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ON THE DIOPHANTINE EQUATION $x^2 + 2^a 3^b 73^c = y^n$

Murat Alan and Mustafa Aydin

Abstract. In this paper, we find all integer solutions (x, y, n, a, b, c) of the equation in the title for non-negative integers a, b and c under the condition that the integers x and y are relatively prime and $n \geq 3$. The proof depends on the famous primitive divisor theorem due to Bilu, Hanrot and Voutier and the computational techniques on some elliptic curves.

1. Introduction

In recent years, many papers deal with the Diophantine equation

$$x^2 + p_1^{\alpha_1} \dots p_k^{\alpha_k} = y^n,\ n \geq 3,\ \gcd(x, y) = 1 \tag{1}$$

in non negative integers $(x, y, \alpha_1, \dots, \alpha_k)$, where p_i's are fixed prime. With the development of modern tools such as primitive divisor theorem, modular approach, or computational techniques, many authors investigated the above equation when $k \geq 1$, see for example [1, 2, 3, 7, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22]. Especially, the cases $(p_1, p_2, p_3) = (2, 3, 11), (2, 11, 19), (2, 3, 19), (2, 3, 17), (5, 13, 17)$ are considered in [6, 9, 10, 11] and [21], respectively. For more information and the rich literature on equation (1), we refer to an excellent survey [12] and the 359 references therein.

In this paper, we study the equation (1) when $k = 3$ with $(p_1, p_2, p_3) = (2, 3, 73)$ and we get the following result.

Theorem 1. *All integer solutions of the equation*

$$x^2 + 2^a 3^b 73^c = y^n,\ n \geq 3,\ a, b, c \geq 0,\ x, y \geq 1,\ \gcd(x, y) = 1 \tag{2}$$

are given by

(1) *n=3: the solutions are given in Table* (1),
(2) *n=4: the solutions are given in Table* (2),
(3) *n=6:* $(x, y, a, b, c) = (2485, 19, 8, 7, 1), (15479, 25, 8, 5, 1), (42389, 35, 5, 5, 2)$,
(4) *n=8:* $(x, y, a, b, c) = (65, 3, 5, 0, 1)$,
(5) *n=9:* $(x, y, a, b, c) = (95, 3, 1, 0, 2)$,
(6) *n=12:* $(x, y, a, b, c) = (15479, 5, 8, 5, 1)$.

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2. Preliminaries

Let α, β be algebraic integers. A pair (α, β) is called a Lucas pair if $\alpha+\beta$ and $\alpha\beta$ are non-zero coprime rational integers and $\frac{\alpha}{\beta}$ is not a root of unity. For any Lucas pair (α, β), the corresponding sequences of Lucas numbers is defined by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ n = 0, 1, 2, \dots .$$

A prime number p is a primitive divisor of $L_n(\alpha, \beta)$ if $p \mid L_n(\alpha, \beta)$ and $p \nmid (\alpha-\beta)^2 L_1(\alpha, \beta) \dots L_{n-1}(\alpha, \beta)$ $(n > 1)$. Among other things a primitive divisor q of $L_n(\alpha, \beta)$ has the property that $q \equiv \left(\frac{(\alpha-\beta)^2}{q}\right) \pmod{n}$ where $\left(\frac{*}{*}\right)$ stands for Legendre symbol [8].

If $n > 4$ and $n \neq 6$ then every $n-$th term of any Lucas sequences $L_n(\alpha, \beta)$ has a primitive divisors except for an explicit finite list of parameters α, β and n [4].

Let S be any finite set of prime numbers. By an $S-$integer we mean a rational number $\frac{r}{s}$ with relatively prime integers r and $s > 0$ such that any prime factor of s belongs to S.

3. Proof of the Theorem 1

Our first approach to equation (2) is to factorize it in $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ as

$$(x + e\sqrt{-d})(x - e\sqrt{-d}) = y^n$$

where $d \in \{1, 2, 3, 6, 73, 146, 219, 438)\}$ and $e = 2^\alpha 3^\beta 73^\gamma$ for some non negative integers α, β and γ. Assume that y is even. Then $a = 0$ and x is odd, and hence $x^2 \equiv 1 \pmod{8}$. Since $3^b 73^c \equiv 1, 3 \pmod{8}$ depending on the parity of b, we get by considering equation (2) modulo 8 that either $2 \equiv 0 \pmod{8}$ or $4 \equiv 0 \pmod{8}$, which is a contradiction in each case. So y is always odd and hence the ideals generated by $x + e\sqrt{-d}$ and $x - e\sqrt{-d}$ are coprime in $\mathbb{K}$. For any choice of d, the class number $h(\mathbb{K})$ are only 1, 2, 4, 8 or 16. So we have that $\gcd(n, h(\mathbb{K})) = 1$ when n is odd. Thus, in the case n is odd, we write

$$\begin{cases} x + e\sqrt{-d} = u_1 \varepsilon^n \\ x - e\sqrt{-d} = u_2 \bar{\varepsilon}^n , \end{cases} \tag{3}$$

for an algebraic integer ε in $\mathbb{K}$ and units u_1, u_2 in the ring of algebraic integers of $\mathbb{K}$.

Let n be odd. Since, for all values of d, except for $d = 3$, the orders of multiplicative group of units in the ring of algebraic integers of $\mathbb{K}$ are either 2 or 4 which are relatively prime to n, the units u_1 and u_2 can be absorbed in the factors ε^n and $\bar{\varepsilon}^n$. So we may omit the factors u_1 and u_2 in the equation (3) when $d \neq 3$. If $d = 3$ then the orders of multiplicative group of units in the ring of algebraic integers of $\mathbb{K}$ is 6. Therefore, in the case $5 \leq n$ is an odd prime, similar argument also valid for $d = 3$.

If $d \in \{1, 2, 6, 73, 146, 438\}$, then $-d \not\equiv 1 \pmod 4$ and hence we take $\{1, \sqrt{-d}\}$ as an integral basis of $\mathbb{K}$, whereas we take the set $\{1, \frac{1+\sqrt{-d}}{2}\}$ as an integral basis of $\mathbb{K}$ when $d \in \{3, 219\}$ since $-d \equiv 1 \pmod 4$ in this case. Thus we may take $\varepsilon = u + v\sqrt{-d}$ or $\varepsilon = \frac{u+v\sqrt{-d}}{2}$ for some integers u and v. More precisely we write

$$\begin{cases} x + e\sqrt{-d} = \varepsilon^n = (u + v\sqrt{-d})^n \\ x - e\sqrt{-d} = \overline{\varepsilon}^n = (u - v\sqrt{-d})^n \end{cases}, \ y = u^2 + dv^2 \tag{4}$$

if $d \in \{1, 2, 6, 73, 146, 438)\}$ and

$$\begin{cases} x + e\sqrt{-d} = \varepsilon^n = \left(\dfrac{u+v\sqrt{-d}}{2}\right)^n \\ x - e\sqrt{-d} = \overline{\varepsilon}^n = \left(\dfrac{u-v\sqrt{-d}}{2}\right)^n \end{cases}, \ u \equiv v \pmod 2, \ y = \frac{u^2+dv^2}{4} \tag{5}$$

if $d \in \{3, 219\}$ and $n \geq 5$ is an odd prime.

Now we continue the proof of Theorem 1 for the cases $n = 3$, $n = 4$ and $n \geq 5$ separately.

3.1. **The Case n=3.**

Lemma 2. *If $n = 3$ then all solutions (x, y, a, b, c) of equation (2) are given in Table 1.*

Proof. Let $n = 3$. We write $a = 6a_1 + i$, $b = 6b_1 + j$, $c = 6c_1 + k$ where $i, j, k \in \{0, 1, \ldots, 5\}$. Thus, we may view equation (2) as an elliptic curve by writing it as $M^2 = N^3 - 2^i 3^j 73^k$ where $M = \left(\frac{x}{2^{3a_1}3^{3b_1}73^{3c_1}}\right)$ and $N = \left(\frac{y}{2^{2a_1}3^{2b_1}73^{2c_1}}\right)$ and therefore the problem of finding positive integer solutions of (2) is reduced to finding all $\{2, 3, 73\}-$integer points of the above 216 elliptic curves for each i , j and k. To find all $S-$integral points on the above curves we use the Magma function `SIntegralPoints` for $S = \{2, 3, 73\}$ [5]. Except for the seven triples (i, j, k) given below, Magma could determine all $S-$integral points of the above curves and taking into account that x and y are coprime positive integers, exactly those which are given in Table (1) lead to solutions of the equation (2).

Those seven triples are $(i, j, k) = (1, 0, 5)$, $(1, 3, 5)$, $(1, 5, 5)$, $(2, 4, 5)$, $(5, 2, 5)$, $(5, 5, 3)$, $(5, 5, 5)$. We consider those values separately. Note that since

$$de^2 = 2^{6a_1+i}3^{6b_1+j}73^{6c_1+k}$$

and d is square free, d can be only 73, 146 or 438 and hence for all possibilities of d, we have that $-d \not\equiv 1 \pmod 4$. Thus, from (4) we get that

$$2e\sqrt{-d} = \varepsilon^3 - \overline{\varepsilon}^3 .$$

Therefore, by expanding the right hand side of above equation and equating the coefficients of $\sqrt{-d}$, we get that $e = v(3u^2 - dv^2)$, in other words

$$3u^2 = dv^2 \pm \frac{e}{v}\,.$$

Note that $\gcd(x, y) = 1$ implies that $\gcd(u, v) = 1$. We will use this fact without further mention in the following cases.

Case 1: $(i, j, k) = (1, 0, 5)$. Then $de^2 = 146(2^{3a_1}3^{3b_1}73^{3c_1+2})^2$, and therefore

$$3u^2 = 146v^2 \pm \frac{2^{3a_1}3^{3b_1}73^{3c_1+2}}{v}\,.$$

We have that either $v = \pm 2^{3a_1}3^{3b_1}73^{3c_1+2}$ or $v = \pm 2^{3a_1}3^{3b_1-1}73^{3c_1+2}$ when $b_1 > 0$. In the first case, we have that $3u^2 = 2^{6a_1+1}3^{6b_1}73^{6c_1+5} \pm 1$. From the congruence $0 \equiv 2 \cdot 3^{6b_1} \cdot 1 \pm 1 \pmod 3$, we deduce that $b_1 = 0$ and only the positive sign can be happen. So, reducing modulo 7, we get that $3u^2 \equiv 4 \pmod 7$, which is not possible in integers. Now assume that $b_1 > 0$ and $v = \pm 2^{3a_1}3^{3b_1-1}73^{3c_1+2}$. Then we have that $u^2 = 2^{6a_1+1}3^{6b_1-3}73^{6c_1+5} \pm 1$. By modulo 3, only the positive sign can occur. Thus reducing modulo 7, we find that $u^2 \equiv 5 \pmod 7$, a contradiction.

Case 2: $(i, j, k) = (1, 3, 5)$. Then $de^2 = 438(2^{3a_1}3^{3b_1+1}73^{3c_1+2})^2$, and therefore

$$3u^2 = 438v^2 \pm \frac{2^{3a_1}3^{3b_1+1}73^{3c_1+2}}{v}\,.$$

We have that either $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$ or $v = \pm 2^{3a_1}3^{3b_1}73^{3c_1+2}$. The case $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$ implies that

$$3u^2 = 438(2^{3a_1}3^{3b_1+1}73^{3c_1+2})^2 \pm 1\,.$$

But this is clearly false since it gives $0 \equiv \pm 1 \pmod 3$. If $v = \pm 2^{3a_1}3^{3b_1}73^{3c_1+2}$ then we get that

$$u^2 = 2^{6a_1+1}3^{6b_1}73^{6c_1+5} \pm 1\,.$$

We could not get the result using small values of congruence consideration to solve this equation. So we write

$$A^2 = B^3 \pm 2^2 73^4\,,$$

where $A = 2 \cdot 73^2 u$ and $B = 2^{2a_1+1}3^{2b_1}73^{2c_1+3}$. Computation in Magma shows that these elliptic curves have no integral points when $uv \neq 0$.

Case 3: $(i, j, k) = (1, 5, 5)$. Then

$$3u^2 = 438v^2 \pm \frac{2^{3a_1}3^{3b_1+2}73^{3c_1+2}}{v}\,.$$

So we have that either $v = \pm 2^{3a_1}3^{3b_1+2}73^{3c_1+2}$ or $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$. We see that the case $v = \pm 2^{3a_1}3^{3b_1+2}73^{3c_1+2}$ is not possible just reducing the resulting equation modulo 3. If $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$ then

$$u^2 = 2^{6a_1+1}3^{6b_1+2}73^{6c_1+4} \pm 1\,.$$

By modulo 3, we see that only the the positive sign occurs. By modulo 13, we find that $u^2 \equiv 6 \pmod{13}$, which is not possible.

Case 4: $(i, j, k) = (2, 4, 5)$. Then $d = 73$ and

$$3u^2 = 73v^2 \pm \frac{2^{3a_1+1}3^{3b_1+2}73^{3c_1+2}}{v}.$$

We easily eliminate the cases $v = \pm 73^{3c_1+2}$, $v = \pm 3^{3b_1+2}73^{3c_1+2}$, $v = \pm 2^{3a_1+1}73^{3c_1+2}$, $v = \pm 2^{3a_1+1}3^{3b_1+2}73^{3c_1+2}$ by congruence consideration modulo 3. We consider two more cases $v = \pm 3^{3b_1+1}73^{3c_1+2}$ and $v = \pm 2^{3a_1+1}3^{3b_1+1}73^{3c_1+2}$. If $v = \pm 3^{3b_1+1}73^{3c_1+2}$ then

$$u^2 = 3^{6b_1+1}73^{6c_1+5} \pm 2^{3a_1+2}.$$

The congruence $1 \equiv 3 \pm 2 \pmod 8$, shows that only the negative sign can occur in the right hand side. So we get that $u^2 \equiv -1 \pmod 7$, a contradiction. If $v = \pm 2^{3a_1+1}3^{3b_1+1}73^{3c_1+2}$ then we get

$$u^2 = 2^{6a_1+2}3^{6b_1+1}73^{6c_1+5} \pm 1.$$

Reducing modulo 4, we see that only the positive sign occurs and hence by modulo 7, we find $u^2 \equiv 5 \pmod 7$, a contradiction.

Case 5: $(i, j, k) = (5, 2, 5)$. In this case $d = 146$ and

$$3u^2 = 146v^2 \pm \frac{2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}}{v}.$$

It is enough to reduce modulo 3 to eliminate the cases $v = \pm 2^{3a_1+2}73^{3c_1+2}$, $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}$. When $b_1 > 0$, we need to check one more possible case $v = \pm 2^{3a_1+2}3^{3b_1}73^{3c_1+2}$. This case implies that

$$u^2 = 2^{6a_1+5}3^{6b_1-1}73^{6c_1+5} \pm 1.$$

By reducing modulo 4, we see that only the positive sign occurs. In this case, congruence consideration modulo 5, 7 or 13 like the previous cases does not give the desired result. So, we multiply both side of the above equation by $2^2 3^4 73^4$, to get the elliptic curve

$$A^2 = B^3 + 2^2 3^4 73^4,$$

where $A = 2 \cdot 3^2 73^2 u$ and $B = 2^{2a_1+2}3^{2b_1+3}73^{2c_1+3}$. A quick computation with Magma shows that this curve has no integral points when $B \neq 0$, and therefore we conclude that this case also does not lead to a solution.

Case 6: $(i, j, k) = (5, 5, 3)$. Then $d = 438$, and

$$3u^2 = 438v^2 \pm \frac{2^{3a_1+2}3^{3b_1+2}73^{3c_1+1}}{v}.$$

In this case we have either $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+1}$ or $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+1}$. If $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+1}$ then $3u^2 = 2^{6a_1+5}3^{6b_1+5}73^{6c_1+3} \pm 1$, which is clearly false because of modulo 3. Let $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+1}$. Then we get that $u^2 = 2^{6a_1+5}3^{6b_1+4}73^{6c_1+2} \pm 1$. Reducing modulo 4, we see that only the positive sign can be occur. Reducing modulo 7, we get that $u^2 \equiv 5 \pmod 7$, a contradiction.

Case 7: $(i, j, k) = (5, 5, 5)$. Then again $d = 438$ and therefore we have that

$$3u^2 = 438v^2 \pm \frac{2^{3a_1+2}3^{3b_1+2}73^{3c_1+2}}{v}.$$

Then either $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+2}$ or $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}$. The case $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+2}$ is not possible because of modulo 3.

If $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}$ then $u^2 = 2^{6a_1+5}3^{6b_1+2}73^{6c_1+5} \pm 1$. As in the previous case, we see that this equation also has no solution in integers.

So, we conclude that these exceptional seven cases do not lead to a solution. □

3.2. **The Case n=4.**

Lemma 3. *If* $n = 4$ *then all solutions of* (2) *are given in Table 4.*

Proof. Let $n = 4$. Write $a = 4a_1 + i$, $b = 4b_1 + j$ and $c = 4c_1 + k$ where i, j, $k \in \{0, 1, 2, 3\}$. Thus, the equation (2) is of the form

$$M^2 = N^4 - 2^i 3^j 73^k$$

where $M = \left(\frac{x}{2^{2a_1}3^{2b_1}73^{2c_1}}\right)$ and $N = \left(\frac{y}{2^{a_1}3^{b_1}73^{c_1}}\right)$. To find all integer solutions of (2) corresponds to find all $S = \{2, 3, 73\}$- integral points of the above 64 quartic curves. We used the subroutine `SIntegralLjunggrenPoints` of Magma to determine all S-Integral Points of the above curves.

Taking into account $\gcd(x, y) = 1$, we see that only the results given in Table (2) lead to the solutions of equation (2). □

3.3. **The Case** $n \geq 5$**.**

Lemma 4. *If* $n \geq 5$ *then the equation* (2) *has no positive integer solution except for* $(x, y, a, b, c) = (2485, 19, 8, 7, 1)$, $(15479, 25, 8, 5, 1)$, $(42389, 35, 5, 5, 2)$ *for* $n = 6$, $(65, 3, 5, 0, 1)$ *for* $n = 8$, $(95, 3, 1, 0, 2)$ *for* $n = 9$, $(15479, 5, 8, 5, 1)$ *for* $n = 12$.

Proof. Let $n \geq 5$. If there exists any solution of equation (2) for $n = 2^k, k \geq 3$ then this solution can be derived from the solutions with $n = 4$ since $y^{2^k} = \left(y^{2^{k-2}}\right)^4$. So, by picking up the solutions which contain the perfect power values of y among the solutions of (2) for $n = 4$ given in Table 4, we see that there exist exactly two such solutions, namely $(65, 9, 5, 0, 1)$ and $(15479, 125, 8, 5, 1)$ for (x, y, a, b, c). Therefore, from these solutions we get that two more solutions $(x, y, a, b, c) = (65, 3, 5, 0, 1)$ and $(x, y, a, b, c) = (15479, 5, 8, 5, 1)$ for $n = 8$ and $n = 12$, respectively.

Similarly, for the case $n = 3k$, $k \geq 2$, we may get all solutions of equation (2) from the solutions given in Table (1) for $n = 3$. Thus, we find that equation (2) has also solutions $(x, y, a, b, c) = (2485, 19, 8, 7, 1)$, $(15479, 25, 8, 5, 1)$ and $(42389, 35, 5, 5, 2)$ for $n = 6$ and $(95, 3, 1, 0, 2)$ for $n = 9$. Hence, without loss of generality, from now on we may assume that $n \geq 5$ is an odd prime.

From the equations (4) and (5), one can find that

$$e = L_n v = \left|\frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}}\right| v \quad \text{where} \quad \begin{cases} \varepsilon = u + v\sqrt{-d} \\ \bar{\varepsilon} = u - v\sqrt{-d} \end{cases}$$

and

$$2e = L_n^{'} v = \left|\frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}}\right| v \quad \text{where} \quad \begin{cases} \varepsilon = \dfrac{u + v\sqrt{-d}}{2} \\ \bar{\varepsilon} = \dfrac{u - v\sqrt{-d}}{2} \end{cases}$$

according to the value of d belonging the sets $\{1, 2, 6, 73, 146, 438\}$ and $\{3, 219\}$, respectively.

It is easy to see that the sequence L_n and $L_n^{'}$ are both Lucas sequences. All Lucas sequences which are having no primitive divisors are explicitly listed in [4] for $n \geq 5$ and neither L_n nor $L_n^{'}$ does not match any of them.

So we need to consider the possibility of L_n or $L_n^{'}$ has a primitive divisor. Assume that L_n or $L_n^{'}$ has a primitive divisor, say q. Then either $q = 2$, 3 or $q = 73$. From the fact that any primitive divisor is congruent to ± 1 modulo n we easily eliminate the possibility $q = 2$ and $q = 3$ since $n \geq 5$. So we continue with $q = 73$. By the definition of primitive divisor, we have that $q \nmid (\varepsilon - \overline{\varepsilon})^2 = -4dv^2$ for the case L_n and $q \nmid (\varepsilon - \overline{\varepsilon})^2 = -dv^2$ for the case $L_n^{'}$ which implies that $d = 1, 2, 3$ or 6. Note that for all these values of d, $\left(\frac{(\alpha - \beta)^2}{q}\right) = 1$, and hence we find that $73 \equiv 1 \pmod{n}$ which is a contradiction since $n \geq 5$ is an odd prime. Thus neither L_n nor $L_n^{'}$ has a primitive divisor. This completes the proof. □

TAB. 1: Solutions for $n = 3$ and $\gcd(x, y) = 1$.

(M, N, i, j, k)	a	b	c	x	y
(181/9, 1430/27, 0, 0, 2)	0	6	2	1430	181
(55/9, 82/27, 0, 1, 1)	0	7	1	82	55
(3961/324, 233875/5832, 0 ,1 ,1)	6	13	1	233875	3961
(283/9, 4744/27, 0,1 ,1)	0	7	1	4744	283
(462745/2304, 291230531/110592 ,0 , 1,3)	24	7	3	291230531	462745
(62281/36, 15542875/216,0,2,2)	6	8	2	15542875	62281
(127/9, 782/27, 0, 3, 1)	0	9	1	782	127
(45025/2304, 8195759/110592, 0, 3, 1)	24	9	1	8195759	45025
(319/9, 5570/27, 0, 3, 1)	0	9	1	5570	319
(93385/1296, 28462213/46656, 0, 3, 1)	12	15	1	28462213	93385
(13, 46, 0, 4, 0)	0	4	0	46	13
(7, 10, 0, 5, 0)	0	5	0	10	7
(67, 532, 0, 5, 1)	0	5	1	532	67
(3501025/16, 6550777007/64, 0, 5, 3)	12	5	3	6550777007	3501025
(3, 5, 1, 0, 0)	1	0	0	5	3
(195, 2723, 1, 0, 1)	1	0	1	2723	195
(27, 95, 1, 0, 2)	1	0	2	95	27
(7, 17, 1, 3, 0)	1	3	0	17	7
(98377/16, 30855995/64, 1, 4, 2)	13	4	2	30855995	98377
(4681/4, 320005/8, 1, 5, 2)	7	5	2	320005	4681
(5, 11, 2, 0, 0)	2	0	0	11	5

(M, N, i, j, k)	a	b	c	x	y
(361/36, 2485/216, 2, 1, 1)	8	7	1	2485	361
(925/81, 18053/729, 2, 1, 1)	2	13	1	18053	925
(5917/9, 455147/27, 2, 1, 1)	2	7	1	455147	5917
(3145/4, 176059/8, 2, 4, 2)	8	4	2	176059	3145
(13, 35, 2, 5, 0)	2	5	0	35	13
(397/9, 3293/27, 2, 5, 1)	2	11	1	3293	397
(913/16, 21689/64, 2, 5, 1)	14	5	1	21689	913
(61, 395, 2, 5, 1)	2	5	1	395	61
(85, 737, 2, 5, 1)	2	5	1	737	85
(6217/64, 470843/512, 2, 5, 1)	20	5	1	470843	6217
(625/4, 15479/8, 2, 5, 1)	8	5	1	15479	625
(1165, 39763, 2, 5, 1)	2	5	1	39763	1165
(74833/36, 20470951/216, 2, 5, 1)	8	11	1	20470951	74833
(380269/81, 234067645/729, 2, 5, 3)	2	17	3	234067645	380269
(937/9, 28135/27, 3, 0, 2)	3	6	2	28135	937
(1493065/36, 1824391621/216, 3, 3, 2)	9	9	2	1824391621	1493065
(97, 955, 3, 4, 0)	3	4	0	955	97
(1568665/85264, 1629420733/24897088, 15, 5, 6)	27	5	12	1629420733	1568665
(73/9, 595/27, 4, 1, 0)	4	7	0	595	73
(145/9, 703/27, 4, 1, 1)	4	7	1	703	145
(4753/36, 308935/216, 4, 1, 2)	10	7	2	308935	4753
(5257/9, 380915/27, 4, 1, 2)	4	7	2	380915	5257
(193, 2681, 4, 4, 0)	4	4	0	2681	193
(265, 3421, 4, 4, 2)	4	4	2	3421	265
(265/4, 667/8, 4, 5, 1)	10	5	1	667	265
(97, 793, 4, 5, 1)	4	5	1	793	97
(649, 16525, 4, 5, 1)	4	5	1	16525	649
(5977/9, 445445/27, 4, 5, 2)	4	11	2	445445	5977
(15697/9, 1371223/27, 5, 1, 4)	5	7	4	1371223	15697
(457, 9035, 5, 4, 2)	5	4	2	9035	457
(1153, 39151, 5, 5, 0)	5	5	0	39151	1153
(84097, 24387695, 5, 5, 1)	5	5	1	24387695	84097
(1225, 42389, 5, 5, 2)	5	5	2	42389	1225

TAB. 2: Solutions for $n = 4$ and $gcd(x, y) = 1$.

(M, N, i, j, k)	a	b	c	x	y
(± 125/12,15479/144, 0, 1, 1)	8	5	1	15479	125
(± 41/8,367/64, 0, 2, 1)	12	2	1	367	41
(± 3/2,7/4, 1, 0, 0)	5	0	0	7	3
(± 9/2,65/4, 1, 0, 1)	5	0	1	65	9
(± 5/2,23/4, 1, 1, 0)	5	1	0	23	5
(± 35/4,1079/16, 1, 2, 1)	9	2	1	1079	35
(± 827/16,94105/256, 1, 2, 3)	17	2	3	94105	827
(± 17/2,143/4, 1, 3, 1)	5	3	1	143	17
(± 37/6,1223/36, 2, 0, 1)	6	4	1	1223	37
(± 137/12,18607/144, 2, 0, 1)	10	4	1	18607	137
(± 77/6,5897/36, 2, 0, 1)	6	4	1	5897	77
(± 7/2,47/4, 2, 1, 0)	6	1	0	47	7
(± 11/2,25/4, 2, 1, 1)	6	1	1	25	11
(± 29/4,695/16, 2, 1, 1)	10	1	1	695	29
(± 5/2,7/4, 2, 2, 0)	6	2	0	7	5
(± 89/6,2737/36, 3, 0, 2)	7	4	2	2737	89
(± 13/2,23/4, 3, 1, 1)	7	1	1	23	13
(± 157/8,24503/64, 3, 1, 1)	15	1	1	24503	157
(± 17/2,287/4, 3, 2, 0)	7	2	0	287	17
(± 19/2,215/4, 3, 2, 1)	7	2	1	215	19
(± 611/12,373175/144, 3, 2, 1)	11	6	1	373175	611
(± 145/2,21023/4, 3, 2, 1)	7	2	1	21023	145

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Yildiz Technical University, Department of Mathematics,
Davutpasa Campus, 34210, Esenler, Istanbul, Turkey
E-mail: `alan@yildiz.edu.tr`, `mustafaaydin86@gmail.com`