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CRITICAL POINTS FOR REACTION-DIFFUSION SYSTEM WITH ONE AND TWO UNILATERAL CONDITIONS

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Abstract. We show the location of so called *critical points*, i.e., couples of diffusion coefficients for which a non-trivial solution of a linear reaction-diffusion system of activator-inhibitor type on an interval with Neumann boundary conditions and with additional non-linear unilateral condition at one or two points on the boundary and/or in the interior exists. Simultaneously, we show the profile of such solutions.

1. Introduction

Let us consider a reaction-diffusion system

$$u_t = d_1 u_{xx} + f(u,v)\,, \quad v_t = d_2 v_{xx} + g(u,v) \qquad \text{in} \quad (0,\ell) \tag{1.1}$$

with Neumann boundary conditions for u

$$u_x(0) = u_x(\ell) = 0 \tag{1.2}$$

and at first with Neumann boundary conditions also for v

$$v_x(0) = v_x(\ell) = 0\,. \tag{1.3}$$

Let us assume there is (U_c, V_c) a stationary and spatially constant solution to (1.1) with (1.2), (1.3), in particular $f(U_c,V_c) = g(U_c,V_c) = 0$. We can assume without loss of generality that the *trivial solution* $(U_c,V_c) = (0,0)$ but keep in mind that in application where u and v represent e.g. concentrations of two chemicals or of two population species they are assumed to be positive.

We will allways assume the Jacobi matrix $B = (b_{ij})$ of (f,g) at (U_c,V_c) satisfies

$$\operatorname{Tr} B < 0 \quad \text{and} \quad \det B > 0\,. \tag{1.4}$$

Then it follows from Hurwitz criteria that the trivial solution (U_c,V_c) is stable as a solution to the corresponding ODE system without diffusion, i.e., for $d_1 = d_2 = 0$.

Finally, we will assume that

$$b_{11} > 0, \quad b_{12}b_{21} < 0, \quad b_{22} < 0\,. \tag{1.5}$$

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Then the RD-system (1.1) is of an activator-inhibitor or a depletion-substrate type if $b_{12} > 0$ or $b_{21} > 0$, respectively. It is well known [7] that under (1.5) an effect of *Turing instability* appears: Only for some diffusion coefficients $(d_1, d_2) \in \mathbb{R}^2_+$ the trivial solution (U_c, V_c) remains stable but becomes unstable for the rest of positive diffusion coefficients. More precisely, there is a system of hyperbolas

$$H_n = \{(d_1, d_2) \in \mathbb{R}^+_2 : d_2 = (\det B - \kappa_n d_1 b_{22})/(b_{21}\kappa_n - d_1 \kappa_n^2)\}\,, \tag{1.6}$$

where $\kappa_n > 0$ is a sequence of *positive* eigenvalues to Neumann BVP

$$u_{xx} + \kappa u = 0 \quad \text{on } (0, \ell)$$

with (1.2). Let us remark that there is no H_0 for $\kappa_0 = 0$. Now, the *domain of stability* D_S of the trivial solution is the set of couples (d_1, d_2) lying to the right from all hyperbolas H_n and the *domain of instability* D_U is the set of (d_1, d_2) lying to the left from at least one hyperbola.

In the rest of this paper we will study only stationary solutions of (1.1) and consider only a linear ODE system

$$d_1 u'' + b_{11} u + b_{12} v = 0\,, \quad d_2 v'' + b_{21} u + b_{22} v = 0 \qquad \text{in } (0, \ell) \tag{1.7}$$

where the prime denotes the derivative w.r.t. to the variable $x \in (0, \ell)$. We will still refer to (1.2) and/or (1.3) where $u_x = u'$ and $v_x = v'$.

It is easy to see that for any $(d_1, d_2) \in \mathbb{R}^2_+$ the pair $(0, 0)$ is a solution to (1.7) with (1.2), (1.3). *Critical points* of a given boundary value problem will be the set of diffusion coefficients $(d_1, d_2) \in \mathbb{R}^2_+$ for which a *nontrivial (spatially nonconstant) solution* exists. It follows from [5] (cf. also Lemma 2.2 below) that the set of critical points of the Neumann BVP (1.7), (1.2), (1.3) is just the system of hyperbolas (1.6).

We will describe and locate the set of critical points if we prescribe, in addition to Neumann BCs, some unilateral condition(s) for the inhibitor v. More precisely, we will describe in the following sections the sets of critical points for the BVPs (1.7) with Neumann boudary conditions (1.2) for activator u and with several types of unilateral conditions for v. Let us remark that we choose the simplest examples in order to be at least partially analytically and numerically tractable. This is the reason to consider only one dimensional space domain and only point-wise unilateral obstacles. This method could be applied for the higher dimensional domain only of a very special form (e.g. a rectangle with unilateral conditions on (a part of) one edge) but we could obtain only a subset of possible critial points only because we can not analytically express all non-trivial solutions of a given unilateral BVP.

Let us finally remark that even the system (1.7) is linear, the unilateral conditions break the linearity, the BVP remains only *positively homogeneous*: only a *non-negative* multiple of a solution is also a solution.

2. A unilateral obstacle for inhibitor

We will start with one point-wise unilateral (one-sided) obstacle.

2.1. **A unilateral obstacle on the boundary.** The simplest unilateral obstacle is given by a Signorini condition prescribed at one boundary point, without loss of generality at $x = \ell$

$$v'(0) = 0, \qquad v(\ell) \geq 0, \quad v'(\ell) \geq 0\,, \quad v(\ell)v'(\ell) = 0\,. \tag{2.1}$$

The last three conditions allow $v(\ell)$ to be non-negative with a non-negative derivative $v'(\ell)$, but only one of them can be positive. If the value is positive, zero Neumann condition must be fulfilled. This BC can be considered as a certain regulation allowing the concentration to be above a prescribed value (here $V_c = 0$) and then the boundary is closed, there is no flux through this part of the boundary. But if $v(\ell)$ decreases below this value, the boudary opens and the inhibitor income from outside is large enough to stop the decrease of $v(\ell)$ below V_c (v satisfies Dirichlet BC in that case). In other words, the simple point of view is that v satifies Signorini BC at $x = \ell$ if and only if it satisfies either Neumann BC with a proper (non-negative) sign of $v(\ell)$ or Dirichlet BC with a proper (non-negative) sign of $v'(\ell)$. Of course, it can exceptionally happen that both $v(\ell) = v'(\ell) = 0$.

Looking for critical points of the BVP (1.7) with (1.2), (2.1), these considerations allow us to decompose this unilateral and hence non-linear Neumann-Signorini BVP onto two problems: on purely Neumann BVP (1.7), (1.2), (1.3) with a proper sign of $v(\ell)$ and on Neumann-Dirichlet BVP (1.7), (1.2),

$$v'(0) = 0, \qquad v(\ell) = 0 \tag{2.2}$$

with a proper sign of $v'(\ell)$.

Lemma 2.1. *Let (u, v) be a solution to one of linear BVPs* (1.7), (1.2), (1.3) *or* (1.7), (1.2), (2.2). *Then (u, v) or $(-u, -v)$ is a solution of the unilateral BVP* (1.7), (1.2), (2.1).

Proof. If (u, v) is a solution to a *linear* BVP then also $(-u, -v)$ is a solution. Now it is necessary to realize that in both BVPs we need to control a sign only of one object. $\square$

Lemma 2.2. *The set of critical points K_N to the BVP* (1.7), (1.2), (1.3) *are just the hyperbolas H_n from* (1.6),

$$K_N = \bigcup_{n=1}^{\infty} H_n\,.$$

The profiles of the corresponding non-trivial solutions for $(d_1, d_2) \in H_n$ are

$$\begin{aligned} u_n(x) &= A(d_2\kappa_n - b_{22})\cos(nx)/b_{21}\,, \\ v_n(x) &= A\cos(nx) \end{aligned} \tag{2.3}$$

with arbitrary $A \in \mathbb{R}$.

Proof. The assertion follows e.g. from [5]. $\square$

Characteristic equation corresponding to the system (1.7) is biquadratic

$$d_1 d_2 r^4 + (d_2 b_{11} + d_1 b_{22})r^2 + \det B = 0$$

and has the (possibly complex) roots $\pm r_1$ and $\pm r_2$. We obtain for any $(d_1, d_2) \in \mathbb{R}_+^2$ with the exception of two half-lines

$$(b_{11}d_2 + b_{22}d_1)^2 - 4d_1d_2 \det B = 0$$

(where the upper one is a joint tangent to all hyperbolas H_n) a general solution

$$\begin{aligned} u(x) &= Ae^{r_1x} + Be^{-r_1x} + Ce^{r_2x} + De^{-r_2x}\,, \\ v(x) &= -(d_1u''(x) + b_{11}u(x))/b_{12} \end{aligned} \tag{2.4}$$

with arbitrary A, B, C, $D \in \mathbb{R}$.

Let us define on $(0, \ell)$ some auxiliary functions

$$\begin{aligned} C_1(x) &:= e^{r_1x} + e^{-r_1x}\,, \qquad & S_1(x) &:= e^{r_1x} - e^{-r_1x}\,, \\ C_2(x) &:= e^{r_2x} + e^{-r_2x}\,, \qquad & S_2(x) &:= e^{r_2x} - e^{-r_2x} \end{aligned}$$

and denote

$$R_1 := r_1^2 + \frac{b_{11}}{d_1}\,, \quad R_2 := r_2^2 + \frac{b_{11}}{d_1}\,.$$

Lemma 2.3. *The set of critical points K_D to the BVP* (1.7), (1.2), (2.2) *are the positive roots of the complex-valued function*

$$F_D(d_1, d_2) = d_1r_1R_2S_1(\ell)C_2(\ell) - d_1r_2R_1S_2(\ell)C_1(\ell)\,.$$

The profiles of the corresponding non-trivial solutions for $d = (d_1, d_2) \in K_D$ are

$$\begin{aligned} u(x) &= A(C_1(x) - C_2(x)\beta(d))\,, \\ v(x) &= -A(d_1(r_1^2C_1(x) - r_2^2C_2(x)\beta(d)) + b_{11}(C_1(x) - C_2(x)\beta(d)))/b_{12} \end{aligned} \tag{2.5}$$

with arbitrary $A \in \mathbb{R}$ and $\beta(d) = r_2S_2(\ell)/(r_1S_1(\ell))$.

Proof. The function F_D corresponds to the determinant of the linear system of 4 equations for coefficients A, B, C, D from (2.4) derived by using BCs (1.2), (2.2). Since these conditions are linear, a nontrivial quadruplet exists if and only if this determinant is zero. The form (2.5) then follows from (2.4). See e.g. [3] or [6] for details. □

Remark 2.4. Let us emphasize that the coefficients r_1, r_2 and therefore also the functions $C_i(x)$ and $S_i(x)$, $i = 1, 2$, and the numbers R_1, R_2 and β are in general complex and depend on diffusion parameters $(d_1, d_2) \in \mathbb{R}_+^2$. The form (2.4) and hence also (2.5) are written in a complex form, nevertheless one can rewrite them to obtain a couple (u, v) of non-trivial real solutions to the corresponding BVP.

Theorem 2.5. *The set of critical points K_S to the unilateral BVP* (1.7), (1.2), (2.1) *is given by*

$$K_S = K_N \cup K_D = \bigcup_{n=1}^{\infty} H_n \cup \{(d_1, d_2) \in \mathbb{R}_+^2 \,:\, F_D(d_1, d_2) = 0\}\,.$$

The profiles of the corresponding non-trivial solutions for (d_1, d_2) lying on some H_n or in K_D are given by (2.3) *or* (2.5) *with any $A \in \mathbb{R}$ having the proper sign, i.e. such that $v(\ell) \geq 0$ or $v'(\ell) \geq 0$, respectively.*

Proof. The assertion follows from Lemmas 2.1, 2.2 and 2.3 and considerations above. □

2.2. **A unilateral obstacle in the interior of the domain.** Let us consider our system (1.7) with (1.2), (1.3) and let us add for v a one-sided obstacle given by a unilateral condition at $x = x_1 \in (0,\ell)$ of the form

$$v(x_1) \geq 0, \quad v'(x_1-) \geq v'(x_1+)\,, \quad v(x_1)\,(v'(x_1-) - v'(x_1+)) = 0\,. \tag{2.6}$$

It is clear that if (u,v) is a non-trivial solution to (1.7), (1.2), (1.3) then (u,v) or $(-u,-v)$ (or exceptionally both) satisfies also (2.6). Such pairs are the C^2-smooth solutions to the unilateral problem (1.7), (1.2), (1.3), (2.6) and hence K_N is one part of the set of corresponding critical points.

The other type of solutions are those, for which the obstacle is 'active' and they are broken in the derivative of v (we write one-sided derivatives in (2.6)). The smoothness of activator u remains 'full', i.e. $u \in C[0,\ell] \cap C^2(0,\ell)$ but

$$v \in C[0,\ell] \cap C^2(0,x_1) \cap C^2(x_1,\ell)$$

and (1.7) separates to two systems, on $(0,x_1)$ and on (x_1,ℓ), and four conditions connecting the left (u_L,v_L) and right (u_R,v_R) solutions appear from (2.6)

$$u_L(x_1) = u_R(x_1)\,, \quad u_L'(x_1-) = u_R'(x_1+)\,, \quad v_L(x_1) = 0\,, \quad v_R(x_1) = 0\,, \tag{2.7}$$

together with the proper sign of the jump of derivatives

$$v'(x_1-) \geq v'(x_1+)\,. \tag{2.8}$$

Expressing general solution on $(0,x_1)$ and on (x_1,ℓ) and using BCs (1.2), (1.3) and conditions (2.7) we obtain a linear system for 8 coefficients A_L, B_L, C_L, D_L and A_R, B_R, C_R, D_R. Determinant of the matrix corresponding to this linear system is the desired function $F_{x_1}(d_1,d_2)$, positive roots of which are critical points corresponding to solutions satisfying $v(x_1) = 0$ (they touch the obstacle) and which can be (obstacle is not active) or are not (obstacle is active and breaks v) C^1-smooth on the whole domain $(0,\ell)$.

Lemma 2.6 ([3])**.** *The set of critical points K_{x_1} to the BVP* (1.7) *on* $(0,x_1)$ *and on* (x_1,ℓ) *with* (1.2), (1.3), (2.7) *are the roots of the complex-valued function*

$$F_{x_1}(d_1,d_2) = \tfrac{r_1}{r_2}\left(S_1(x_1) + S_1(\ell - x_1)\tfrac{C_1(x_1)}{C_1(\ell-x_1)}\right) - \tfrac{R_1}{R_2}C_1(x_1)\left(\tfrac{S_2(\ell-x_1)}{C_2(\ell-x_1)} + \tfrac{S_2(x_1)}{C_2(x_1)}\right).$$

The profiles of the corresponding non-trivial solutions for $(d_1,d_2) \in K_{x_1}$ are

$$\begin{aligned} u_L(x) &= A_L(C_1(x) - \beta_1(d)C_2(x))\,, \\ v_L(x) &= -A_L\tfrac{d_1\left(r_1^2C_1(x) - \beta_1(d)r_2^2C_2(x)\right) + b_{11}(C_1(x) - \beta_1(d)C_2(x))}{b_{12}} \end{aligned} \tag{2.9}$$

on $(0,x_1)$ *and*

$$\begin{aligned} u_R(x) &= A_L\beta_3(d)(C_1(\ell - x) - \beta_2(d)C_2(\ell - x))\,, \\ v_R(x) &= -A_L\beta_3(d)\tfrac{d_1\left(r_1^2C_1(\ell-x) - \beta_2(d)r_2^2C_2(x)\right) + b_{11}(C_1(\ell-x) - \beta_2(d)C_2(\ell-x))}{b_{12}} \end{aligned} \tag{2.10}$$

on (x_1,ℓ) *with arbitrary* $A_L \in \mathbb{R}$ *and*

$$\beta_1(d) = \frac{R_1C_1(x_1)}{R_2C_2(x_1)}\,, \quad \beta_2(d) = \frac{R_1C_1(\ell - x_1)}{R_2C_2(\ell - x_1)}\,, \quad \beta_3(d) = \frac{C_1(x_1)}{C_1(\ell - x_1)}\,.$$

Proof. The expressions follow from [3, Section 5.5]. □

Theorem 2.7. *The set of critical points U_{x_1} to the unilateral BVP* (1.7), (1.2), (1.3) *and* (2.6) *is given by*

$$U_{x_1} = K_N \cup K_{x_1} = \bigcup_{n=1}^{\infty} H_n \cup \{(d_1, d_2) \in \mathbb{R}_+^2 \,:\, F_{x_1}(d_1, d_2) = 0\}\,.$$

The profiles of the corresponding non-trivial solutions for (d_1, d_2) lying on H_n or in K_{x_1} are given by (2.3) *or by* (2.9), (2.10), *with any A or A_L having the proper sign, i.e. such that $v(x_1) \geq 0$ or* (2.8) *holds, respectively.*

Proof. The assertion follows from the analogy of Lemma 2.1 together with Lemmas 2.2 and 2.6. □

3. Two unilateral obstacles for inhibitor

3.1. **Two obstacles from below.** Let us focus now on the example of two one-sided obstacles at $x = x_1$ and at $x = \ell$ (both *acting from below*) for v, i.e., we will consider the BVP (1.7), (1.2), (2.1) and (2.6). Two obstacles mean that there is no analogy of Lemma 2.1. We can still decompose the task: the critical points are such pairs (d_1, d_2) for which the corresponding solutions have no active contact with obstacles or for which only one or even both obstacles are active. In the last cas we have

Lemma 3.1. *The set of critical points $K_{x_1\ell}$ to the BVP* (1.7) *on $(0, x_1) \cup (x_1, \ell)$ with* (1.2), (2.2), (2.7) *are positive pairs (d_1, d_2) for which the algebraic linear system*

$$\begin{aligned}
&r_1(A_R - B_R) + r_2(C_R - D_R) = 0\,,\\
&R_1(A_R + B_R) + R_2(C_R + D_R) = 0\,,\\
&R_1(A_R e^{r_1 x_1} + B_R e^{-r_1 x_1}) + R_2(C_R e^{r_2 x_1} + D_R e^{-r_2 x_1}) = 0\,,\\
&A_R e^{r_1 x_1} + B_R e^{-r_1 x_1} + C_R e^{r_2 x_1} + D_R e^{-r_2 x_1} = A_L C_1(x_1)\left(1 - \tfrac{R_1}{R_2}\right)\,,\\
&\tfrac{r_1}{r_2}(A_R e^{r_1 x_1} - B_R e^{-r_1 x_1}) + C_R e^{r_2 x_1} - D_R e^{-r_2 x_1} = A_L C_1(x_1)\left(\tfrac{r_1}{r_2} - \tfrac{R_1}{R_2}\right)\,,
\end{aligned} \tag{3.1}$$

has a non-trivial solution $(A_L, A_R, B_R, C_R, D_R)$. Then the nontrivial left and right solutions (u_L, v_L) and (u_R, v_R) of our BVP are given by (2.9) *and* (2.4) *with this A_L and (A_R, B_R, C_R, D_R), respectively.*

Proof. We obtain (3.1) by using boundary and inner conditions (1.2), (2.2), (2.7) for general solution (2.4) considered on $(0, x_1)$ and on (x_1, ℓ). □

Theorem 3.2. *The set of critical points $U_{x_1\ell}$ to the unilateral BVP* (1.7), (1.2), (2.1) *and* (2.6) *is given by*

$$U_{x_1\ell} \subset (K_N \cup K_{x_1} \cup K_D \cup K_{x_1\ell}) \tag{3.2}$$

such that the profiles of the corresponding non-trivial solutions satisfy both (2.1) *and* (2.6).

Remark 3.3. Nodal properties of the v-part of corresponding non-trivial solutions are preserved along the individual branches of critical points only to purely Neumann BVP (i.e. only along hyperbolas H_n). This is not true in general for the unilateral

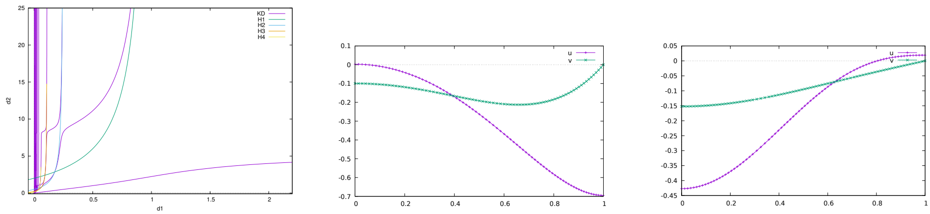

Fig. 1: Critical points K_S (H_n for $n = 1, 2, 3, 4$, and violet K_D) for BVP (1.7), (1.2), (2.1). Profile of solution (u, v) for $(d_1, d_2) = (0.913, 2) \in K_D$ on the horizontal branch and for $(d_1, d_2) = (0.407, 10) \in K_D$ on the second violet branch.

BVPs. Therefore, it strongly depends on the location of $x_1 \in (0, \ell)$ which parts of these branches are simultaneously critical points also for the 2-obstacles BVP. Hence we can not characterize explicitely $U_{x_1\ell}$ by the equality in (3.2).

Remark 3.4. Numerically it seems that the nodal properties of v are preserved along a large part of the right-most branches (going to the right which seems to be bounded in d_2) of K_D, K_{x_1} as well as of $K_{x_1\ell}$. This boundedness perfectly fits with the theoretical results for BVPs with unilateral conditions prescribed on the boundary, see [1, 2, 4]. As far as (more precisely, as close as to the origin) we can go with $d = (d_1, d_2)$ along the right-most branches while the profile of v satisfies simultaneously sharp inequality in (2.8) and $v(\ell) \geq 0$, such d belongs also to $U_{x_1\ell}$.

3.2. **Two obstacles from opposite sides.** Let us consider the similar BVP but with obstacles acting from the opposite sides and without loss of generality take

$$v(x_1) \leq 0, \quad v'(x_1-) \leq v'(x_1+), \quad v(x_1)(v'(x_1-) - v'(x_1+)) = 0 \tag{3.3}$$

instead of (2.6). We obtain an analogue of Theorem 3.2 with different subset $U^-_{x_1\ell}$ of $K_N \cup K_{x_1} \cup K_D \cup K_{x_1\ell}$. Irrespectively to Remark 3.4, if $d \in K_D$ or $d \in K_{x_1}$ lies on the righ-most branch and close enough to the origin, v with proper sign of A or A_L, resp., satisfies both (3.3) and $v(\ell) \geq 0$, hence such $d \in U^-_{x_1\ell}$.

Remark 3.5. The second right-most branch of $K_{x_1\ell}$ lies completely to the right from all H_n, i.e., in the domain of stability D_S of the trivial solution.

Let $d \in K_{x_1\ell}$ be from the second right-most branch. Let the corresponding v satisfy (2.8). Then numerically we observe that this inequality is sharp. Moreover, $v'(\ell) \geq 0$ (hence $d \in U_{x_1\ell}$) or $v'(\ell) \leq 0$ (hence $(-u, -v)$ satisfies (2.1) and (3.3), so $d \in U^-_{x_1\ell}$) for d being sufficiently close to or far from, respectively, the origin.

4. Examples and numerical results for given obstacles

Let us consider unilateral BVP (1.7), (1.2), (2.1) with a matrix $B = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}$. The set of critical points K_S from Lemma 2.3 and Theorem 2.5 is visible on Fig. 1. One can observe just one branch going to the right and being bounded in d_2. This branch are the only critical points from K_D and hence from K_S lying in D_S, i.e. to the right from all hyperbolas H_n. The other branches belong to D_U.

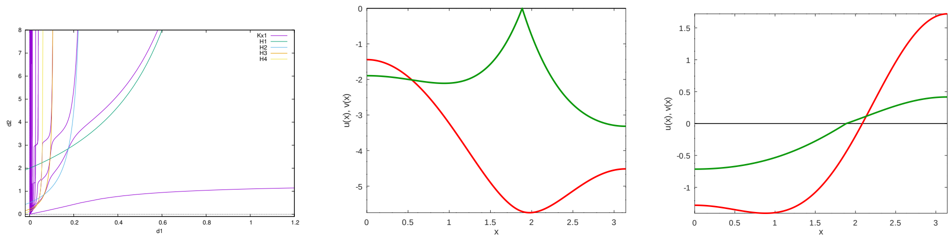

Fig. 2: Critical points K_{x_1} (in violet) for BVP (1.7), (1.2), (1.3), (2.6) with $x_1 = 0.6\pi$. Profile of solution u in red and v in green for $(d_1, d_2) = (0.6, 0.944) \in K_{x_1}$ on the horizontal branch and for $(d_1, d_2) = (0.218, 3.52) \in K_{x_1}$ on the second violet branch.

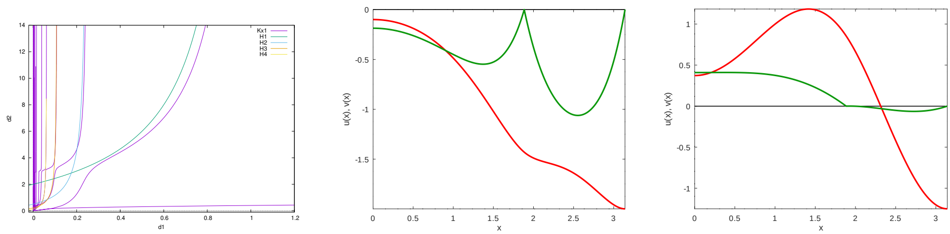

Fig. 3: Critical points $K_{x_1\ell}$ for BVP (1.7), (1.2), (2.1), (2.6) with $x_1 = 0.6\pi$. Profile of solution u in red and v in green for $(d_1, d_2) = (0.4, 0.286) \in K_{x_1\ell}$ on the horizontal branch and for $(d_1, d_2) = (0.251, 2.7) \in K_{x_1\ell}$ on the second violet branch.

References

[1] Eisner, J., Kučera, M., Väth, M., *Global bifurcation of a reaction-diffusion system with inclusions*, J. Anal. Appl. **28** (4) (2009), 373–409.

[2] Eisner, J., Väth, M., *Degree, instability and bifurcation of reaction-diffusion systems with obstacles near certain hyperbolas*, Nonlinear Anal. **135** (2016), 158–193.

[3] Kouba, P., *Existence of nontrivial solutions for reaction-diffusion systems of activator-inhibitor type with dependence on parameter*, Master's thesis, Č. Budějovice, Faculty of Science, University of South Bohemia, 2015, (in Czech).

[4] Kučera, M., Väth, M., *Bifurcation for reaction-diffusion systems with unilateral and Neumann boundary conditions*, J. Differential Equations **252** (2012), 2951–2982.

[5] Mimura, M., Nishiura, Y., Yamaguti, M., *Some diffusive prey and predator systems and their bifurcation problems*, Ann. N.Y. Acad. Sci. **316** (1979), 490–510.

[6] Pšenicová, M., *Newton boundary value problem for reaction-diffusion system of activator-inhibitor type with parameter*, Bachelor thesis, Č. Budějovice (2018), Faculty of Science, University of South Bohemia, 2018, (in Czech).

[7] Turing, A.M., *The chemical basis of morphogenesis*, Philos. Trans. Roy. Soc. London Ser. B **237** (641) (1952), 37–72.

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