i^+(t+\delta_i T_{wi}(1-\xi_i)) = w_i^+(t-\delta_i T_{wi}\xi_i)\,,\\ &r_i^-(\xi_i,t) = y_i^-(t+\delta_i T_{wi}\xi_i) = w_i^-(t+\delta_i T_{wi}(\xi_i-1))\,.\end{aligned} \tag{3.10}$$

Summarizing, the mathematical result reads as follows

Theorem 3.1. *Consider the system of the Riemann invariants* (3.5) *with the initial conditions* $\{z(0), r_{io}^\pm(\xi_i)\}$, *where*

$$r_{io}^\pm(\xi_i) = \frac{1}{2}(\delta_i\chi_i^o(\xi_i) \pm q_i^o(\xi_i))\;,\;0\le\xi_i\le 1\,. \tag{3.11}$$

If $\{z(t), r_i^\pm(\xi_i,t)\}$ *is a classical solution of* (3.5), *then* $\{z(t), w_i^\pm(t)\}$ *is a piecewise continuous solution of* (3.9) *with the initial conditions defined by* $\{z(0), w_{io}^\pm(\theta)\}$, *where*

$$w_{io}^+(\theta) = r_{io}^+(-\theta/(\delta_i T_{wi}))\;,\;w_{io}^-(\theta) = r_{io}^-(1+\theta/(\delta_i T_{wi}))\;;\;-\delta_i T_{wi}\le\theta\le 0\,. \tag{3.12}$$

Conversely, let $\{z(t), w_i^\pm(t)\}$ *be a solution of* (3.9) *with the initial conditions* $\{z(0), w_{io}^\pm(\theta)\}$. *Then* $\{z(t), r_i^\pm(\xi_i,t)\}$, *where* $r_i^\pm(\xi_i,t)$ *are given by* (3.7), *is a (possibly discontinuous) classical solution of* (3.5) *with the initial conditions* $r_{io}^\pm(\xi_i)$ *obtained by letting* $t=0$ *in* (3.7).

4. The Lyapunov functional and the stability analysis

We refer firstly to system (3.2) and to the energy identities (3.3). The energy identities suggest the following Lyapunov functional

$$\mathcal{V}\big(z,\phi_i(\cdot),\psi_i(\cdot)\big) = \frac{1}{2}\left\{T_s z^2 + \sum_1^2 T_{wi}\int_0^1[\phi_i^2(\xi_i) + \delta_i^2\psi_i^2(\xi_i)]\,\mathrm{d}\xi_i\right\}, \tag{4.1}$$

written as a quadratic functional on the state space $\mathbb{R} \times \mathcal{L}^2(0,1;\mathbb{R}^4)$. We write down (4.1) along the solutions of (3.2), differentiate it with respect to t and take into account the energy identities and the boundary conditions in (3.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^\star(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\big(z(t), q_i(\cdot,t), \chi_i(\cdot,t)\big) = -R_sT_s\left(\frac{\mathrm{d}z}{\mathrm{d}t}(t)\right)^2 \leq 0\,. \tag{4.2}$$

Inequality (4.2) gives the Lyapunov stability of the zero solution of (3.2) *in the sense of the metrics induced by the Lyapunv functional itself*

$$\mathcal{V}\big(z(t), q_i(\cdot,t), \chi_i(\cdot,t)\big) \leq \mathcal{V}\big(z_0, q_i^{\mathrm{o}}(\cdot), \chi_i^{\mathrm{o}}(\cdot)\big)\,. \tag{4.3}$$

Inequality (4.2) also shows that asymptotic stability might be obtained *via* the invariance principle of Barbashin-Krasovskii-LaSalle. For this we shall turn to system (3.9). Using the representation formulae (3.10), also (3.4), the Lyapunov functional of (4.2) becomes, after some simple manipulation and with a slight abuse of notation

$$\begin{aligned}\mathcal{V}\big(z(t), w_1^-(t+\cdot), w_2^+(t+\cdot)\big) &= \frac{1}{2}T_sz^2(t) + \frac{1}{\delta_1}\int_{-\nu\vartheta}^{\mathrm{o}} w_1^-(t+\lambda)^2\mathrm{d}\lambda \\ &+\frac{1}{\delta_2}\int_{-\vartheta}^{\mathrm{o}} w_2^+(t+\lambda)^2\mathrm{d}\lambda\,,\end{aligned} \tag{4.4}$$

the derivative of $\mathcal{V}$ remaining unchanged. This derivative vanishes for $\mathrm{d}z/\mathrm{d}t = 0$, that is on the set where

$$-(\delta_1+\delta_2)z(t) - 2w_1^-(t-\nu\vartheta) + 2w_2^+(t-\vartheta) = 0. \tag{4.5}$$

On this set the difference subsystem of (3.9) takes the form, after substituting $z(t)$ from (4.5))

$$\begin{aligned} w_1^-(t) &= \frac{1}{\delta_1+\delta_2}[(\delta_2-\delta_1)w_1^-(t-\nu\vartheta) + 2\delta_1w_2^+(t-\vartheta)], \\ w_2^+(t) &= \frac{1}{\delta_1+\delta_2}[-2\delta_2w_1^-(t-\nu\vartheta) + (\delta_2-\delta_1)w_2^+(t-\vartheta)]. \end{aligned} \tag{4.6}$$

The invariant set of (4.5) is composed of the only constant solution $\{0,0\}$ and $\bar{z} = 0$. *The only invariant set included in the set where the derivative of the Lyapunov functional vanishes is the zero solution.* The theorem of Barbashin-Krasovskii-LaSalle for system (3.9) – Theorem 9.8.2 of [7] – would give asymptotic stability and, therefore, asymptotic stability for (3.5) and, *via* (3.4), for system (3.2). However, there is a certain aspect to be taken into account: in the case of NFDE (and system (3.9) *is* neutral - see [7, Section 9, p.301]) the invariance principle is proven under the assumption that the difference operator $\mathcal{D}$ is asymptotically stable. This is not quite true for (3.9). If the difference subsystem of (3.9) is considered, its asymptotic stability is equivalent to the location of the roots of the characteristic equation

$$(1-\rho_1\mathrm{e}^{-\lambda\nu\vartheta})(1+\rho_2\mathrm{e}^{-\lambda\vartheta}) + (1-\rho_1)(1-\rho_2)\mathrm{e}^{-\lambda(\nu+1)\vartheta} = 0 \tag{4.7}$$

in $\mathbb{C}^-$. Since the two delays are, generally speaking, rationally independent (ν is a real number), (4.7) ought have its roots with $\Re e(\lambda) \leq -\alpha < 0$ for some $\alpha > 0$. Denoting $\mu := \mathrm{e}^{\lambda\vartheta}$, the condition above reduces to the location of the roots of

$$(\mu^\nu - \rho_1)(\mu+\rho_2) + (1-\rho_1)(1-\rho_2) = 0 \tag{4.8}$$

inside the unit disk $\mathbb{D}_1 \subset \mathbb{C}$. As it was rigorously and completely proven in [11], this condition is fulfilled *only for ν rational with both numerator and denominator – odd numbers*. For ν rational with even numerator and odd denominator, $\mu = -1$ is a simple root of (4.8). Moreover, since the spectral radius of the difference operator equals 1, *there is no irrational ν such that* (4.8) *should have its roots inside* $\mathbb{D}_1 \subset \mathbb{C}$ - see [7]. We call this kind of asymptotic stability *fragile* since it holds for a countable set of rational ratios ν of the propagation delays.

Summarizing, the mathematical result is as follows

Theorem 4.1. *Consider the system* (3.2) *with the associated Lyapunov functional* (4.1)*, together with systems* (3.9)*,* (3.10) *and with the rewritten Lyapunov functional* (4.4)*. Systems* (3.2) *and* (3.9)*,* (3.10) *are stable in the sense of Lyapunov with respect to the metrics defined by their associated Lyapunov functionals. If the delay ratio $\nu = (\delta_1 T_{w1})(\delta_2 T_{w2})^{-1}$ is rational with both numerator and denominator - odd numbers, then this stability is also asymptotic.*

5. An even more critical system

We mention here another system arising from hydraulics, describing a hydroelectric plant supplied through two independent tunnels starting from the same reservoir (lake), endowed with surge tank, under water hammer [4]. Under lumped parameters i.e. described by ODE and under the same description as our previous structure (all losses neglected except the surge tank throttling):

$$T_{wi}\frac{\mathrm{d}q_i}{\mathrm{d}t} + z + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \ (i = 1, 2) \ , \ T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1 + q_2 \, , \tag{5.1}$$

having an invariant set defined by

$$T_{w1}q_1(t) - T_{w2}q_2(t) \equiv T_{w1}q_1(0) - T_{w2}q_2(0) \, . \tag{5.2}$$

The steady state is uniquely determined on the invariant set only. In the case of the distributed parameters (PDE description) nothing is known about the invariant set while the steady state is not uniquely determined. Other considerations on this model can be seen in [10].

6. Some conclusions

It was mentioned in the Introduction that the outcome of the paper is twofold. The first outcome refers to the mathematical aspects. Here (and not only) there are displayed applications for which the difference operator associated to the NFDE is only critically stable. The assumption on the asymptotic, even the strong stability (i.e. stability with respect to the delays) turned to be very fruitful (productive) in the sense that it allowed an immediate extension to NFDE of the results of the stability theory obtained for the R(etarded)FDE. The price to be paid was that several papers dealing with the aforementioned critical cases were obscured and forgotten. We stress that returning to their results might be useful (their list is given in [10]. Another approach to be taken within the mathematical studies would be the one suggested in [12], page 341. It is specified there that the assumption on the asymptotic stability for the difference operator is necessary to obtain the compactness of the positive orbits whenever the solution is bounded.It is then suggested to embed the resulting semi-dynamical system in a space wherein the positive orbits are pre-compact. To illustrate this approach the reader is sent to an application in Chapter V, Section 4, page 252: the application there is a BVP for a hyperbolic PDE. With the one-to-one correspondence between the solutions of the BVP for the hyperbolic PDE and those of the associated system of NFDE, the problem becomes one of choosing the state space for the NFDE - other than $\mathcal{C}$ [6]. A good reference for the role of the pre-compactness is [8]. On the other hand, the aforementioned models of hydraulics are *strongly idealized* by neglecting almost all static energy dissipation. Re-introducing some of them means changing the model and restarting the entire analysis. *Too much idealization can turn harmful!.*

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Department of Automation and Electronics,
Faculty of Automation, Computers and Electronics University of Craiova,
A. I. Cuza str. 13, RO-200585 Craiova, Romania
and
Romanian Academy of Engineering Sciences ASTR
E-mail: `vladimir.rasvan@edu.ucv.ro`

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UNIQUE SOLVABILITY OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION ON THE BASIS OF VALLÉE-POUSSIN THEOREM

Satyam Narayan Srivastava, Alexander Domoshnitsky, Seshadev Padhi, and Vladimir Raichik

Abstract. We propose explicit tests of unique solvability of two-point and focal boundary value problems for fractional functional differential equations with Riemann-Liouville derivative.

1. Introduction

In this paper we consider the fractional functional differential equation

$$(D_{0+}^{\alpha}x)(t) + \sum_{i=0}^{m}(T_i x^{(i)})(t) = f(t)\,, \quad t \in [0,1],\ m \le n-2,\ n \ge 2\,, \tag{1.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of the order $n-1 < \alpha \le n$ (see [11], [14]), n is integer, the operators $T_i\colon C \to L_\infty$ are linear continuous operators acting from the space of the continuous functions C to the space of essentially bounded functions L_∞, $i = 0, \dots, m$, and $f \in L_\infty$.

We consider also the auxiliary equation

$$(D_{0+}^{\alpha}x)(t) + \sum_{i=0}^{m}(|T_i| x^{(i)})(t) = f(t)\,, \quad t \in [0,1],\ m \le n-2,\ n \ge 2\,, \tag{1.2}$$

where the positive operator $|T_i|$ is such that the following inequalities hold:

$$-\,(|T_i|1)(t) \le (T_i 1)(t) \le (|T_i|1)(t)\,, \qquad t \in [0,1]\,. \tag{1.3}$$

Of course, it will be clear below, that we are interested in the operators $|T_i|$ with the minimal norms in the space of continuous functions C.

The operators $T_i\colon C \to L_\infty$ and $|T_i|\colon C \to L_\infty$ can be, for example, of the following forms:

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1) Operators with deviations

$$(T_i x^{(i)})(t) = \sum_{j=0}^{m_i} q_{ij}(t) x^{(i)}\big(t - \tau_{ij}(t)\big)\,, \tag{1.4}$$

$$(|T_i| x^{(i)})(t) = \sum_{j=0}^{m_i} |q_{ij}(t)| x^{(i)}\big(t - \tau_{ij}(t)\big)\,,$$

where $\tau_{ij}\colon [0,1] \to \mathbb{R}$, $q_{ij}\colon [0,1] \to \mathbb{R}$, are measurable bounded functions, $\mathbb{R} = (-\infty, +\infty)$. To complete the description of these operators, we have to define what has to be substituted into (1.4) instead of $x^{(i)}(t - \tau_{ij}(t))$ in the case of $t - \tau_{ij}(t) \notin [0,1]$. Let us assume that

$$x^{(i)}(\xi) = 0 \quad \text{for} \quad \xi \notin [0,1],\ i = 0, \dots, m\,, \tag{1.5}$$

that allows us to preserve the n-dimensional fundamental system for the homogeneous equation

$$(D_{0+}^{\alpha} x)(t) + \sum_{j=0}^{m_i} q_{ij}(t) x^{(i)}(t - \tau_{ij}(t)) = 0\,. \tag{1.6}$$

2) Integral operators

$$(T_i x^{(i)})(t) = \int_0^1 K_i(t,s) x^{(i)}(s)\, ds\,, \tag{1.7}$$

$$(|T_i| x^{(i)})(t) = \int_0^1 |K_i(t,s)| x^{(i)}(s)\, ds\,,$$

under the standard assumptions on the kernels $K_i(t,s)$ implementing that $T_i\colon C \to L_\infty$, for example, $K_i(t,s)$ is a continuous function $[0,1] \times [0,1] \to \mathbb{R}$ (see, [12]).

3) Linear combinations and superpositions of the deviations and integral operators, for example, the operators

$$(T_i x^{(i)})(t) = \int_0^1 \sum_{j=1}^{m_i} K_{ij}(t,s) x^{(i)}\big(s - \tau_{ij}(s)\big)\, ds\,. \tag{1.8}$$

$$(|T_i| x^{(i)})(t) = \int_0^1 \sum_{j=1}^{m_i} |K_{ij}(t,s)| x^{(i)}\big(s - \tau_{ij}(s)\big)\, ds\,.$$

We consider the boundary value problem consisting of equation (1.1) and the boundary conditions

$$x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \dots, n-2,\ x^{(k)}(1) = 0\,, \tag{1.9}$$

where k is an integer which is between 0 and $n-1$. In the case of $k=0$, we have the classical two-point $(n-1,1)$- problem. In the case of $k \le n-1$, we have the sort of focal problems. We assume below that $m \le k$.

We consider equation (1.1) in the space D of functions $x\colon [0,1] \to \mathbb{R}$ such that $x^{(n-1)}$ is absolutely continuous on every interval $[\varepsilon, 1]$, where $\varepsilon > 0$ and summable on $[0,1]$ and $x^{(n)}$ such that $tx^{(n)}$ is summable. The norm in the space D define as $\|x\|_D = \sum\limits_{i=0}^{n-2} \max\limits_{0\le t\le 1} \left|x^{(i)}(t)\right| + \int\limits_0^1 \left|x^{^{(n-1)}}(t)\right| dt + \int\limits_0^1 t \left|x^{^{(n)}}(t)\right| dt$. Considering this space D looks naturally when fractional equations with the Riemann-Liouville derivatives and the boundary conditions (1.9) are considered. We say that $x \in D$ is a solution of (1.1) if it satisfies this equation for almost every $t \in [0,1]$. If the problem consisting of the homogeneous equation $(D^{\alpha}_{0+}x)(t) + \sum\limits_{i=0}^{m} (T_i x^{(i)})(t) = 0$ and condition (1.9) has only the trivial solution, then problem (1.1), (1.9) has a unique solution which can be represented in the form [2]

$$x(t) = \int_0^1 G(t,s) f(s) ds\,. \tag{1.10}$$

For applications of fractional differential equations in various field of science and engineering one can refer the classical books [11, 14].

The main reason for the study of fractional functional differential equations could be, in our opinion, around the following idea for the study of systems of fractional equations. Consider a boundary value problem consisting, for example, of a system of two “ordinary fractional differential equations". For its analysis, we can use the integral representations of solutions of the first equation and obtain $x_1(t)$ through $x_2(t)$. Then we substitute this representation instead of $x_1(t)$ into the second equation and obtain a scalar fractional functional differential equation. In the simplest case of a system of “ordinary” fractional equations, the equation, we get, includes the integral operator of type 2). If we start with a system of delay fractional differential equations, the equation, we get after the substitution into the second equation, is a fractional functional differential equation that includes the superpositions of deviation and integral operators. Thus, operators of type 3) appear. Examples of such systems can be found in [7, 8, 9].

Positivity of solutions is one of the most important properties in applications (see, for example, the book by Henderson and Luca [7]). Concerning problem (1.4),(1.9), in the case of so called ordinary linear equations, (i.e. $\tau_{ij}(t) \equiv 0$, $t \in [0,1]$, $j = 0, \ldots, m_i$, $i = 1, \ldots, m$ in (1.4)) and its nonlinear generalizations, we can note the following papers [3, 8, 9, 10, 13, 15].

One of the motivations for our research is Lyapunov's inequalities for fractional differential equations which have been presented in Chapter 5 of the recent book by Agarwal, Bohner, and Ozbekler [1]. Note the following assertion was presented for the first time in [5]. Actually, the result in [5] is more general than Theorem 1.1 as the solution need not be assumed to be different from zero on $(0,1)$.

α	In inequality (1.13)	In inequality (1.15)
1.6	2.052759111	4.120246548
1.5999	2.05244883	4.119533208
1.5998	2.052138367	4.11819636
1.597	2.043474592	4.098884212
1.58	1.991943084	3.97506386
1.5	1.7724538	3.45372767

TAB. 1

Theorem 1.1 ([1, 5]). *Let $1 < \alpha \leq 2$ and x be a solution of the boundary value problem*

$$\begin{cases} (D_{0+}^{\alpha}x)(t) + q_0(t)x(t) = 0 \quad on \quad [0,1]\,, \\ x(0) = x(1) = 0\,. \end{cases} \tag{1.11}$$

If $x(t) \neq 0$ for all $t \in (0,1)$, then the inequality

$$\int_0^1 |q_0(t)|\,dt > \Gamma(\alpha)4^{\alpha-1} \tag{1.12}$$

holds.

Note that in [5], it was not assumed that $x(t) \neq 0$ for $t \in (0,1)$. For (1.11) with a constant coefficient $q_0(t) = q_0$, we have (1.12) in the form

$$|q_0| \geq \Gamma(\alpha)4^{\alpha-1}\,. \tag{1.13}$$

Using Corollary 2.3 (one can refer [4] for proof), we get that the inequality

$$|q_0| < \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}}\Gamma(\alpha+1) \tag{1.14}$$

guarantees that the problem (1.11) has only the trivial solution. Note that the part on unique solvability coincides with the known result of [6]. Inequality (1.14) means that in the case of zeros of solution $x(t)$ at the points 0 and 1, we obtain that

$$|q_0(t)| \geq \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}}\Gamma(\alpha+1) \tag{1.15}$$

since in the case of the coefficient q_0 satisfying inequality (1.11) we exclude the existence of zero at the point 1, i.e. $x(1) \neq 0$. Let us compare (1.13) and (1.15), computing the right-hand sides in them, we have values in Table 1.

Table 1 demonstrates the advances of our results if we compare the results of [1, 5] and ours.

2. Main Results

Lemma 2.1. *Using the technique of* [13], *one can obtain the uniqueness of solution to the problem*

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t)\,, \\ x(0) = x'(0) = \ldots = x^{(n-2)}(0) = 0\,, \\ x^{(k)}(1) = 0\,, \end{cases} \tag{2.1}$$

where k is an integer number which is between 0 and $n-1$, in the form

$$x(t) = \int_0^1 G_k(t,s)f(s)\,ds\,, \tag{2.2}$$

where $G_k(t,s)$ is Green's function of problem (2.1) *defined by*

$$G_k(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (t-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1-k}\,, & 0 \le s \le t \le 1\,, \\ -t^{\alpha-1}(1-s)^{\alpha-1-k}\,, & 0 \le t < s \le 1 \end{cases} \tag{2.3}$$

and its j-th derivative is defined by

$$\frac{\partial^j}{\partial t^j}G_k(t,s) = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-j-1} - t^{\alpha-j-1}(1-s)^{\alpha-1-k}\,, & 0 \le s \le t \le 1\,, \\ -t^{\alpha-j-1}(1-s)^{\alpha-1-k}\,, & 0 \le t < s \le 1\,. \end{cases} \tag{2.4}$$

Let us define the operator $K\colon L_\infty \to L_\infty$ and $|K|\colon L_\infty \to L_\infty$ by the equalities

$$\begin{aligned} (Kz)(t) &= -\sum_{i=0}^{m} T_i\left[\int_0^1 \frac{\partial^i}{\partial t^i}G_k(t,s)z(s)\,ds\right](t) = f(t)\,, \\ (|K|z)(t) &= -\sum_{i=0}^{m} |T_i|\left[\int_0^1 \frac{\partial^i}{\partial t^i}G_k(t,s)z(s)\,ds\right](t) = f(t)\,. \end{aligned} \tag{2.5}$$

We use the notation $T_i[\gamma(t)]$, $(|T_i|[\gamma(t)])$ meaning that the operator T_i and $|T_i|$ acts on the continuous function $\gamma(t)$, i.e. $T_i[\gamma(t)] = (T_i\gamma)(t)$, $|T_i|[\gamma(t)] = (|T_i|\gamma)(t)$.

Theorem 2.2. *Assume that there exist a function $v \in D$ such that $v(t) > 0$, $v'(t) > 0$, $\cdots$, $v^{(k)}(t) > 0$ for $t \in (0,1)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$ and*

$$(D_{0+}^{\alpha}v)(t) + \sum_{i=0}^{m}(|T_i|v^{(i)})(t) \equiv \psi(t) \le -\varepsilon < 0 \quad \text{for} \quad t \in (0,1)\,; \tag{2.6}$$

then the problem (1.1), (1.9) *is uniquely solvable for any essentially bounded f and the spectral radius of $|K|\colon L_\infty \to L_\infty$ is less than one.*

Proof. Consider the auxiliary problem

$$\begin{cases} (D_{0+}^{\alpha}x)(t) = z(t)\,, \\ x^{(i)}(0) = v^{(i)}(0),\ x^{(k)}(1) = v^{(k)}(1)\,, \quad i = 0,1,\ldots,n-2\,, \end{cases} \tag{2.7}$$

where $z(t)$ is a function in L_∞ and such that there exists a positive number δ such that $z(t) \le -\delta$ for $t \in [0,1]$. It is clear that

$$\begin{cases} x(t) = \int_0^1 G_k(t,s)z(s)\,ds + u_k(t)\,, \\ x'(t) = \int_0^1 \frac{\partial}{\partial t} G_k(t,s)z(s)\,ds + u_k'(t)\,, \\ x''(t) = \int_0^1 \frac{\partial^2}{\partial t^2} G_k(t,s)z(s)\,ds + u_k''(t)\,, \\ \vdots \\ x^{(m)}(t) = \int_0^1 \frac{\partial^m}{\partial t^m} G_k(t,s)z(s)\,ds + u_k^{(m)}(t)\,, \end{cases} \tag{2.8}$$

where $u(t)$ is a solution of the homogeneous equation $D^\alpha_{0+}u(t) = 0$ satisfying the conditions $u^{(i)}(0) = v^{(i)}(0)$, $i = 0, \dots, n-2$, $u^{(k)}(1) = v^{(k)}(1)$. Let us substitute these representations instead of $v(t)$ and its derivatives into inequality (2.6):

$$z(t) + \sum_{i=0}^{m} T_i \left[\int_0^1 \frac{\partial^i}{\partial t^i} G_k(t,s)z(s)\,ds \right] + \sum_{i=0}^{m} (T_i u^i)(t)) = \psi(t)\,. \tag{2.9}$$

It is clear that $|T_i|\colon C \to L_\infty$ are positive operators for $i = 0, 1, \dots, m$, and this imply that the operator $|K|\colon L_\infty \to L_\infty$ defined by equality (2.5) is positive. Thus, we have the equation

$$z(t) - (|K|z)(t) = \Psi(t)\,, \quad t \in [0,1]\,, \tag{2.10}$$

where

$$\Psi(t) \equiv \psi(t) - \sum_{i=0}^{m} (|T_i|u^{(i)})(t)\,. \tag{2.11}$$

It is clear that $u^{(i)}(t) > 0$ for $t \in (0,1]$. This implies that $\Psi(t) \le -\varepsilon < 0$. The function $w(t) = -z(t)$ satisfies the inequality $w(t) - (|K|w)(t) = -\Psi(t) > 0$ for $t \in [0,1]$. From equality (2.10), according to [12, Theorem 5.3 on page 76] it follows that $\rho(|K|) < 1$. This completes the proof of the theorem. □

Corollary 2.3. *If $n-1 < \alpha \le n$ and the following inequality is fulfilled*

$$\begin{aligned} &|T_0| \left[t^{\alpha-1} \left(\frac{\alpha}{\alpha-k} - t \right) \right] \\ &+ \sum_{i=1}^{m} \alpha(\alpha-1)\cdots(\alpha-i+1)|T_i| \left[t^{\alpha-i-1} \left(\frac{\alpha-i}{\alpha-k} - t \right) \right] < \Gamma(\alpha+1),\ t \in [0,1]\,, \end{aligned} \tag{2.12}$$

then problem (1.1), (1.9) *is uniquely solvable for any $f \in L_\infty$.*

Proof. The proof follows from Corollary 4 of [4]. □

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Department of Mathematics, Ariel University,
Ariel 40700, Israel
E-mail: `satyamsrivastava983@gmail.com`, `adom@ariel.ac.il`, `vladimir.raichik@gmail.com`

Department of Mathematics,
Birla Institute of Technology Mesra,
Ranchi 835215, India
E-mail: `spadhi@bitmesra.ac.in`

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RICCATI MATRIX DIFFERENTIAL EQUATION AND THE DISCRETE ORDER PRESERVING PROPERTY

Viera Štoudková Růžičková

Abstract. In this paper we recall discrete order preserving property related to the discrete Riccati matrix equation. We present results obtained by applying this property to the solutions of the Riccati matrix differential equation.

1. Notation

In the whole paper we denote by $\mathbf{S}$ the set of real symmetric $n \times n$ matrices. For any two symmetric matrices Q, $\hat{Q} \in \mathbf{S}$, by the inequality $Q \le \hat{Q}$ we mean that the symmetric matrix $\hat{Q} - Q$ is non-negative definite. For any two $n \times n$ symmetric matrix functions $Q(t)$, $\hat{Q}(t)$, by the inequality $Q(t) \le \hat{Q}(t)$, $t \in M$ we mean that both functions are defined for all $t \in M$ and that $\hat{Q}(t) - Q(t)$ is non-negative definite on M.

2. Discrete order preserving property

In this section we recall the order preserving property of the discrete Riccati matrix equation and its modifications. By the *discrete Riccati matrix equation* we mean the difference equation

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0\,, \tag{2.1}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, and Q_k are real $n \times n$ matrices, Q_k are symmetric and the $2n \times 2n$ matrices $\mathcal{S}_k$ with block entries $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are symplectic. This means that

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If a matrix $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ is symplectic, then its inverse exists, $S^{-1} = \left(\begin{smallmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ -\mathcal{C}^T & \mathcal{A}^T \end{smallmatrix}\right)$, it is also symplectic matrix and the identities

$$\begin{aligned} \mathcal{A}^T\mathcal{C} = \mathcal{C}^T\mathcal{A}\,, \quad \mathcal{B}^T\mathcal{D} = \mathcal{D}^T\mathcal{B}\,, \quad \mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} = I\,, \\ \mathcal{A}\mathcal{B}^T = \mathcal{B}\mathcal{A}^T\,, \quad \mathcal{C}\mathcal{D}^T = \mathcal{D}\mathcal{C}^T\,, \quad \mathcal{A}\mathcal{D}^T - \mathcal{B}\mathcal{C}^T = I \end{aligned} \tag{2.2}$$

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hold. Other known properties of symplectic matrices are formulated in the next lemma.

Lemma 2.1. *Let* $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ *be a* $2n \times 2n$ *symplectic matrix and* $Q_0 \in \mathbf{S}$ *such that* $\mathcal{A} + \mathcal{B}Q_0$ *is invertible and denote* $Q_1 := (\mathcal{C} + \mathcal{D}Q_0)(\mathcal{A} + \mathcal{B}Q_0)^{-1}$*. Then* Q_1 *is symmetric,* $(\mathcal{A} + \mathcal{B}Q_0)^{-1} = \mathcal{D}^T - \mathcal{B}^T Q_1$ *and* $Q_0 = (-\mathcal{C}^T + \mathcal{A}^T Q_1)(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}$*. Further,* $(\mathcal{A} + \mathcal{B}Q_0)^{-1}\mathcal{B} \geq 0$ *iff* $(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}\mathcal{B}^T \geq 0$*.*

Proof. All can be done by direct calculation, using the properties (2.2). □

The following result about order preserving property of the Riccati equation (2.1) is from [3].

Proposition 2.2 (Proposition 2.4 from [3])**.** *Assume that* Q *and* $\hat{Q}$ *are symmetric solutions of the Riccati equation* (2.1) *on* $[0, N]_{\mathbb{Z}} := [0, N] \cap \mathbb{Z}$ *such that* $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0$ *on* $[0, N]_{\mathbb{Z}}$*. If* $Q_0 \leq \hat{Q}_0$ *(*$Q_0 < \hat{Q}_0$*), then* $Q_k \leq \hat{Q}_k$ *(*$Q_k < \hat{Q}_k$*) on* $[0, N+1]_{\mathbb{Z}}$*. Moreover, in this case* $(\mathcal{A}_k + \mathcal{B}_k \hat{Q}_k)^{-1}\mathcal{B}_k \geq 0$ *on* $[0, N]_{\mathbb{Z}}$ *as well.*

Without the assumption $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$, the conclusion of Proposition 2.2 does not hold in general. See example in [3, Remark 2.5]. It can be shown that this assumption is necessary.

The following result is a generalization of the order preserving property from Proposition 2.2. It contains equivalence instead of implication and it is formulated in a more general way that omits the notion of Riccati equation.

Theorem 2.3. *Let* $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ *be a* $2n \times 2n$ *symplectic matrix and* $Q \in \mathbf{S}$ *be such that the inverse* $(\mathcal{A} + \mathcal{B}Q)^{-1}$ *exists. The following statements are equivalent:*

(i) $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0\,,$

(ii) $\forall \hat{Q} \in \mathbf{S} : Q \leq \hat{Q} \implies (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1},$

(iii) $\forall \hat{Q} \in \mathbf{S} : (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \geq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} \implies Q \geq \hat{Q}\,.$

Further, the following statements are equivalent:

(iv) $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \leq 0\,,$

(v) $\forall \hat{Q} \in \mathbf{S} : (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} \implies Q \leq \hat{Q}\,,$

(vi) $\forall \hat{Q} \in \mathbf{S} : Q \geq \hat{Q} \implies (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \geq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}.$

Before the proof of Theorem 2.3 we present the following two lemmas, which are used in the proof of this theorem.

Lemma 2.4. *Let* $\mathcal{A}$*,* $\mathcal{B}$ *be* $n \times n$ *matrices,* $\mathcal{A}\mathcal{B}^T$ *symmetric, and* $Q \in \mathbf{S}$ *be such that the inverse* $(\mathcal{A} + \mathcal{B}Q)^{-1}$ *exists and* $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0$*. If* $\hat{Q} \in \mathbf{S}$ *and* $Q \leq \hat{Q}$*, then the inverse* $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$ *exists as well and* $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$*.*

Proof. First notice that $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0$ is equivalent with $\mathcal{B}(\mathcal{A} + \mathcal{B}Q)^T \geq 0$ and further $Q \leq \hat{Q}$ implies $B(\mathcal{A} + \mathcal{B}\hat{Q})^T \geq 0$. Now let v be an n-vector such that $(\mathcal{A} + \mathcal{B}\hat{Q})^T v = 0$. The inequality $Q \leq \hat{Q}$ implies $0 \leq v^T \mathcal{B}(\hat{Q} - Q)\mathcal{B}^T v$ and from this we further get

$$\begin{aligned}0 &\le v^T\mathcal{B}(\hat{Q}-Q)\mathcal{B}^Tv = v^T(\mathcal{B}\mathcal{A}^T+\mathcal{B}\hat{Q}\mathcal{B}^T-\mathcal{B}\mathcal{A}^T-\mathcal{B}Q\mathcal{B}^T)v\\ &= v^T\mathcal{B}(\mathcal{A}+\mathcal{B}\hat{Q})^Tv - v^T\mathcal{B}(\mathcal{A}+\mathcal{B}Q)^Tv = -v^T\mathcal{B}(\mathcal{A}+\mathcal{B}Q)^Tv\,.\end{aligned}$$

From positive semidefinity of $\mathcal{B}(\mathcal{A}+\mathcal{B}Q)^T$ we have that $v^T\mathcal{B}(\hat{Q}-Q)\mathcal{B}^Tv=0$ and from positive semidefinity of $\hat{Q}-Q$ we further have that $(\hat{Q}-Q)\mathcal{B}^Tv=0$. From this relationship together with $(\mathcal{A}+\mathcal{B}\hat{Q})^Tv=0$ we get that $(-\mathcal{A}^T-Q\mathcal{B}^T)v=0$ and hence $v=0$. This proves that the inverse $(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}$ exists. Now, inequality $B(\mathcal{A}+\mathcal{B}\hat{Q})^T\ge 0$ implies that $(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}\mathcal{B}\ge 0$. □

Lemma 2.5. *Let $S=\left(\begin{smallmatrix}\mathcal{A}&\mathcal{B}\\ \mathcal{C}&\mathcal{D}\end{smallmatrix}\right)$ be a $2n\times 2n$ symplectic matrix. For any matrices $Q,\hat{Q}\in\mathbf{S}$, such that the inverses $(\mathcal{A}+\mathcal{B}Q)^{-1}$ and $(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}$ exist, we have the identity*

$$\begin{aligned}&(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\\ &= (\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q+(\hat{Q}-Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}(\hat{Q}-Q)\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}.\end{aligned}\tag{2.3}$$

Proof. We use the identities (2.2) and Lemma 2.1 and we get the identity by direct calculation:

$$\begin{aligned}&(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\\ &\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}(\mathcal{C}^T+\hat{Q}\mathcal{D}^T)-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\\ &\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\\ &\qquad\times\left[(\mathcal{C}^T+\hat{Q}\mathcal{D}^T)(\mathcal{A}+\mathcal{B}Q)-(\mathcal{A}^T+\hat{Q}\mathcal{B}^T)(\mathcal{C}+\mathcal{D}Q)\right](\mathcal{A}+\mathcal{B}Q)^{-1}\\ &\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q\right](\mathcal{A}+\mathcal{B}Q)^{-1}\\ &\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q\right](\mathcal{A}+\mathcal{B}Q)^{-1}(\mathcal{A}+\mathcal{B}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}\\ &\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q\right](\mathcal{A}+\mathcal{B}Q)^{-1}\left[\mathcal{A}+\mathcal{B}Q+\mathcal{B}(\hat{Q}-Q)\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}\\ &\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q+(\hat{Q}-Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}(\hat{Q}-Q)\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}.\end{aligned}$$

□

Proof of Theorem 2.3. The implication (i) $\Longrightarrow$ (ii) and the implication (i) $\Longrightarrow$ (iii) follows immediately from the identity (2.3) and Lemma 2.4.

Now we prove the implication (ii) $\Longrightarrow$ (i). Let's suppose (i) does not hold, that is, there exists such n-vector v that $v^T(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}v<0$. We now show that then also (ii) does not hold. We take $\hat{Q}=Q+tI$, where t is a positive real number such that the inverse $(\mathcal{A}+\mathcal{B}Q+t\mathcal{B})^{-1}$ exists. Then $Q\le\hat{Q}$. Further we get from the identity (2.3) that

$$\begin{aligned}&(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\\ &\qquad= (\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[tI+t^2(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}.\end{aligned}$$

Now we take vector $u = (\mathcal{A} + \mathcal{B}\hat{Q})v$ and we get

$$u^T\big[(\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} - (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1}\big]u \\ = tv^Tv + t^2v^T(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B}v\,,$$

which is negative for sufficiently large t. Hence, there exists $\hat{Q} = Q + tI$ such that the matrix $(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1} - (\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}$ is not positive semi-definite.

In the proof of the implication (iii) $\implies$ (i) we use Lemma 2.1. Let $Q_1 := (\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}$. Then $\mathcal{D}^T - \mathcal{B}^TQ_1$ is invertible and $Q = (-\mathcal{C}^T + \mathcal{A}^TQ_1)(\mathcal{D}^T - \mathcal{B}^TQ_1)^{-1}$. Now again we suppose that (i) does not hold. Then also the inequality $(\mathcal{D}^T - \mathcal{B}^TQ_1)^{-1}\mathcal{B}^T \geq 0$ does not hold, that is, there exists such n-vector v that $v^T(\mathcal{D}^T - \mathcal{B}^TQ_1)^{-1}\mathcal{B}^Tv < 0$. We now show that then also (iii) does not hold. We take $\hat{Q}_1 = Q_1 - tI$, where t is a positive real number such that the inverse $(\mathcal{D}^T - \mathcal{B}^TQ_1 + t\mathcal{B}^T)^{-1}$ exists. Then $Q_1 \geq \hat{Q}_1$. Denote $\hat{Q} := (-\mathcal{C}^T + \mathcal{A}^T\hat{Q}_1)(\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1}$. Now we get from the identity (2.3), applied on the symplectic matrix $\left(\begin{smallmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ -\mathcal{C}^T & \mathcal{A}^T \end{smallmatrix}\right)$, that

$$\begin{aligned}\hat{Q} - Q &= (-\mathcal{C}^T + \mathcal{A}^T\hat{Q}_1)(\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1} - (-\mathcal{C}^T + \mathcal{A}^TQ_1)(\mathcal{D}^T - \mathcal{B}^TQ_1)^{-1} \\ &= (\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{T-1}\left[-tI - t^2(\mathcal{D}^T - \mathcal{B}^TQ_1)^{-1}\mathcal{B}^T\right](\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1}.\end{aligned}$$

Now we take vector $u = (\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)v$ and we get

$$u^T\left[\hat{Q} - Q\right]u = -tv^Tv - t^2v^T(\mathcal{D}^T - \mathcal{B}^TQ_1)^{-1}\mathcal{B}^Tv\,,$$

which is positive for sufficiently large t. Hence, there exists $\hat{Q} \in \mathbf{S}$ such that the matrix $(\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} - (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} = \hat{Q}_1 - Q_1 = -tI$ is negative semidefinite but $\hat{Q} - Q$ is not negative semidefinite.

The proof of the equivalence of (iv)–(vi) is analogous. □

Remark 2.6. In Theorem 2.3 in the statements (ii), (iii), (v) and (vi), we can replace the set $\mathbf{S}$ with the set $\mathbf{M} = \{Q \in \mathbf{S} : \mathcal{A} + \mathcal{B}Q$ is invertible and $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0\}$. In the proof of the implication (ii) $\implies$ (i) there exists also a sufficiently large t such that $(\mathcal{A} + \mathcal{B}Q)^{-1}$ exists and $B(\mathcal{A}^T + Q\mathcal{B}^T + t\mathcal{B}^T) \geq 0$, so $\hat{Q} = Q + tI \in \mathbf{M}$, and in the proof of the implication (iii) $\implies$ (i) there exists also a sufficiently large t such that $(\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1}$ exists and $\mathcal{B}^T(\mathcal{D} - Q_1\mathcal{B} + t\mathcal{B}) \geq 0$, which is equivalent with $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$.

The following corollary we get directly from Theorem 2.3.

Corollary 2.7. *Let* $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ *be a* $2n \times 2n$ *symplectic matrix and* $Q, \hat{Q} \in \mathbf{S}$ *be such that both inverses* $(\mathcal{A} + \mathcal{B}Q)^{-1}$ *and* $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$ *exist and both inequalities* $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \leq 0, (\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \leq 0$ *hold, or both inequalities* $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0, (\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$ *hold. Then*

$$Q \leq \hat{Q} \quad \Leftrightarrow \quad (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}.$$

3. Riccati matrix differential equation

In this section we present application of the discrete order preserving property to the continuous case, that is to the Riccati matrix differential equation. By the *Riccati matrix differential equation* we mean the equation

$$Q'(t) + A^T(t)Q(t) + Q(t)A(t) + Q(t)B(t)Q(t) - C(t) = 0\,, \tag{3.1}$$

where $A(t)$, $B(t)$, $C(t)$ and $Q(t)$ are real $n \times n$ matrix functions of t and $B(t)$, $C(t)$, $Q(t)$ are symmetric.

At first we introduce the classic version of the order preserving property of the Riccati matrix differential equation.

Proposition 3.1 (Proposition 6, Chapter 2 in [1])**.** *Let $Q(t)$, $\hat{Q}(t)$ be symmetric solutions of the Riccati matrix differential equation* (3.1) *on an interval $\mathcal{I}$. If, for some a in $\mathcal{I}$, $Q(a) \le \hat{Q}(a)$, then $Q(t) \le \hat{Q}(t)$ for all t in $\mathcal{I}$. If $Q(a) < \hat{Q}(a)$, then $Q(t) < \hat{Q}(t)$ for all t in $\mathcal{I}$.*

The relation between the Riccati matrix differential equation and the discrete Riccati matrix equation can be seen from the form of the solution of the differential equation, which is shown in the next lemma.

Lemma 3.2. *Let*

$$S(t) = \begin{pmatrix} \tilde{X}(t) & \bar{X}(t) \\ \tilde{U}(t) & \bar{U}(t) \end{pmatrix}$$

be the solution of linear Hamiltonian system corresponding to the equation (3.1)*,*

$$S'(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & -A^T(t) \end{pmatrix} \cdot S(t),$$

with the initial condition $S(0) = I$ on an interval $\mathcal{I}_0$ and let $\tilde{X}(t) + \bar{X}(t)Q_0$ be invertible on $\mathcal{I}_0$. Then

$$Q(t) = (\tilde{U}(t) + \bar{U}(t)Q_0)(\tilde{X}(t) + \bar{X}(t)Q_0)^{-1} \tag{3.2}$$

is solution of the Riccati equation (3.1) *with the initial condition*

$$Q(0) = Q_0 \tag{3.3}$$

on $\mathcal{I}_0$. Moreover, the matrix $S(t)$ is symplectic.

Proof. We can directly substitute the right side of (3.2) into the equation (3.1) and verify the result. See also the proof of [4, Lemma 2]. □

Now follows the main result, a modification of the order preserving property of the Riccati matrix differential equation.

Theorem 3.3. *Let $Q(t)$ be unique symmetric solution of the Riccati matrix differential equation* (3.1) *with the initial condition $Q(0) = Q_0$ on an interval $\mathcal{I}_0$. Let $S(t) = \left(\begin{smallmatrix} \tilde{X}(t) & \bar{X}(t) \\ \tilde{U}(t) & \bar{U}(t) \end{smallmatrix}\right)$ be the solution of the corresponding Hamiltonian system as in*

Lemma 3.2. The following statements are equivalent:

$$\text{(i)}\quad (\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t)\geq 0\,,\ t\in M\,,$$

$$\text{(ii)}\quad \forall \hat{Q}_0\in \mathbf{S}:\ Q_0\leq \hat{Q}_0 \implies Q(t)\leq \hat{Q}(t)\,,\ t\in M\,,$$

$$\text{(iii)}\quad \forall \hat{Q}_0\in \mathbf{S}:\ Q(t)\geq \hat{Q}(t)\,,\ t\in M \implies Q_0\geq \hat{Q}_0\,,$$

where M is any subset of $\mathcal{I}_0$ and $\hat{Q}(t)$ is unique symmetric solution of the Riccati matrix differential equation (3.1) *with the initial condition $\hat{Q}(0)=\hat{Q}_0$.*

Further, the following statements are equivalent:

$$\text{(iv)}\quad (\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t)\leq 0\,,\ t\in M\,,$$

$$\text{(v)}\quad \forall \hat{Q}_0\in \mathbf{S}:\ Q(t)\leq \hat{Q}(t)\,,\ t\in M \implies Q_0\leq \hat{Q}_0\,,$$

$$\text{(vi)}\quad \forall \hat{Q}_0\in \mathbf{S}:\ Q_0\geq \hat{Q}_0 \implies Q(t)\geq \hat{Q}(t)\,,\ t\in M\,.$$

Proof. By Lemma 3.2 and the assumption that the solutions $Q(t)$ and $\hat{Q}(t)$ are unique, we have that $Q(t)=(\tilde{U}(t)+\bar{U}(t)Q_0)(\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}$, $\hat{Q}(t)=(\tilde{U}(t)+\bar{U}(t)\hat{Q}_0)(\tilde{X}(t)+\bar{X}(t)\hat{Q}_0)^{-1}$ and the matrix $S(t)$ is symplectic on $\mathcal{I}_0$. Thus, the statements (i)–(iii) and the statements (iv)–(vi) are equivalent for any fixed $t\in M$ because of Theorem 2.3. □

Corollary 3.4. *Let $Q(t),\hat{Q}(t)$ be unique symmetric solutions of the Riccati matrix differential equation* (3.1) *with the initial conditions $Q(0)=Q_0$ and $\hat{Q}(0)=\hat{Q}_0$ on an interval $\mathcal{I}_0$. Let $\left(\begin{smallmatrix}\tilde{X}(t) & \bar{X}(t)\\ \tilde{U}(t) & \bar{U}(t)\end{smallmatrix}\right)$ be the solution of the corresponding Hamiltonian system as in Lemma 3.2 and let M be a subset of $\mathcal{I}_0$. If inequalities*

$$(\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t)\leq 0\,,\ (\tilde{X}(t)+\bar{X}(t)\hat{Q}_0)^{-1}\bar{X}(t)\leq 0\,,\ t\in M$$

hold, or inequalities

$$(\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t)\geq 0\,,\ (\tilde{X}(t)+\bar{X}(t)\hat{Q}_0)^{-1}\bar{X}(t)\geq 0\,,\ t\in M$$

hold, then

$$Q_0\leq \hat{Q}_0 \quad\Leftrightarrow\quad Q(t)\leq \hat{Q}(t)\,,\ t\in M\,.$$

Proof. The proof is analogous to the proof of Theorem 3.3, we use Corollary 2.7. □

In the last part of this section we present a simple example to illustrate the difference between the order preserving property from Proposition 3.1 and Theorem 3.3.

Example 3.5. Let us have the Riccati equation

$$Q'(t)-Q^2(t)-I=0\,. \tag{3.4}$$

The corresponding linear Hamiltonian system is

$$S'(t)=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\cdot S(t)$$

and its unique solution with the initial condition $S(0) = I$ is

$$S(t) = \begin{pmatrix} I\cos t & -I\sin t \\ I\sin t & I\cos t \end{pmatrix}.$$

The solution of the equation (3.4) with $Q(0) = Q_0$ is

$$Q(t) = (I\sin t + Q_0\cos t)(I\cos t - Q_0\sin t)^{-1} = \tan(It + \arctan Q_0)$$

and it is defined on the interval $\mathcal{I}_0 = \left(-\frac{\pi}{2} - \arctan\lambda_{\min}, \frac{\pi}{2} - \arctan\lambda_{\max}\right)$, where $\lambda_{\min}$ is the smallest and $\lambda_{\max}$ is the largest eigenvalue of Q_0. The inequality in the statement (i) from Theorem 3.3 is $(I\cos t - Q_0\sin t)^{-1}\sin t \le 0$ and the largest subset of $\mathcal{I}_0$ where it is true is the interval $M = \left(-\frac{\pi}{2} - \arctan\lambda_{\min}, 0\right]$.

Now, from Theorem 3.3, we get, that for all $\hat{Q}_0 \in \mathbf{S}$: $Q_0 \le \hat{Q}_0$ implies that the solution $\hat{Q}(t) = \tan(It + \arctan\hat{Q}_0)$ is defined for all $t \in M$ and that the inequality $Q(t) \le \hat{Q}(t)$ holds for all $t \in M$. Moreover, M is the largest subset of $\mathcal{I}_0$ with this property.

For a comparison, from Proposition 3.1 we get, that for all $\hat{Q}_0 \in \mathbf{S}$: $Q_0 \le \hat{Q}_0$ implies $Q(t) \le \hat{Q}(t)$ on any subset of $\mathcal{I}_0$ such that the solution $\hat{Q}(t)$ is defined there.

From Theorem 3.3 we may further get other results regarding this equation (3.4), analogous to the one presented above.

The converse statement to that one in Proposition 3.1 was proven in [2] and it says that if a symmetric matrix equation has the order preserving property and the matrix dimension is $n \ge 2$, then it is the Riccati equation. Therefore it is possible, that also converse statements to those in Theorem 3.3 or in Corollary 3.4 can be proven, as well as converse statements to Theorem 2.3 and Corollary 2.7, which represent the discrete case. This remains an open problem. There is only the result published in [4] that deals with the continuous version of the order preserving property and the discrete Riccati equation.

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Institute of Mathematics, Faculty of Mechanical Engineering,
Brno University of Technology,
Technická 2, 616 69 Brno, Czech Republic
E-mail: `ruzickova@fme.vutbr.cz`

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EQUIVALENCE OF ILL-POSED DYNAMICAL SYSTEMS

Tomoharu Suda

Abstract. The problem of topological classification is fundamental in the study of dynamical systems. However, when we consider systems without well-posedness, it is unclear how to generalize the notion of equivalence. For example, when a system has trajectories distinguished only by parametrization, we cannot apply the usual definition of equivalence based on the phase space, which presupposes the uniqueness of trajectories.

In this study, we formulate a notion of "topological equivalence" using the axiomatic theory of topological dynamics proposed by Yorke [7], where dynamical systems are considered to be shift-invariant subsets of a space of partial maps. In particular, we study how the type of problems can be regarded as invariants under the morphisms between systems and how the usual definition of topological equivalence can be generalized.

This article is intended to also serve as a brief introduction to the axiomatic theory of ordinary differential equations (or topological dynamics) based on the formalism presented in [6].

1. Introduction

The purpose of the present article is to explain what an axiomatic theory of ordinary differential equations is and how it enables us to classify "flows" without well-posedness assumptions.

In the first place, it is natural to ask why we need an axiomatic theory of ordinary differential equations here. A short answer is that the usual criteria of classification require too much to be applicable to those without well-posedness.

In the study of dynamical systems of flows, we classify systems according to the notion of topological equivalence, which is defined as follows [5].

Definition 1.1 (Topological equivalence)**.** Let X and Y be topological spaces. Two flows $\Phi\colon \mathbb{R}\times X \to X$ and $\Psi\colon \mathbb{R}\times Y\to Y$ are *topologically equivalent* if there exists a homeomorphism $h\colon X\to Y$ such that each orbit of Φ is mapped to an orbit of Ψ preserving the orientation of the orbit.

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However, it poses an inherent difficulty to generalize it to the systems without well-posedness in the sense of Hadamard, in particular, uniqueness. Let us illustrate this point with examples.

Example 1.2. If the uniqueness of orbits is not assumed, topological equivalence as defined above does not define an equivalence relation. For example, let us consider two "flows" defined on $\mathbb{R}$ by

(1) $\dot{x} = 1$

(2) $\dot{x} = 3x^{\frac{2}{3}}$.

The identity map is a homeomorphism that sends each orbit of (1) to an orbit of (2), preserving the orientation. However, the number of equilibria is clearly different between these two systems.

The problem of the last example can be amended straightforwardly by requiring that the inverse of homeomorphisms also preserve the orbits. However, even if we require so, problems remain.

Example 1.3. The following systems on $\mathbb{R}$ are indistinguishable if we use the same criteria as in Definition 1.1:

(1) $\dot{x} = 1$.

(2) $\dot{x} \in \{1/2, 1\}$.

(3) $\dot{x} \in [1/2, 1]$.

Here, systems (2) and (3) are differential inclusions (the definition and details can be found, for example, in [1]). Even if we require that the inverse of homeomorphisms also preserve the orbits, we still cannot distinguish them. This is because there exists only one orbit if we ignore the parametrization.

Thus, in the classification of systems without well-posedness, it is necessary to consider a kind of "topological equivalence", which does not entirely ignore the parametrization. One of the valuable properties of an axiomatic theory of ODE is that we may consider the space of solutions without mentioning problems. This enables us to construct a general framework to treat such classification problems.

In general, an axiomatic theory of ODE consists of two ingredients. One is a space of partial maps, later regarded as a space of "solutions". Another is a set of axioms to be satisfied by such "solutions". Depending on the selection of the above two elements, possibly we obtain many different theories. However, there are mainly two formalisms of the axiomatic theory of ODE. In J.A. Yorke's formalism, partial maps with open domains are considered [7]. On the other hand, V.V. Filippov's theory is based on partial maps with closed domains [3, 4]. This difference in the choice of the class of partial maps results in a significant difference in the treatment. Here we consider a generalization of Yorke's formalism since it is easier to consider the generalization of flows and the problem of their classification within this framework, although V.V. Filippov's theory is much more developed (actually, the details of J.A. Yorke's formalism have not been published except for a small portion).

In this article, we consider the problem of classification of general dynamical systems based on Yorke's formalism of axiomatic theory of ODE. While the theory given here is based on [6], we use an improved formulation in this article, and new results on the description of dynamics are also presented.

In what follows, we assume that X is a second-countable metric space and G is a locally compact second-countable metrizable topological group, e.g., $\mathbb{R}$.

2. Yorke's formalism

First, let us define the notion of partial maps as used here.

Definition 2.1 (Partial maps)**.** A continuous map $\phi\colon D \to X$ is a *partial map from G to X* if $D \subset G$ is a nonempty open set.

The set of all partial maps is denoted by $C_p(G, X)$. For each $\phi\colon D \to X$, we set $\operatorname{dom}\phi := D$.

A partial map $\phi \in C_p(G, X)$ with a connected domain is *maximally defined* if, for all $\psi \in C_p(G, X)$ with a connected domain, the condition $\operatorname{dom}\phi \subset \operatorname{dom}\psi$ and $\phi = \psi$ on $\operatorname{dom}\phi$ implies $\phi = \psi$.

The set of all maximally defined partial maps is denoted by $C_s(G, X)$.

Remark 2.2. It is convenient to define the inverse image of a subset $A \subset X$ under a partial map $\phi\colon G \to X$ by

$$\phi^{-1}(A) := \{g \in G \mid g \in \operatorname{dom}\phi \text{ and } \phi(g) \in A\}.$$

In particular, we have $\operatorname{dom}\phi = \phi^{-1}(X)$. By the continuity on the domain, the inverse image of an open set under a partial map is always open.

We topologize $C_s(G, X)$ by introducing the topology of compact convergence (with modifications). That is, we define $\phi_n \to \phi$ as $n \to \infty$ in $C_s(G, X)$ if and only if, for all compact subsets $K \subset \operatorname{dom}\phi$, we have $K \subset \operatorname{dom}\phi_n$ for sufficiently large n and $\sup_{t\in K} d(\phi_n(t), \phi(t)) \to 0$ as $n \to \infty$.

This topology can also be described using the compact-open topology (Lemma 2.3 in [6]). In this description, subbases are the sets of the form

$$W(K, V) := \{\phi \in C_s(G, X) \mid K \subset \operatorname{dom}\phi \text{ and } \phi(K) \subset V\},$$

where $K \subset G$ is compact and $V \subset X$ is open.

The problem here is that $C_s(G, X)$ need not be Hausdorff in this topology.

Example 2.3 (Yorke [7])**.** Consider a sequence of maps $\{\phi_n\}_{n\in\mathbb{N}} \subset C_s(\mathbb{R}, \mathbb{R})$ given by

$$\phi_n(t) := \frac{1}{t^2 + \frac{1}{n}},$$

and partial maps $\phi^{\pm}$ defined by $\phi^{\pm}(t) = \frac{1}{t^2}$, $\operatorname{dom}\phi^+ = (0, \infty)$ and $\operatorname{dom}\phi^- = (-\infty, 0)$. Then the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to both ϕ^+ and ϕ^- in $C_s(\mathbb{R}, \mathbb{R})$. Consequently, $C_s(\mathbb{R}, \mathbb{R})$ is not Hausdorff.

It is worth noting that $C_s(G, X)$ satisfies separation axioms weaker than Hausdorff.

Proposition 2.4. *The space $C_s(G,X)$ is T_1.*

Proof. As X is a second-countable metric space, it is in particular T_1. Let ϕ and ψ be two distinct partial maps in $C_s(G,X)$. Since ϕ and ψ are maximally defined, either $\operatorname{dom}\phi \cap \operatorname{dom}\psi$ is empty or there exists $g \in \operatorname{dom}\phi \cap \operatorname{dom}\psi$ with $\phi(g) \neq \psi(g)$. In the former case, we may take any $g' \in \operatorname{dom}\psi$ to obtain $\phi \notin W(\{g'\},X)$. In the latter case, there exists an open neighborhood V of $\psi(g)$ such that $\phi(g) \notin V$. Therefore, we have $\phi \notin W(\{g\},V)$. □

Remark 2.5. If X is discrete, $C_s(G,X)$ is Hausdorff.

To justify the use of sequences in the analysis, we consider the following construction originally due to Yorke.

Definition 2.6. For a subset $S \subset C_s(G,X)$, we define a partial map $e_S \colon G \times S \to G \times X$ by

$$e_S(g,\phi) := \big(g,\phi(g)\big)\,.$$

For each subset $W \subset G \times X$, we define

$$S^*W := e_S^{-1}(W) = \big\{(g,\phi) \mid \phi \in S, g \in \operatorname{dom}\phi, \big(g,\phi(g)\big) \in W\big\}\,.$$

We call S^*W the *star-construction* defined by S and W.

Lemma 2.7. *For a nonempty subset $S \subset C_s(G,X)$, the partial map $e_S \colon G \times S \to G \times X$ is well-defined, that is, it is continuous on the domain, which is nonempty and open.*

Proof. It is sufficient to show that $e_S^{-1}(W)$ is open if $W \subset G \times X$ is open. Let W be open and $(g,\phi) \in e_S^{-1}(W)$. Then, we can find an open neighborhood U_0 of g and V of $\phi(g)$ with $U_0 \times V \subset W$. Since $U_0 \cap \phi^{-1}(V)$ is an open neighborhood of g and G is locally compact, there exists another open neighborhood U of g such that $U \subset \bar{U} \subset U_0 \cap \phi^{-1}(V)$ and $\bar{U}$ is compact. Then $U \times W(\bar{U},V)$ is an open neighborhood of (g,ϕ) contained in $e_S^{-1}(W)$. □

The map e_S can be seen as an extended evaluation map, and consequently, the star-construction S^*W is an abstraction of the initial value problem on W with solutions in S. We can show that the space S^*W is Hausdorff and second-countable under our assumptions on X and G (Theorem 2.8 in [6]).

In the next definition, we introduce the main additional axioms, which are an abstraction of the conditions for well-posedness.

Definition 2.8. Let $S \subset C_s(G,X)$.

(1) The subspace S satisfies the *compactness axiom* if e_S is a proper map.
(2) The subspace S satisfies the *existence axiom* on W if $e_S \colon e_S^{-1}(W) \to W$ is surjective.
(3) The subspace S satisfies the *uniqueness axiom* on W if $e_S \colon e_S^{-1}(W) \to W$ is injective.
(4) The subspace S has a *domain* D if e_S is defined on $D \times S$.

The interpretation of the existence and uniqueness axioms is straightforward. The compactness axiom is an abstraction of the continuous dependence on the initial conditions (see Theorem 2.3 in [6]). If a space of solutions S has a domain D, we may regard S to be globally defined on D.

Remark 2.9. The formulation of the theory given here is somewhat different from that in [6] or [7], which does not involve the extended evaluation map e_S. However, it is easily observed to be equivalent.

The apparatuses introduced so far enable us to describe an initial value problem and the corresponding space of solutions. Based on this framework, the dynamics are described using the shift map.

Theorem 2.10 (Shift map, Theorem 3.1 in [6]). *The shift map* $\sigma\colon G\times C_p(G,X)\to C_p(G,X)$*, which is defined by*

$$\sigma(g,\phi)(x) := \phi(xg)$$

for $x\in \mathrm{dom}(\phi)g^{-1}$*, is continuous and satisfies the following conditions:*

(1) *For each* $\phi\in C_p(G,X)$ *we have* $\sigma(e,\phi)=\phi$.

(2) *For all* g, $h\in G$ *and* $\phi\in C_p(G,X)$*, we have* $\sigma(g,\sigma(h,\phi))=\sigma(gh,\phi)$.

That is, σ *is a left* G*-action.*

The correspondence with the usual theory of dynamical systems is given by the following theorem, which claims that flows can be identified with well-behaved subspaces of $C_s(G,X)$. We may regard this to be one of the fundamental theorems of Yorke's theory.

Theorem 2.11 (Theorem 2.3 in [7] and Theorem 3.3 in [6]). *Let* X *be locally compact. Then a* σ*-invariant subset* $S\subset C_s(G,X)$ *satisfies the compactness, existence, and uniqueness axioms and has domain* G *if and only if it is given by a left* G*-action* $\pi_S : G\times X\to X$ *on* X *via*

$$S := \{\pi_S(\cdot,x)\mid x\in X\}\,. \tag{2.1}$$

Thus, our theory subsumes that of flows, and in this sense, it is a generalization of the theory of topological dynamics.

3. Concatenation of solutions and conditional evolution of trajectories

For the description of dynamics in the case $G=\mathbb{R}$, an interesting question is when the concatenation of solutions is admissible. In Yorke's formalism, this property is formulated as follows.

Definition 3.1. A subspace $S\subset C_s(\mathbb{R},X)$ satisfies the *switching axiom* if S contains the map defined by

$$\psi(t)=\begin{cases}\phi_1(t) & (t\le\tau)\\ \phi_2(t) & (t\ge\tau)\end{cases}$$

whenever ϕ_1, $\phi_2 \in S$ satisfy

$$\phi_1(\tau) = \phi_2(\tau)$$

for some $\tau \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$.

Remark 3.2. Compared to other axiomatic theories of ODE or semiflows, such as Filippov's theory or Ball's theory of generalized semiflows [2], it is one of the characteristics of Yorke's formalism that it does not require the concatenation property by default.

The rules of time evolution for a system $S \subset C_s(\mathbb{R}, X)$ are described in terms of the conditional evolution of trajectories. For example, an autonomous ODE can be regarded to describe how a trajectory may be extended given the present position in the phase space. Therefore, we introduce the following notion of conditional solution spaces, which represent the rules of time evolution as inferred from the past data.

Definition 3.3. Let $S \subset C_s(\mathbb{R}, X)$. For $\phi \in S$ and $\tau \in \operatorname{dom}\phi$, we define

$$S(\phi|_{(-\infty,\tau]}) := \{\psi|_{[\tau,\infty)} \mid \psi \in S \text{ and } \psi(t) = \phi(t) \ \text{ for } t \leq \tau\}$$
$$S(\tau, x) := \{\psi|_{[\tau,\infty)} \mid \psi \in S \text{ and } \psi(\tau) = x\}$$

The following result makes it clear that the switching axiom is actually an axiom restricting the rules of time evolution. In short, knowing all the past makes no difference if and only if the switching axiom holds.

Proposition 3.4. *A subspace $S \subset C_s(\mathbb{R}, X)$ satisfies the switching axiom if and only if*

$$S\left(\phi|_{(-\infty,\tau]}\right) = S\big(\tau, \phi(\tau)\big)$$

for all $\phi \in S$ and $\tau \in \operatorname{dom}\phi$.

Proof. Let S satisfy the switching axiom, and fix $\phi \in S$ and $\tau \in \operatorname{dom}\phi$. By definition, we have

$$S\left(\phi|_{(-\infty,\tau]}\right) \subset S\big(\tau, \phi(\tau)\big).$$

If $\psi|_{[\tau,\infty)} \in S(\tau, \phi(\tau))$, we have $\phi(\tau) = \psi(\tau)$ and therefore we may apply the switching axiom to deduce that $\psi|_{[\tau,\infty)} \in S\left(\phi|_{(-\infty,\tau]}\right)$. Therefore $S\left(\phi|_{(-\infty,\tau]}\right) = S(\tau, \phi(\tau))$.

Conversely, let

$$S\left(\phi|_{(-\infty,\tau]}\right) = S\big(\tau, \phi(\tau)\big)$$

for all $\phi \in S$ and $\tau \in \operatorname{dom}\phi$ and fix ϕ_1, $\phi_2 \in S$ with $\phi_1(\tau) = \phi_2(\tau)$ for some $\tau \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$. Then we have

$$\phi_2|_{[\tau,\infty)} \in S\big(\tau, \phi_2(\tau)\big) = S\big(\tau, \phi_1(\tau)\big) = S\big(\phi_1|_{(-\infty,\tau]}\big).$$

Therefore there exists $\psi \in S$ with $\psi|_{[\tau,\infty)} = \phi_2|_{[\tau,\infty)}$ and $\psi|_{(-\infty,\tau]} = \phi_1|_{(-\infty,\tau]}$. Consequently, S satisfies the switching axiom. $\square$

The next result is obvious.

Corollary 3.5. *If a subspace $S \subset C_s(\mathbb{R}, X)$ satisfies the uniqueness axiom on $\mathbb{R} \times X$, S satisfies the switching axiom.*

Thus, the concatenation property can be seen as an analog for the Markov property, and we may assume it if the state of the time evolution is completely determined by the position in the phase space. Also, it follows that if the rule of the time evolution involves other state variables, such as the history of the trajectory, then we cannot expect the concatenation property to hold.

4. Generalizations of topological equivalence

So far, we have considered individual systems. At this point, we may ask how the relationship between them is described under this framework. In general, to consider the relationship between mathematical objects, it is necessary to introduce the notion of morphisms. For the star-constructions, we may define it as follows.

Definition 4.1 (Morphisms of the star-construction)**.** Let $S \subset C_s(G, X)$, $S' \subset C_s(G', X')$, $W \subset G \times X$ and $W' \subset G' \times X'$. A *morphism* between the star-constructions S^*W and $(S')^*W'$ is a triplet of continuous maps

$$H\colon S^*W \to (S')^*W' , \qquad k\colon W \to W' , \qquad \eta\colon S \to S'$$

such that following diagrams commute:

$$\begin{array}{ccc} S^*W & \xrightarrow{H} & (S')^*(W') \\ {\scriptstyle e_S}\big\downarrow & & \big\downarrow{\scriptstyle e_{S'}} \\ W & \xrightarrow{k} & W' \end{array} \qquad \begin{array}{ccc} S^*W & \xrightarrow{H} & (S')^*(W') \\ {\scriptstyle p_S}\big\downarrow & & \big\downarrow{\scriptstyle p_{S'}} \\ S & \xrightarrow{\eta} & S' \end{array}$$

where p_S and $p_{S'}$ are projections to the map component. We denote a morphism by $\langle H, k, \eta\rangle\colon S^*W \to (S')^*W'$.

If there exists a morphism such that H, k and η are homeomorphisms, then $\langle H, k, \eta\rangle$ is an *isomorphism* and S^*W and $(S')^*W'$ are *isomorphic.*

The axioms listed in Definition 2.8 are preserved by isomorphisms.

Example 4.2. It can be shown that the three systems in Example 1.3 are not isomorphic. Indeed, system (1) satisfies the uniqueness axiom and the compactness axiom. The other systems lack uniqueness. While system (3) satisfies the compactness axiom, system (2) does not.

The equivalence class of subsets of $C_s(G, X)$ under the isomorphism relation is rather large. For example, continuous flows are identified:

Theorem 4.3 (Theorem 4.5 in [6])**.** *Let X be locally compact, and G be connected. If a σ-invariant subset $S \subset C_s(G, X)$ satisfies the compactness, existence, and uniqueness axioms on $G \times X$ and has domain G, $S^*(G \times X)$ is isomorphic to $S_0^*(G \times X)$, where*

$$S_0 := \{\psi_x \in C_s(G, X) \mid x \in X \quad \textit{and} \quad \psi_x(g) = x \quad \textit{for all} \quad g \in G\} .$$

As Yorke's axioms are abstraction of the well-posedness properties of the initial value problems, the classification induced by the isomorphism notion can be seen as that of the types of problems based on how well-posed they are. Considering this point, a more useful notion is defined as follows.

Definition 4.4 (Phase space preserving morphism)**.** Let $S \subset C_s(G, X)$, $S' \subset C_s(G', X')$, $W \subset G \times X$ and $W' \subset G' \times X'$. A morphism $\langle H, k, \eta \rangle$ between the star-constructions S^*W and $(S')^*W'$ *preserves phase space* if k has a form $k = (\tau, h)$, where $\tau \colon W \to G'$ and $h \colon W \to X'$ are continuous and $h(g_1, x) = h(g_2, x)$ for all $(g_1, x), (g_2, x) \in W$.

S^*W and $(S')^*W'$ are *isomorphic via phase space preserving isomorphisms* if there exists a phase space preserving isomorphism $\langle H, k, \eta \rangle$ between S^*W and $(S')^*W'$ such that $\langle H^{-1}, k^{-1}, \eta^{-1} \rangle$ also preserves phase space.

The notion of being isomorphic via phase space preserving isomorphisms respects basic dynamical properties, although the direction of time may be reversed.

Theorem 4.5 (Theorem 4.14 in [6])**.** *Let the star-constructions $S^*(G \times X)$ and $(S')^*(G' \times X')$ be isomorphic via a phase space-preserving isomorphism $\langle H, k, \eta \rangle : S^*(G \times X) \to (S')^*(G' \times X')$. Then we have*

$$\hat{h}\left(\mathcal{O}(\phi)\right) = \mathcal{O}\left(\eta(\phi)\right)$$

for all $\phi \in S$, where $k = (\tau, h)$. Here an orbit $\mathcal{O}(\phi)$ of $S^*(G \times X)$ is the set of the form $\mathcal{O}(\phi) := \{\phi(g) \mid g \in \operatorname{dom} \phi\}$, where $\phi \in S$, and the map $\hat{h}$ is defined by $\hat{h}(x) := h(g, x)$ for some $g \in W_x := \{g \in G \mid (g, x) \in W\}$.

The notion of isomorphisms can be improved if we require an additional isotopy condition. Then we obtain another, more stringent generalization of the usual topological equivalence.

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Department of Mathematics, Faculty of Science and Technology,
Keio University, Yagami Campus: 3-14-1 Hiyoshi, Kohoku-ku,
Yokohama, Kanagawa 2238522, Japan
E-mail: `tomoharu.suda@keio.jp`

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SOLUTIONS OF AN ADVANCE-DELAY DIFFERENTIAL EQUATION AND THEIR ASYMPTOTIC BEHAVIOUR

Gabriela Vážanová

Abstract. The paper considers a scalar differential equation of an advance-delay type

$$\dot{y}(t) = -\Big(a_0 + \frac{a_1}{t}\Big)\, y(t-\tau) + \Big(b_0 + \frac{b_1}{t}\Big)\, y(t+\sigma)\,,$$

where constants a_0, b_0, τ and σ are positive, and a_1 and b_1 are arbitrary. The behavior of its solutions for $t \to \infty$ is analyzed provided that the transcendental equation

$$\lambda = -a_0 \mathrm{e}^{-\lambda\tau} + b_0 \mathrm{e}^{\lambda\sigma}$$

has a positive real root. An exponential-type function approximating the solution is searched for to be used in proving the existence of a semi-global solution. Moreover, the lower and upper estimates are given for such a solution.

1. Preliminaries

In [6, Section 2.1] a general scalar linear equation

$$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)y(t+\sigma(t)) \tag{1.1}$$

is considered with Lipschitz continuous $c, d\colon [t_0,\infty) \to [0,\infty)$, $\tau\colon [t_0,\infty) \to [0, r_1]$ and $\sigma\colon [t_0,\infty) \to [0,r_2]$, where $r_i > 0$, $i = 1,2$, and the existence of right semi-global solutions to (1.1) is proved . The right semi-global solution is defined as follows. A continuous function $y\colon [t_0 - r_1, \infty) \to \mathbb{R}$ is a right semi-global solution to (1.1) on $[t_0 - r_1,\infty)$ if it is continuously differentiable on $[t_0,\infty)$ and satisfies (1.1) on $[t_0,\infty)$.

The present paper considers a particular case of equation (1.1), specifying c and d as

$$c(t) = a_0 + \frac{a_1}{t}\,, \quad d(t) = b_0 + \frac{b_1}{t}\,,$$

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where a_0, b_0 are positive constants, a_1 and b_1 are arbitrary reals, $\tau > 0$ and $\sigma > 0$. Then (1.1) can be written as

$$\dot{y}(t) = -\Big(a_0 + \frac{a_1}{t}\Big)y(t-\tau) + \Big(b_0 + \frac{b_1}{t}\Big)y(t+\sigma)\,. \tag{1.2}$$

Equation (1.2) is now analyzed. An approximate solution of (1.2) is found in the form

$$y_{as}(t) = \mathrm{e}^{\lambda t} t^{-r}\Big(1 - \frac{A}{t}\Big) \tag{1.3}$$

where λ is a positive real root of the transcendental equation

$$\lambda = -a_0 \mathrm{e}^{-\lambda\tau} + b_0 \mathrm{e}^{\lambda\sigma} \tag{1.4}$$

and the coefficients A and r will be specified dependending on a_i, b_i, $i = 0, 1$, τ, σ and λ. Then, it is proved that there exists a right semi-global solution to (1.2) which can be estimated from below and from above by functions having a form of the suggested approximate solution y_{as}.

2. Auxiliary lemma

Being a part of Theorem 2 in [6], the following auxiliary result will be used in the paper to prove the existence of a solution with properties described above in Section 1.

Lemma 2.1. *Consider bounded continuous functions $\mathcal{L}, \mathcal{R}\colon [t_0 - r_1, \infty) \to \mathbb{R}$, $\mathcal{L}(t) \leqslant \mathcal{R}(t)$, $t \in [t_0 - r_1, \infty)$ and a Lipschitz continuous function $\varphi\colon [t_0 - r_1, t_0] \to \mathbb{R}$ satisfying $\varphi(t_0) = 0$. Moreover, let*

$$\mathcal{L}(t) \leqslant -c(t)\exp\left(\int_t^{t-\tau(t)} \mathcal{L}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_t^{t+\sigma(t)} \mathcal{L}(s)\,\mathrm{d}s\right), \tag{2.1}$$

$$\mathcal{R}(t) \geqslant -c(t)\exp\left(\int_t^{t-\tau(t)} \mathcal{R}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_t^{t+\sigma(t)} \mathcal{R}(s)\,\mathrm{d}s\right) \tag{2.2}$$

on $[t_0, \infty)$ and

$$\mathcal{L}(t) \leqslant -c(t_0)\exp\left(\int_{t_0}^{t_0-\tau(t_0)} \mathcal{L}(s)\,\mathrm{d}s\right) + d(t_0)\exp\left(\int_{t_0}^{t_0+\sigma(t_0)} \mathcal{L}(s)\,\mathrm{d}s\right) + \varphi(t), \tag{2.3}$$

$$\mathcal{R}(t) \geqslant -c(t_0)\exp\left(\int_{t_0}^{t_0-\tau(t_0)} \mathcal{R}(s)\,\mathrm{d}s\right) + d(t_0)\exp\left(\int_{t_0}^{t_0+\sigma(t_0)} \mathcal{R}(s)\,\mathrm{d}s\right) + \varphi(t) \tag{2.4}$$

on $[t_0 - r_1, t_0]$. Then, there exists a right semi-global solution $y(t)$ of (1.1) on $[t_0 - r_1, \infty)$ such that $y(t_0 - r_1) = 1$ and

$$\exp\left(\int_{t_0-r_1}^t \mathcal{L}(s)\,\mathrm{d}s\right) \leqslant \ y(t) \leqslant \exp\left(\int_{t_0-r_1}^t \mathcal{R}(s)\,\mathrm{d}s\right), \quad t \in [t_0 - r_1, \infty)\,. \tag{2.5}$$

Remark 2.2. In applications of Lemma 2.1, a crucial role is played by a proper choice of functions $\mathcal{L}$ and $\mathcal{R}$ because this is often not an easy task. From this point of view, an important contribution of the paper is, among others, in the construction of such functions.

3. Existence of approximate solutions

In this section we will look for approximate solutions of the equation (1.2) in the form (1.3).

Consider transcendental equation (1.4). In the rest of the paper, a typical assumption is that this equation, which we rewrite in the form

$$f(\lambda) := \lambda + a_0 \mathrm{e}^{-\lambda\tau} - b_0 \mathrm{e}^{\lambda\sigma} = 0\,,$$

has a real root $\lambda = \lambda^*$ such that $f'(\lambda^*) \neq 0$ or $f'(\lambda^*) > 0$ where

$$f'(\lambda) = 1 - a_0 \tau \mathrm{e}^{-\lambda\tau} - b_0 \sigma \mathrm{e}^{\lambda\sigma}\,.$$

The lemma below gives sufficient conditions for the existence of such a real root.

Lemma 3.1. *Let μ, ν be positive numbers such that $\mu < \nu$, $f(\mu) < 0$ and $f(\nu) > 0$. If, moreover,*

$$V(\mu, \nu) := 1 - a_0 \tau \mathrm{e}^{-\mu\tau} - b_0 \sigma \mathrm{e}^{\nu\sigma} > 0\,,$$

then there exists a positive root λ^* of equation (1.4) *such that $f'(\lambda^*) > 0$.*

Proof. It may be seen that there exists a root $\lambda = \lambda^*$ of equation $f(\lambda) = 0$ such that $\lambda^* \in (\mu, \nu)$. Since, for $\lambda \in (\mu, \nu)$, $f'(\lambda) > V(\mu, \nu) > 0$, the root is the only one in the interval (μ, ν) and $f'(\lambda^*) > 0$. □

Example 3.2. Let $a_0 = 1$, $b_0 = 2$, $\sigma = 1/10$ and $\tau = 1$. Then,

$$f(\lambda) := \lambda + \mathrm{e}^{-\lambda} - 2\mathrm{e}^{0.1\lambda}.$$

All the hypotheses of Lemma 3.1, where $\mu = 2$, $\nu = 3$, are satisfied since

$$f(2) = 2 + \mathrm{e}^{-2} - 2\mathrm{e}^{0.2} \doteq -0.307 < 0, \quad f(3) := 3 + \mathrm{e}^{-3} - 2\mathrm{e}^{0.3} \doteq 0.350 > 0$$

and

$$V(2,3) := 1 - \mathrm{e}^{-2} - 2 \cdot 10^{-1} \mathrm{e}^{0.3} \doteq 0.595 > 0\,.$$

Therefore, there exists a positive root $\lambda = \lambda^* \in (2,3)$ of the equation $f(\lambda) = 0$ such that $f'(\lambda^*) > 0$. We refer to Figure 1, where the positive root $\lambda^* \doteq 2.479$, shown in red, has the property $f'(\lambda^*) > 0$. The remaining real roots are $\lambda^{**} \doteq -1.047$ and $\lambda^{***} \doteq 25.426$.

The formula, used below is a consequence of the binomial one: For $t \to \infty$ and $\alpha, \beta \in \mathbb{R}$, the asymptotic representation

$$(t-\alpha)^\beta = t^\beta \left[1 - \frac{\alpha\beta}{t} + \frac{\beta(\beta-1)\alpha^2}{2t^2} + o\left(\frac{1}{t^2}\right)\right] \tag{3.1}$$

holds.

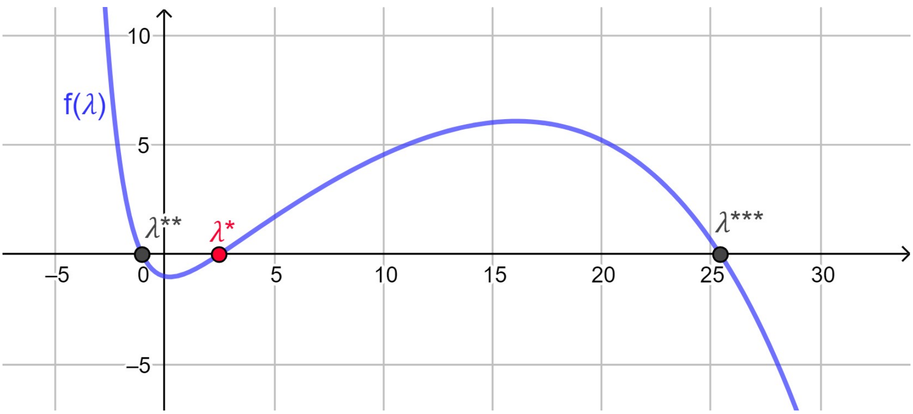

FIG. 1: To Example 3.2: Function $f(\lambda)$ and its real roots

Let positive constants a_0, b_0, τ and σ be given. Suppose that the equation (1.4) has a real root λ^* such that $f'(\lambda^*) \neq 0$. The calculation below indicates that taking $\lambda = \lambda^*$ the parameters r and A in formula (1.3) for y_{as} should be chosen as

$$r = \frac{a_1 \mathrm{e}^{-\lambda^* \tau} - b_1 \mathrm{e}^{\lambda^* \sigma}}{f'(\lambda^*)}\,, \tag{3.2}$$

$$A = \frac{r\left(-a_1 \tau \mathrm{e}^{-\lambda^* \tau} - b_1 \sigma \mathrm{e}^{\lambda^* \sigma}\right) + 0.5 r(-r-1)\left(a_0 \tau^2 \mathrm{e}^{-\lambda^* \tau} - b_0 \sigma^2 \mathrm{e}^{\lambda^* \sigma}\right)}{f'(\lambda^*)}\,. \tag{3.3}$$

Substituting the assumed form of the solution (1.3) into equation (1.2) we obtain an approximate expression

$$\begin{aligned}
&\lambda \mathrm{e}^{\lambda t} t^{-r} - (r + A\lambda)\mathrm{e}^{\lambda t} t^{-r-1} + A(r+1)\mathrm{e}^{\lambda t} t^{-r-2} \propto \\
&-\left(a_0 + \frac{a_1}{t}\right) \mathrm{e}^{\lambda(t-\tau)} (t-\tau)^{-r} \left(1 - \frac{A}{t-\tau}\right) + \left(b_0 + \frac{b_1}{t}\right) \mathrm{e}^{\lambda(t+\sigma)} (t+\sigma)^{-r} \left(1 - \frac{A}{t+\sigma}\right).
\end{aligned}$$

Using formula (3.1), we have

$$\begin{aligned}
\lambda \mathrm{e}^{\lambda t} t^{-r} &- (r + A\lambda)\mathrm{e}^{\lambda t} t^{-r-1} + A(r+1)\mathrm{e}^{\lambda t} t^{-r-2} \propto \\
&- \left(a_0 + \frac{a_1}{t}\right) \mathrm{e}^{\lambda(t-\tau)} t^{-r} \left[1 + \frac{r\tau}{t} - \frac{r(-r-1)\tau^2}{2t^2}\right] \left(1 - \frac{A}{t} - \frac{A\tau}{t^2}\right) \\
&\quad + \left(b_0 + \frac{b_1}{t}\right) \mathrm{e}^{\lambda(t+\sigma)} t^{-r} \left[1 - \frac{r\sigma}{t} - \frac{r(-r-1)\sigma^2}{2t^2}\right] \left(1 - \frac{A}{t} + \frac{A\sigma}{t^2}\right)
\end{aligned}$$

and, consequently,

$$
\begin{aligned}
&\lambda - (r + A\lambda)t^{-1} + A(r+1)t^{-2} \propto \\
&\qquad - \left(a_0 + \frac{a_1}{t}\right) \mathrm{e}^{-\lambda\tau} \left[1 + \frac{r\tau}{t} - \frac{r(-r-1)\tau^2}{2t^2}\right] \left(1 - \frac{A}{t} - \frac{A\tau}{t^2}\right) \\
&\qquad\qquad + \left(b_0 + \frac{b_1}{t}\right) \mathrm{e}^{\lambda\sigma} \left[1 - \frac{r\sigma}{t} - \frac{r(-r-1)\sigma^2}{2t^2}\right] \left(1 - \frac{A}{t} + \frac{A\sigma}{t^2}\right).
\end{aligned}
$$

Matching the coefficients corresponding at the powers of t, we derive

$$
\begin{aligned}
t^0 \ :& \qquad\quad \lambda &=&\ -a_0\mathrm{e}^{-\lambda\tau} + b_0\mathrm{e}^{\lambda\sigma}, \\
t^{-1} :& \ -(r + A\lambda) &=&\ a_0A\mathrm{e}^{-\lambda\tau} - a_0\mathrm{e}^{-\lambda\tau}r\tau - a_1\mathrm{e}^{-\lambda\tau} - b_0A\mathrm{e}^{\lambda\sigma} - b_0r\sigma\mathrm{e}^{\lambda\sigma} + b_1\mathrm{e}^{\lambda\sigma}, \\
t^{-2} :& \quad\ A(r+1) &=&\ a_0A\tau\mathrm{e}^{-\lambda\tau} + a_1A\mathrm{e}^{-\lambda\tau} + b_0A\sigma\mathrm{e}^{\lambda\sigma} - b_1A\mathrm{e}^{\lambda\sigma} \\
&&& + r\left(a_0A\tau\mathrm{e}^{-\lambda\tau} - a_1\tau\mathrm{e}^{-\lambda\tau} + b_0A\sigma\mathrm{e}^{\lambda\sigma} - b_1\sigma\mathrm{e}^{\lambda\sigma}\right) \\
&&& - \frac{r(r+1)}{2}\left(a_0\tau^2\mathrm{e}^{-\lambda\tau} - b_0\sigma^2\mathrm{e}^{\lambda\sigma}\right).
\end{aligned}
$$

From the second equation (after substituting the first one), we get

$$
r = \frac{a_1\mathrm{e}^{-\lambda\tau} - b_1\mathrm{e}^{\lambda\sigma}}{1 - a_0\tau\mathrm{e}^{-\lambda\tau} - b_0\sigma\mathrm{e}^{\lambda\sigma}} = \frac{a_1\mathrm{e}^{-\lambda\tau} - b_1\mathrm{e}^{\lambda\sigma}}{f'(\lambda)} \tag{3.4}
$$

and, from the third one,

$$
A = \frac{1}{f'(\lambda)}\left(r\left(-a_1\tau\mathrm{e}^{-\lambda\tau} - b_1\sigma\mathrm{e}^{\lambda\sigma}\right) + \frac{r(-r-1)}{2}\left(a_0\tau^2\mathrm{e}^{-\lambda\tau} - b_0\sigma^2\mathrm{e}^{\lambda\sigma}\right)\right)
$$

which corresponds to (3.2) and (3.3) for $\lambda = \lambda^*$.

4. Existence of semi-global solutions

In this section we formulate and prove the main result of the paper. As mentioned in Section 2, its proof is based on Lemma 2.1.

Theorem 4.1. *Let the transcendental equation* (1.4) *have a positive real root* λ^* *such that* $f'(\lambda^*) > 0$. *Then, for every fixed* $\varepsilon \in (0,1)$ *and* $t \geqslant t_0 - \tau$, *there exist a* $t_0 \in \mathbb{R}$ *and a right semi-global positive solution* $y(t)$ *of equation* (1.2) *on* $[t_0 - \tau, \infty)$ *satisfying the inequalities*

$$
K_1\mathrm{e}^{\lambda^* t}t^{-r}\left(1 - \frac{A-\varepsilon}{t}\right) \leqslant y(t) \leqslant K_2\mathrm{e}^{\lambda^* t}t^{-r}\left(1 - \frac{A+\varepsilon}{t}\right), \quad t \geqslant t_0 - \tau \tag{4.1}
$$

with the coefficients A, r *defined by formulas* (3.2), (3.3) *and*

$$
K_1 = (t_0 - \tau)^r \exp\left(-\lambda^*(t_0 - \tau) + \frac{A-\varepsilon}{t_0 - \tau}\right),
$$

$$
K_2 = (t_0 - \tau)^r \exp\left(-\lambda^*(t_0 - \tau) + \frac{A+\varepsilon}{t_0 - \tau}\right).
$$

Proof. Let $\varepsilon \in (0,1)$ be fixed and let t_0 be large enough for the asymptotic relations discussed below to hold. Let us take

$$
\mathcal{L}(t) := \lambda^* - \frac{r}{t} + \frac{A-\varepsilon}{t^2}, \tag{4.2}
$$

$$\mathcal{R}(t) := \lambda^* - \frac{r}{t} + \frac{A+\varepsilon}{t^2}\,, \tag{4.3}$$
$$\varphi(t) := \mathcal{L}(t) - \mathcal{L}(t_0)\,.$$

We will verify all the hypotheses of Lemma 2.1 and prove that there exists a solution of (1.2) satisfying inequalities (4.6). Obviously, $\mathcal{L}(t) \leqslant \mathcal{R}(t)$ on $[t_0-\tau,\infty)$. We need to show, that (2.1) holds, i.e., that

$$\begin{aligned}\lambda^* - \frac{r}{t} + \frac{A-\varepsilon}{t^2} \leqslant &-\Big(a_0 + \frac{a_1}{t}\Big)\exp\Big(\int_t^{t-\tau}\Big(\lambda^* - \frac{r}{s} + \frac{A-\varepsilon}{s^2}\Big)\,\mathrm{ds}\Big)\\ &+\Big(b_0 + \frac{b_1}{t}\Big)\exp\Big(\int_t^{t+\sigma}\Big(\lambda^* - \frac{r}{s} + \frac{A-\varepsilon}{s^2}\Big)\,\mathrm{ds}\Big)\,.\end{aligned} \tag{4.4}$$

Let us denote the right-hand side of the inequality (4.4) by $T\mathcal{L}(t)$. Then, after integration, we get

$$\begin{aligned}T\mathcal{L}(t) &= -\Big(a_0 + \frac{a_1}{t}\Big)\cdot\exp\Big(-\lambda^*\tau - r\ln(t-\tau) + r\ln(t) - (A-\varepsilon)\Big(\frac{1}{t-\tau} - \frac{1}{t}\Big)\Big)\\ &\quad+\Big(b_0 + \frac{b_1}{t}\Big)\cdot\exp\Big(\lambda^*\sigma - r\ln(t+\sigma) + r\,ln(t) - (A-\varepsilon)\Big(\frac{1}{t+\sigma} - \frac{1}{t}\Big)\Big)\\ &= -\Big(a_0 + \frac{a_1}{t}\Big)\cdot\exp\Big(-\lambda^*\tau + \ln\Big(\frac{t-\tau}{t}\Big)^{-r} - \frac{(A-\varepsilon)\tau}{(t-\tau)t}\Big)\\ &\quad+\Big(b_0 + \frac{b_1}{t}\Big)\cdot\exp\Big(\lambda^*\sigma + \ln\Big(\frac{t+\sigma}{t}\Big)^{-r} + \frac{(A-\varepsilon)\sigma}{(t+\sigma)t}\Big)\\ &= -\Big(a_0 + \frac{a_1}{t}\Big)\mathrm{e}^{-\lambda^*\tau}\cdot\Big(\frac{t-\tau}{t}\Big)^{-r}\cdot\exp\Big(-\frac{(A-\varepsilon)\tau}{(t-\tau)t}\Big)\\ &\quad+\Big(b_0 + \frac{b_1}{t}\Big)\mathrm{e}^{\lambda^*\sigma}\cdot\Big(\frac{t+\sigma}{t}\Big)^{-r}\cdot\exp\Big(\frac{(A-\varepsilon)\sigma}{(t+\sigma)t}\Big)\,.\end{aligned}$$

Using formula (3.1) and the Maclaurin series for e^x, we obtain

$$\begin{aligned}\mathcal{E}_\tau(t) := \exp\Big(-\frac{(A-\varepsilon)\tau}{(t-\tau)t}\Big) &= \exp\Big(-(A-\varepsilon)\tau\frac{1}{t^2}\Big(1 + \frac{\tau}{t} + o\Big(\frac{1}{t^2}\Big)\Big)\Big)\\ &= 1 - \frac{(A-\varepsilon)\tau}{t^2} + o\Big(\frac{1}{t^2}\Big)\,,\\ \mathcal{E}_\sigma(t) := \exp\Big(\frac{(A-\varepsilon)\sigma}{(t+\sigma)t}\Big) &= \exp\Big((A-\varepsilon)\sigma\frac{1}{t^2}\Big(1 - \frac{\sigma}{t} + o\Big(\frac{1}{t^2}\Big)\Big)\Big)\\ &= 1 + \frac{(A-\varepsilon)\sigma}{t^2} + o\Big(\frac{1}{t^2}\Big)\,.\end{aligned}$$

By formula (3.1),

$$T\mathcal{L}(t) = -\Big(a_0 + \frac{a_1}{t}\Big)\mathrm{e}^{-\lambda^*\tau}\cdot\Big[1 + \frac{r\tau}{t} - \frac{r(-r-1)\tau^2}{2t^2} + o\Big(\frac{1}{t^2}\Big)\Big]\cdot\mathcal{E}_\tau$$

$$
\begin{aligned}
&+\left(b_0+\frac{b_1}{t}\right)\mathrm{e}^{\lambda^*\sigma}\cdot\left[1-\frac{r\sigma}{t}-\frac{r(-r-1)\sigma^2}{2t^2}+o\left(\frac{1}{t^2}\right)\right]\cdot\mathcal{E}_\sigma\\
=&-\left(a_0+\frac{a_1}{t}\right)\mathrm{e}^{-\lambda^*\tau}\cdot\left(1+\frac{r\tau}{t}-\frac{r(-r-1)\tau^2}{2t^2}-\frac{(A-\varepsilon)\tau}{t^2}+o\left(\frac{1}{t^2}\right)\right)\\
&+\left(b_0+\frac{b_1}{t}\right)\mathrm{e}^{\lambda^*\sigma}\cdot\left(1-\frac{r\sigma}{t}-\frac{r(-r-1)\sigma^2}{2t^2}+\frac{(A-\varepsilon)\sigma}{t^2}+o\left(\frac{1}{t^2}\right)\right).
\end{aligned}
$$

Matching the coefficients corresponding at the powers of t, we have

$$
\begin{aligned}
t^0\ &:\ \lambda^*=-a_0\mathrm{e}^{-\lambda^*\tau}+b_0\mathrm{e}^{\lambda^*\sigma},\\
t^{-1}&:-r\ =-a_0\mathrm{e}^{-\lambda^*\tau}r\tau-a_1\mathrm{e}^{-\lambda^*\tau}-b_0r\sigma\mathrm{e}^{\lambda^*\sigma}+b_1\mathrm{e}^{\lambda^*\sigma}.
\end{aligned}
$$

The first equation holds due to (1.4). The second one, after simplication, is equivalent to (3.4). For the validity of $\mathcal{L}(t)\leqslant T\mathcal{L}(t)$ on $[t_0-\tau,\infty)$ with t_0 sufficiently large, we need that

$$
\begin{aligned}
t^{-2}:A-\varepsilon\leqslant -a_0\mathrm{e}^{-\lambda^*\tau}&\left(-\frac{r(-r-1)\tau^2}{2}-(A-\varepsilon)\tau\right)-a_1\mathrm{e}^{-\lambda^*\tau}r\tau\\
&+b_0\mathrm{e}^{\lambda^*\sigma}\left(-\frac{r(-r-1)\sigma^2}{2}-(A-\varepsilon)\sigma\right)-b_1\mathrm{e}^{\lambda^*\sigma}r\sigma.
\end{aligned}
$$

This may be rewritten as

$$
A-\varepsilon\leqslant(A-\varepsilon)(1-f'(\lambda^*))+Af'(\lambda^*),
$$

which holds because $0<\varepsilon f'(\lambda^*)$. Therefore, $\mathcal{L}(t)\leqslant T\mathcal{L}(t)$ on $[t_0-\tau,\infty)$.

Since inequalities (2.2), (2.3) and (2.4) may be proved in much the same way, the computations are omitted. The estimates (4.1) follow from (2.5) with $r_1=\tau$ and $\mathcal{L}$, $\mathcal{R}$ defined by (4.2) and (4.3). □

Example 4.2. Consider a particular case of equation (1.2) where $a_0=\mathrm{e}/2$, $b_0=3\mathrm{e}^{-0.1}/2$, $a_1=\mathrm{e}$, $b_1=\mathrm{e}^{-0.1}$, $\tau=1$ and $\sigma=0.1$, i.e.,

$$
\dot{y}(t)=\mathrm{e}\left(\frac{1}{2}+\frac{1}{t}\right)y(t-1)+\mathrm{e}^{-0.1}\left(\frac{3}{2}+\frac{1}{t}\right)y(t+0.1). \tag{4.5}
$$

Then

$$
f(\lambda):=\lambda+\frac{1}{2}\mathrm{e}^{1-\lambda}-\frac{3}{2}\mathrm{e}^{0.1(-1+\lambda)},
$$

equation $f(\lambda)=0$ has a positive root $\lambda=1$ and

$$
f'(\lambda)|_{\lambda=1}=\left(1-\frac{1}{2}\mathrm{e}^{1-\lambda}-\frac{0.3}{2}\mathrm{e}^{0.1(-1+\lambda)}\right)\Big|_{\lambda=1}=0.35>0.
$$

Let $\varepsilon\in(0,1)$ be fixed and let t_0 be sufficiently large. By formulas (3.2), (3.3) we compute $r=A=0$. Then

$$
K_1=\exp\left(-(t_0-1)-\frac{\varepsilon}{t_0-1}\right),\quad K_2=\exp\left(-(t_0-1)+\frac{\varepsilon}{t_0-1}\right).
$$

By Theorem 4.1, there exist a right semi-global positive solution $y(t)$ of equation (4.5) on $[t_0-\tau,\infty)$ satisfying the inequalities

$$K_1\mathrm{e}^t\left(1+\frac{\varepsilon}{t}\right)\leqslant y(t)\leqslant K_2\mathrm{e}^t\left(1-\frac{\varepsilon}{t}\right),\qquad t\geqslant t_0-\tau. \tag{4.6}$$

Note that equation (4.5) has a family of exact solutions $y(t)=c\exp t$ where c is an arbitrary constant. If $K_1<c<K_2$, then these solutions satisfy inequalities (4.6) for $t\to\infty$.

5. Concluding remarks

The paper proves the existence of right semi-global solutions to equation (1.2) deriving their upper and lower estimates, suggested by the form of approximate solutions. The research was motivated by investigations [2, 3, 4] and [6]. The auxiliary Lemma 2.1 is a particular case of Theorem 2 in [6] where this result was proved by the method of monotone iterative sequences, we refer, e.g., to [11]. The investigation carried out is close to [9] dealing with asymptotic properties of solutions of similar classes of equations. In [5] asymptotic properties of solutions are studied for the so-called p-type retarded functional differential equations. For the class of p-type advanced-delayed differential equations a further study may be envisaged of applying (generalizing) the results achieved to such equations. Referring to [10], let us also remark that the subject of the paper is closely related to the Hartman-Wintner theorem for retarded functional differential equations that deals with L_2-perturbations of autonomous delay equations. For rudiments of delayed, advanced, and some classes of advanced-delayed equations, we refer to [1, 7, 8].

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Department of Mathematics,
Faculty of Electrical Engineering and Communication,
Brno University of Technology, Technická 3058/10,
616 00 Brno, Czech Republic
E-mail: `vazanova@vut.cz`

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