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This issue of Archivum mathematicum
is dedicated to Professor František Neuman
on the occasion of his 60th birthday

Preface

The Conference on Differential Equations and Their Applications (EQUADIFF 9) was held in Brno, August 25–29, 1997. It was organized by the Masaryk University, Brno in cooperation with Mathematical Institute of the Czech Academy of Sciences, Technical University Brno, Union of Czech Mathematicians and Physicists, Union of Slovak Mathematicians and Physicists and other Czech scientific institutions with support of the International Mathematical Union. EQUADIFF 9 was attended by 269 participants from 32 countries and more than 50 accompanying persons and other guests.

This volume contains 20 papers by invited speakers in the conference. Together with this issue the following EQUADIFF 9 publications have been prepared:

- Proceedings of EQUADIFF 9 containing 12 survey papers mainly by the plenary speakers published by the Electronic Publishing House in both electronic and hard copy forms.
- CD ROM containing, in electronic form, a special EQUADIFF 9 issue of Archivum mathematicum, the Proceedings and 31 other papers submitted by the participants of the conference as well as other conference material (e.g. Abstracts, List of participants, and Program) — available to any participant of EQUADIFF 9.

This EQUADIFF 9 special issue of Archivum mathematicum is dedicated to Professor František Neuman, Chairman of the Conference, on the occasion of his sixtieth birthday. Professor Neuman obtained the Bolzano medal, an honor awarded to distinguished scientists by the Presidium of the Czech Academy of Sciences. Detailed information concerning the achievements of Professor Neuman as well as a list of his scientific publications can be found in the paper by O. Došlý "Sixty years of Professor František Neuman", published in Mathematica Bohemica 123 (1998), No. 1, 101–107 and in Czechoslovak Mathematical Journal 48 (1998), No. 1, 177–183.

The printed version is identical to the electronic one on CD ROM in spite of slight changes in the usual Archivum mathematicum style. Our aim was to harness the possibilities of new computer technologies, and for this reason all EQUADIFF 9 publications on CD ROM were prepared in hypertext PDF form.

We would like to thank Professor Jaromír Kuben, who made this CD ROM a reality. We would also like to thank Professor Zuzana Došlá for her help during the preparation of this publication.

Brno, April 1998 — Editors

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On Existence of Oscillatory Solutions of nth Order Differential Equations with Quasiderivatives

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Abstract. Sufficient conditions are given under which the nonlinear n-th order differential equation with quasiderivatives has oscillatory solutions.

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1 Introduction

Consider a nonlinear differential equation

$$y^{[n]} = f(t, y^{[0]}, \dots, y^{[n-1]}) \qquad \text{in} \quad D, \tag{1}$$

where $n \geq 3$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$, $D = R_+ \times R^n$, $y^{[i]}$ is the ith quasiderivative of y defined by

$$y^{[0]} = y,\ y^{[i]} = \frac{1}{a_i(t)} \left(y^{[i-1]}\right)',\ i = 1, 2, \dots, n-1,\ y^{[n]} = \left(y^{[n-1]}\right)', \tag{2}$$

the functions $a_i : R_+ \to (0, \infty)$ are continuous, $f : D \to R$ fulfills the local Carathéodory conditions and

$$f(t, x_1, \dots, x_n)x_1 \leq 0, \quad f(t, 0, x_2, \dots, x_n) = 0 \quad \text{in} \quad D\,. \tag{3}$$

Let $y : [0, b) \to R$, $b \le \infty$ be continuous, have the quasi-derivatives up to the order $n-1$ and let $y^{[n-1]}$ be absolutely continuous. Then y is called a *solution* of (1) if (1) is valid for almost all $t \in [0, b)$ and either $b = \infty$ or $b < \infty$ and $\limsup_{t\to b_-} \sum_{i=0}^{n-1} |y^{[i]}(t)| = \infty$. It is called *proper* if $b = \infty$ and $\sup_{\tau \le t < \infty} |y(t)| > 0$ holds for an arbitrary number $\tau \in R_+$. A proper solution is called *oscillatory* if there exists a sequence of its zeros tending to ∞.

Notation 1. Let $t_0 \in R_+, a_n, b \in C^0(R_+)$. Put

$$a_{n+i}(t) = a_i(t), i \in \{1, \dots, n - 1\}, I_0(t, t_0; a_s, b) \equiv 1,$$

$$I_k(t, t_0; a_s, b) = \int_{t_0}^{t} a_s(\tau_s) \int_{t_0}^{\tau_s} a_{s+1}(\tau_{s+1}) \cdots \int_{t_0}^{\tau_{s+k-3}} a_{s+k-2}(\tau_{s+k-2}) \times \times \int_{t_0}^{\tau_{s+k-2}} b(\tau_{s+k-1}) \quad d_{\tau_{s+k-1}} \dots d_{\tau_s},$$

$$J(t, t_0; a_s) = \int_{t_0}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, \tau_s; a_{s+2}, a_{n+s-1}) d\tau_{s+1} d\tau_s.$$

We will assume the following hypotheses (not all simultaneously):

(H1): Let $\frac{a_1}{a_2} \in C^1(R_+)$ for $n = 3$; let $a_2 \in C^1(R_+)$, $a_j \in C^2(R_+)$, $j = 1, 3$ for $n = 4$; let an index $l \in \{1, 2, \dots, n-4\}$ exist such that $a'_{l+j} \in L_{\text{loc}}(R_+)$, $j = 1, 2$ are locally bounded from bellow a.e. on R_+ for $n > 4$.

(H2): Let $b \in L_{loc}(R_+)$ and $g \in C_0(R_+)$ exist such that $g(x) > 0$ for $x > 0$, $\int_1^\infty \frac{dt}{g(t)} = \infty$ and

$$|f(t, x_1, \dots, x_n)| \le b(t) g\Big(\sum_{i=1}^{n} |x_i|\Big) \quad \text{on} \quad D\,.$$

(H3): Let constants $\bar{t} \in R_+$, $K \ge 0, 0 \le \lambda \le 1$ and functions $a_n \in L_{loc}(R_+)$ and $g \in C^0(R_+)$ exist such that $a_n \ge 0$, $g(x) > 0$ for $x > 0$, $g(x) = x^\lambda$ for $x \ge K$,

$$a_n(t) g(|\, x_1 \,|) \le |\, f(t, x_1, \dots, x_n)\,| \quad \text{on} \quad R_+ \times R^n, \tag{4}$$

$$\int_0^\infty a_1(t) dt = \infty, \tag{5}$$

and

$$I_{n-s}(\infty, \bar{t}; a_{s+1}, d_s) = \infty, \quad s = 1, 2, \dots, n-1, \tag{6}$$

where $\quad d_s(t) = a_n(t)\ [I_s(t,\bar{t};a_1,a_s)]^\lambda$.
Further, let in case $\lambda = 1$ for $s = 1, 2, \ldots, n-1$ either

$$\liminf_{t\to\infty}\ e^{-J(t,\bar{t};a_s)} \int_{\bar{t}}^{t} a_s(\tau) e^{-I_n(\tau,\bar{t};a_{s+1},a_s)} d\tau = 0 \tag{7}$$

or

$$I_{n-1}(\infty,\bar{t};a_{s+1},a_{n+s-1}) = \infty \tag{8}$$

hold.

(H4): Let the hypothesis (H3) holds with $K = 0, \lambda \in [0,1)$ and with the exception of (5) and let, moreover,

$$I_n(\infty,0;a_1,a_n) = \infty. \tag{9}$$

A great effort has been devoted to the study of oscillatory solutions of Eq. (1) in the canonical form, i.e if

$$\int_0^\infty a_i(t)dt = \infty, \quad i = 1, 2, \ldots, n-1. \tag{10}$$

Definition 2. Eq. (1) is said to have Property A if every proper solution y is oscillatory for n even, and it is either oscillatory, or

$$\lim_{t\to\infty} y^{[i]}(t) = 0, \quad i = 0, \ldots, n-1$$

holds eventually on R_+ if n is odd.

Chanturia [5] proved the following theorem.

Theorem A ([5]). *Let* $f(t,t_1,\ldots,x_n) \equiv \bar{f}(t,x_1), \bar{f} \in C(R_+ \times R), (1)$ *have Property* A*. Let* (10) *and*

$$|\bar{f}(t,x_1)| \le b(t)|x_1| \qquad \text{on} \quad R_+ \times R$$

be valid where $b \in C^0(R_+)$*. Then* (1) *has an oscillatory solution.*

Sufficient conditions, under the validity of which, (1) has Property A were studied e.g. in [5], [7]. Generalizations of Th. A are stated in [3] and in [6] (for $n = 3$). Apart from other things

$$\int_0^\infty a_1(t)dt = \int_0^\infty a_2(t)dt = \infty \tag{11}$$

is supposed instead of (10).

In some applications of Eq. (1) the conditions (10) and (11) are not fulfilled. Although every Eq. (1) can be transformed into the canonical form by sequence of transformations preserving oscillations (see [8]) it is difficult to realize them. E.g. consider the third order differential equation

$$y''' + q(t)y' + r(t)g(y) = 0, \tag{12}$$

where $q \in C^0\ (R+)$, $\quad r \in L_{loc}\ (R_+)$, $\quad g \in C^0\ (R)$, $\quad r \le 0$ on R_+, $g(x)x > 0$ for $x \ne 0$.

Let h be a positive solution on $[T, \infty), T \in R_+$ of the equation

$$h'' + q(t)h = 0 \tag{13}$$

Then (12) is equivalent with (see [4])

$$\left(h^2\left(\frac{1}{h}\,y'\right)'\right)' + rhg(y) = 0 \tag{14}$$

on $[T, \infty)$, where

$$y^{[1]} = \frac{y'}{h}, \qquad y^{[2]} = h^2\left(y^{[1]}\right)'.$$

If we define $h(t) \equiv h(T)$ on $[0, T]$, then (14) is defined on $R_+ \times R^3$ and it has the form (1) with

$$a_1 = h, \qquad a_2 = \frac{1}{h^2}, \quad f(t, x_1, x_2, x_3) \equiv -r(t)h(t)g(x_1) \tag{15}$$

and (3) holds.
If e.g. $q(t) \le const. < 0$, then it is clear that (10) and (11) for $n = 3$ are not valid.

Our main goal is to prove the existence of oscillatory solutions of (1) without the validity of either (10) or (11) and to apply the results to Eq. (12).

2 Main results

In this section, a special set of oscillatory solutions will be investigated. Consider the Cauchy initial conditions:

$$\begin{aligned} l \in \{0, 1, \ldots, n-1\}, &\quad \sigma \in \{-1, 1\}, \\ \sigma\, y^{[i]}(0) > \ 0 &\quad \text{for } i = 0, 1, \ldots, l-1, \\ \le \ 0 &\quad \text{for } i = l, \\ > \ 0 &\quad \text{for } i = l+1, \ldots, n-1. \end{aligned} \tag{16}$$

We will show that a solution y of (1), fulfilling (16) is oscillatory under some assumptions posed on f and a_i.

Theorem 3. *Let (H1) and (H2) be valid. Then every solution y of (1) satisfying (16) is proper.*

Proof. See [2, Lemmas 4 and 9]. □

Theorem 4. *Let (H3) be valid. Then every proper solution y of (1) satisfying (16) is oscillatory.*

Proof. It follows from [2, Lemma 2] that every proper solution y satisfying (16) is either oscillatory or nonoscillatory, $s \in \{0, 1, \ldots, n-1\}$ and T exists such that $T \geq \max(\bar{t}, 1)$,

$$\begin{aligned} y^{[j]}(t) y^{[s]}(t) &\geq 0 \quad \text{for } j = 0, 1, \ldots, s, \\ &\leq 0 \quad \text{for } j = s+1, \ldots, n, \\ y^{[m]}(t) \neq 0, &\quad m = 0, 1, \ldots, n-2, \quad t \in [T, \infty). \end{aligned} \tag{17}$$

Let y fulfills (17). First, we prove that $s \neq 0$ and

$$\lim_{t \to \infty} |y\ (t)| = \infty. \tag{18}$$

Let, on the contrary, $s = 0$. Then (17) and (2) yield

$$y^{[0]}\ y^{[1]} < 0, \quad |y^{[1]}| \quad \text{is nondecreasing on} \quad [T, \infty]$$

and

$$\infty > |y(\infty) - y(T)| = \int_T^\infty a_1(t)\ |y^{[1]}(t)|\ dt\ \geq\ y^{[1]}(T)\ \int_{\bar{t}}^\infty a_1(t)\ dt = \infty.$$

Thus $s \in \{1, \ldots, n-1\}$.

Let $s = 1$. Suppose, without loss of generality, that $y > 0$. Then (17) yields

$$\left.\begin{aligned} y > 0, &\qquad y \text{ increasing}, \\ y^{[1]} > 0, &\qquad y^{[1]} \text{ decreasing}, \\ y^{[i]} < 0, &\qquad |y^{[i]}| \text{ increasing for } i = 2, \ldots, n-1. \end{aligned}\right\} \tag{19}$$

We prove that (18) holds. Thus, suppose, indirectly, that

$$\lim_{t \to \infty} y(t) = C < \infty. \tag{20}$$

If $y^{[1]}(\infty) > 0$, then

$$\infty > y(\infty) - y(T) = \int_T^\infty a_1(t) y^{[1]}(t) dt \geq y^{[1)}(\infty) \int_T^\infty a_1(t) dt = \infty.$$

The contradiction proves that

$$\lim_{t \to \infty} y^{[1]}(t) = 0. \tag{21}$$

It follows from (19), (2) and (4) that

$$\begin{aligned} |y^{[i]}(t)| &= |\, y^{[i]}(T)| + \int_T^t a_{i+1}(\tau)|y^{[i+1]}(\tau)|d\tau \\ &\geq \int_T^\infty a_{i+1}(\tau)|y^{[i+1]}(\tau)|d\tau,\ i = 2,\dots,n-2, \\ y^{[n-1]}(t) &\geq \int_T^t |y^{[n]}(\tau)|d\tau \geq \int_T^t a_n(\tau)g(y(\tau))d\tau \\ &\geq C_1 \int_T^t a_n(\tau)d\tau, \quad C_1 = \max_{y(T)\leq\tau\leq C} g(\tau) > 0. \end{aligned} \tag{22}$$

From this and from (19), (20) and (21)

$$\begin{aligned} &\infty > y(\infty) - y(T) = \int_T^\infty a_1(\tau_1)y^{[1]}(\tau_1)d\tau_1 \\ &= \int_T^\infty a_1(\tau_1) \int_{\tau_1}^\infty a_2(\tau_2)|y^{[2]}(\tau_2)|d\tau_2 d\tau_1 \\ &\geq C_1 \int_T^\infty a_1(\tau_1) \int_{\tau_1}^\infty a_2(\tau_2) I_{n-2}(\tau_2, T; a_3, a_n) d\tau_2 d\tau_1 \\ &= C_1 \int_T^\infty a_2(\tau_2) I_{n-2}(\tau_2, T; a_3, a_n) \int_T^{\tau_2} a_1(\tau_1) d\tau_1 d\tau_2 \\ &\geq C_1 I_n(\infty, T; a_2, a_1) = \infty \end{aligned}$$

as according to (6), $i = 1$

$$I_{n-1}(\infty, \bar{t}; a_2, d_1) = \infty \Longrightarrow I_{n-1}(\infty, T; a_2, d_1) = \infty$$

and thus

$$I_n(\infty, T; a_2, a_1) \geq I_{n-1}(\infty, T; a_2, d_1) = \infty.$$

The contradiction proves that (18) is valid for $s = 1$.
Let $s > 1$. Then (17) and (2) yield

$$y(t)y^{[1]}(t) > 0, \quad |y^{[1]}| \text{ is nondecreasing on } [T, \infty),$$

$$|y(t) - y(T)| = \int_T^\infty a_1(\tau)|y^{[1]}(\tau)|d\tau \geq |y^{[1]}(\tau)| \int_T^t a_1(\tau)d\tau \quad \underset{t\to\infty}{\longrightarrow} \quad \infty.$$

Thus (18) is valid for all $s \in \{1, \dots, n-1\}$.

Let $0 \leq \lambda < 1$. The statement of the theorem was proved in [3, Ths 1-3] if the more restrictive assumption (H4) is supposed instead of (H3). In this case the inequality (4) was used only for $x_1 = y(t)$, $t \in [T, \infty]$ where y fulfills (17). From this, using (18), the statement is valid under the validity of (H3), too (note, that (9) follows from (5)).

Finally, suppose $\lambda = 1$.

Let $s \in \{1, \dots, n-1\}$. We prove that the solution y, fulfilling (17) does not exist.

First, we estimate $y^{[s]}$. Let, for the simplicity, $y > 0$ for large t. According to (18) there exists $T_1 \ge T$ such that

$$y(t) \ge K, \qquad t \in [T_1, \infty) \tag{23}$$

and (17) yields

$$\left.\begin{array}{lll} y^{[j]}(t) > 0, & y^{[j]} \text{ is increasing}, & j = 0, 1, \dots, s-1, \\ y^{[s]}(t) > 0, & y^{[s]} \text{ is decreasing}, & \\ y^{[m]}(t) < 0, & |y^{[m]}| \text{ is nondecreasing}, & m = s+1, \dots, n-1, \\ & & t \in [T_1, \infty). \end{array}\right\} \tag{24}$$

From this, from (24), (2) and (4) we have

$$\begin{aligned} |y^{[i]}(t)| &\ge \int_{T_1}^{t} a_{i+1}(\tau)|y^{[i+1]}(\tau)|d\tau, \; i = 0, \dots, n-2, \; i \ne s, \\ |y^{[n-1]}(t)| &\ge \int_{T_1}^{t} |y^{[n]}(\tau)|d\tau \ge \int_{T_1}^{t} a_n(\tau)y(\tau)d\tau \;\text{ if }\; s \ne n-1 \end{aligned} \tag{25}$$

and thus, using (24),

$$\begin{aligned} |y^{[s+1]}(t)| &\ge I_{n-1}(t, T_1; a_{s+2}, a_s y^{[s]}) \\ &\ge y^{[s]}(t) I_{n-1}(t, T_1; a_{s+2}, a_s), \quad s \in \{1, \dots, n-2\}, \\ |y(t)| &\ge y^{[n-1]}(t) I_{n-1}(t, T_1; a_1, a_{n-1}) \;\text{ for }\; s = n-1. \end{aligned}$$

Further, using (2) and (24), it follows from this that

$$\begin{aligned} \left(y^{[s]}(t)\right)' &= a_{s+1}(t) y^{[s+1]}(t) = -a_{s+1}(t)|y^{[s+1]}(t)| \\ &\le -a_{s+1}(t) I_{n-1}(t, T_1; a_{s+2}, a_s) y^{[s]}(t) \\ &\quad \text{for } s \in \{1, \dots, n-2\}, \\ \left(y^{[n-1]}(t)\right)' &= -|y^{[n]}(t)| \le -a_n(t)y(t) \le -a_n(t) I_{n-1}(t, T_1; a_1, a_{n-1}) \\ &\quad \times y^{[n-1]}(t) \quad \text{for } \; s = n-1, \; t \ge T_1. \end{aligned}$$

Thus

$$y^{[s]}(t) \le y^{[s]}(T_1) e^{-I_n(t, T_1; a_{s+1}, a_s)}. \tag{26}$$

Especially, using (6),

$$\lim_{t \to \infty} y^{[s]}(t) = 0. \tag{27}$$

Let the assumption (7) be valid. Using (24), (25) and (27)

$$
\begin{aligned}
y^{[s-1]}(t) &= y^{[s-1]}(T_1) + \int_{T_1}^{t} a_s(\tau_s) y^{[s]}(\tau_s) d\tau_s \\
&= y^{[s-1)}(T_1) + \int_{T_1}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) |y^{[s+1]}(\tau_{s+1})| d\tau_{s+1} d\tau_s \\
\geq y^{[s-1]}(T_1) &+ \int_{T_1}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, T_1; a_{s+2}, a_{s-1}\ y^{[s-1]}) d\tau_{s+1} d\tau_s \\
\geq y^{[s-1]}(T_1) &+ \int_{T_1}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, \tau_s; a_{s+2}, a_{s-1}\ y^{[s-1]}) d\tau_{s+1} d\tau_s \\
\geq y^{[s-1]}(T_1) &+ \int_{T_1}^{t} y^{[s-1]}(\tau_s) a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, \tau_s; a_{s+2}, a_{s-1}) d\tau_{s+1} d\tau_s, \\
& t \geq T_1.
\end{aligned}
$$

Thus Gronwall's inequality yields

$$
y^{[s-1]}(t) \geq y^{[s-1]}(T_1) e^{J\ (t, T_1; a_s)}, \quad t \geq T_1. \tag{28}
$$

On the other side, using (26), we have

$$
y^{[s-1]}(t) \leq y^{[s-1]}(T_1) + y^{[s]}(T_1) \int_{T_1}^{t} a_s(\tau) e^{-I_n(\tau, T_1; a_{s+1}, a_s)}\ d\tau.
$$

From this and from (28)

$$
\begin{aligned}
1 \leq e^{-J(t,T_1;a_s)} + \frac{y^{[s]}(T_1)}{y^{[s-1]}(T_1)} e^{-J(t,T_1;a_s)} \int_{T_1}^{t} a_s(\tau) \\
\times \quad e^{-I_n(\tau, T_1; a_{s+1}, a_s)}\ d\tau, \quad t \geq T_1
\end{aligned}
$$

that contradicts to (7).

Let the assumption (8) be valid. Then (24) and (25) yield

$$
\begin{aligned}
\infty > |y^{[s]}(\infty) - y^{[s]}(T_1)| = \\
= \int_{T_1}^{\infty} a_{s+1}(\tau) |y^{[s+1]}(\tau)| d\tau \geq I_{n-1}(\infty, T_1; a_{s+1}, a_{s-1} y^{[s-1]}) \geq \\
\geq y^{[s-1]}(T_1) I_{n-1}(\infty, T_1; a_{s+1}, a_{s-1}) = \infty.
\end{aligned}
$$

Thus, the solution y, fulfilling (17), does not exist. □

Remark 5. (i) Theorem 4 generalizes results of [3], [6] and Theorem A.
(ii) The statements of Theorems 3 and 4 are valid for a solution y on $[\alpha, \infty)$ if the Cauchy conditions (16) are taken in $t = \alpha$ and $\bar{t} \geq \alpha$ (see (H3)).

3 Applications

We apply the previous results to Eq. (12)

$$y''' + q(t)y' + r(t)g(y) = 0 \tag{12}$$

under the validity of the assumption

$$\lambda \in [0,1], \qquad |x|^{\lambda} \le |g(x)| \quad \text{for large } |x|. \tag{29}$$

Let

$$q^{+}(t) = \max(q(t),0),\ \bar{q}(t) = \min(q(t),0),\ t \in R_{+}.$$

Cecchi and Marini [6] studied Eq. (12) under the following hypothesis:

(H5): Let $\int_0^\infty tq^{-}(t)dt = -K > -\infty$, and let the equation

$$h'' + e^{-2K}q^{+}(t)h = 0$$

be disconjugate on R_+ (i.e. every its solution has at most one zero on R_+).

They proved the following theorem.

Theorem B ([6]). *Let (H5) and g be nondecreasing for large $|y|$. Let*

$$\int_0^\infty |g(kt)|r(t)dt = \infty \quad \textit{for every } k \in (0,1). \tag{30}$$

Then every proper solution of Eq. (12) with a zero is oscillatory.

Note, that if the estimation (29) holds, then (30) has the form

$$\int_0^\infty t^{\lambda}r(t)dt = \infty. \tag{31}$$

In case

$$\int_0^\infty tq^{+}(t)dt < \infty, \tag{32}$$

using our previous results, the statement of Th. B can be proved under weaker assumption than (31).

Theorem 6. *Let (H5), (32) and (29) be valid. Further, let*

$$\int_0^\infty t^{2\lambda}r(t)dt = \infty \quad \textit{if} \quad \lambda \in [0,1) \tag{33}$$

and let

$$r(t) \ge \frac{\sigma}{t^3} \qquad \textit{for large } t \qquad \textit{if} \quad \lambda = 1, \tag{34}$$

where $\sigma > 1$ is a constant. Then every proper solution with a zero is oscillatory.

Proof. Let y be a proper solution of (12) with a zero $T \in R_+, y(T) = 0$. If $\sum_{i=0}^{2} |y^{[i]}(T)| = 0$, then according to [1] there exists $t_0 > T$ such that the Cauchy initial conditions at t_0 fulfill (16). In the opposite case it is evident that (16) holds in some right neighbourhood of $t = T$. Thus, in all cases, there exists $t_0 > T$ such that (16) is valid in $t = t_0$.

In [6, Proposition 1] it is proved that (H5) and (32) yield the existence of a solution $h : R_+ \to R$ of Eq. (13) which is positive on $(0, \infty)$, increasing and

$$\lim_{t\to\infty} h(t) = h_0 \in (0, \infty). \tag{35}$$

Thus, (12) is equivalent to (14) on $(0, \infty)$ and (15) yields

$$a_1 = h, \quad a_2 = \frac{1}{h^2}, \quad a_3 = rh \quad \text{on} \quad (0, \infty) \tag{36}$$

and

$$\int_{t_0}^{\infty} a_1(s)ds = \int_{t_0}^{\infty} a_2(s)ds = \infty. \tag{37}$$

Let $\varepsilon > \sqrt[4]{\sigma}$ and let $\tau > t_0$ be such that

$$\frac{h_0}{\varepsilon} \le h(t) \le \varepsilon h_0, \qquad t \ge \tau. \tag{38}$$

We will verify hypothesis (H3) with $\bar{t} = \tau$ (see Remark 5 (ii)). According to (37), (5), (6) for $i = 1$ and (8) for $i = 1$ (in case $\lambda = 1$) are valid. Thus it is necessary to verify (6) for $i = 2$ and, in case $\lambda = 1$, the condition (7) for $i = 2$.

Condition (6), $i = 2$: Using (38) we have

$$I_1(\infty, \tau; a_3) = \int_{\tau}^{\infty} r(t)h(t) \left[\int_{\tau}^{t} h(\alpha) \int_{\tau}^{\alpha} \frac{d\beta}{h^2(\beta)}\, d\alpha\right]^{\lambda} dt$$
$$\ge \varepsilon^{-1-3\lambda}\; h_0^{1-\lambda}\; 2^{-\lambda} \int_{\tau}^{\infty} r(t)\; (t-\tau)^{2\lambda} dt = \infty.$$

Condition (7), $i = 2,\ \lambda = 1$:

$$J\ (t, \tau; a_2) = \int_{\tau}^{t} \frac{1}{h^2(s)} \int_{s}^{\infty} h(s_1)r(s_1) \int_{s}^{s_1} h(s_2)ds_2 ds_1 ds$$
$$\ge \varepsilon^{-4} \int_{\tau}^{t} \int_{s}^{\infty} (s_1 - s)r(s_1)ds_1 ds \ge \sigma_1\ \ln\ \frac{t}{\tau}, \quad \sigma_1 = \frac{\sigma}{2}\ \varepsilon^{-4} > \frac{1}{2},$$

$$I_3(t, \tau; a_3, a_2) = \int_{\tau}^{t} r(s)h(s) \int_{\tau}^{s} h(s_1) \int_{\tau}^{s_1} \frac{ds_2}{h^2(s_2)} ds_1 ds$$
$$\ge \sigma_1 \int_{\tau}^{t} \frac{(s-\tau)^2}{s^3} ds \ge \sigma_1\ \left[\ln \frac{t}{\tau} - 2\right].$$

From this, according to (36), (37) and (38)

$$0 \le \liminf_{t\to\infty} e^{-J(t,\tau;a_2)} \int_\tau^t a_2(s) e^{-I_3(s,\tau;a_3,a_2)}\, ds$$
$$\le \liminf_{t\to\infty} \Big(\frac{\tau}{t}\Big)^{\sigma_1} \int_\tau^t \frac{\varepsilon^2}{h_0^2}\, e^{2\sigma_1} \Big(\frac{\tau}{s}\Big)^{\sigma_1} ds = 0\,.$$

□

Remark 7. Let the assumptions of Th. 6 and hypotheses (H1) and (H2) hold. Then, using Th. 3, it is evident that (12) has an oscillatory solution.

The following example shows that (33) is not sufficient condition for the existence of oscillatory solutions in case $\lambda = 1$ and it shows how far is condition (34) from necessary one.

Example 8. Consider the equation

$$y''' + \frac{\sigma}{t^3}\, y = 0, \qquad \sigma \ge 0. \tag{39}$$

Lemma 9. *Eq.* (39) *has an oscillatory solution if, and only if*

$$\sigma > \frac{2\sqrt{3}}{9} \sim 0,385.$$

Proof. (sketch) Eq. (39) can be transformed into the equation with constant coefficients $\dddot{Y} - 3\ddot{Y} + 2\dot{Y} + \sigma Y = 0$ by $t = e^x, \quad y(t) = Y(x)$. □

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The Nonlinear Limit-Point/Limit-Circle Problem for Higher Order Equations

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Abstract. We describe the nonlinear limit-point/limit-circle problem for the n-th order differential equation

$$y^{(n)} + r(t)f(y, y', \dots, y^{(n-1)}) = 0.$$

The results are then applied to higher order linear and nonlinear equations. A discussion of fourth order equations is included, and some directions for further research are indicated.

AMS Subject Classification. 34C10, 34C15, 34B15

Keywords. Higher order equations, nonlinear limit-point, nonlinear limit-circle

1 Background

In 1910, H. Weyl [21] studied eigenvalue problems for second order linear differential equations of the form

$$(p(t)y')' + r(t)y = \lambda y, \quad \text{Im } \lambda \neq 0,$$

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and he classified this linear equation to be of the *limit-circle type* if every solution y belongs to the class L^2, and to be of the *limit-point type* if at least one solution does not belong to L^2. This notion has been generalized to include even order self-adjoint linear differential equations and operators (see, for example, [5,6,7,8,9,14,15,16,17,18]), and more recently to nonlinear second order equations of the form

$$(a(t)y')' + q(t)f(y) = 0$$

(see the papers of Graef and Spikes [10,11,12,13,19,20]).

Here, we consider the n-th order nonlinear differential equation

$$y^{(n)} + r(t)f(y, y', \dots, y^{(n-1)}) = 0, \tag{E}$$

where $r \in L_{\text{loc}}[0, \infty)$,

$$r \text{ does not change sign on } [t_0, \infty),\ t_0 \geq 0, \tag{1}$$

$f : \mathbf{R}^n \to \mathbf{R}$ is continuous, and

$$x_1 f(x_1, \dots, x_n) \geq 0 \text{ on } \mathbf{R}^n. \tag{2}$$

We consider only those solutions of (E) that are continuable to all of $\mathbf{R}_+ = [0, \infty)$ and are not eventually identically zero. Such a solution is said to be *oscillatory* if it is has arbitrarily large zeros, and it is said to be *nonoscillatory* otherwise.

Definition 1. Equation (E) is of the *nonlinear limit-circle type* if every continuable solution y satisfies

$$\int_0^\infty y(t)f(y(t), y'(t), \dots, y^{(n-1)}(t))\, dt < \infty;$$

if there is at least one continuable solution y of (E) such that

$$\int_0^\infty y(t)f(y(t), y'(t), \dots, y^{(n-1)}(t))\, dt = \infty,$$

then equation (E) is said to be of the *nonlinear limit-point type*.

In this paper, we describe what is known for the higher order nonlinear limit-point/limit-circle problem and indicate a number of open questions for future research.

2 Motivation

Kauffman, Read, and Zettl [14, p. 95] noted that *there are no known examples of functions r such that*

$$y^{(4)} + r(t)y = 0. \tag{L$_4$}$$

is limit-circle, i.e., all solutions of (L$_4$) *are in* L^2. This leads to the following conjecture.

Conjecture 2. *The equation*

$$y^{(4k)} + r(t)y = 0 \qquad (\mathrm{L}_{4k})$$

always has a solution $y \notin L^2[0, \infty)$*, i.e.,* (L_{4k}) *is never of the limit-circle type.*

As a consequence of our results, we will show that as long as r does not change sign, or r is an oscillatory function that is either bounded from above or bounded from below, then (L_{4k}) can never be a limit-circle equation. In addition, we will apply our results to the sublinear Emden-Fowler equation

$$y^{(4k)} + r(t)|y|^{\lambda} \operatorname{sgn} y = 0, \quad \lambda \in (0, 1]$$

and show that this equation always has a solution $y \notin L^{1+\lambda}[0, \infty)$ provided r satisfies (1).

3 Main Results

We begin by presenting some sufficient conditions for equation (E) to be of the nonlinear limit-point type (see [4]).

3.1 The Case $r \leq 0$

Theorem 3. *Suppose* $r(t) \leq 0$ *on* $[t_0, \infty)$*,* (2) *holds, and there exist constants* $M > 0$ *and* $M_1 > 0$ *such that*

$$\frac{1}{x_1} \leq f(x_1, \ldots, x_n) \leq M_1(1 + x_1) \qquad (\mathrm{C}_1)$$

for $x_1 \geq M, \ x_i \in \mathbf{R}, \ i = 2, \ldots, n$*. Then* (E) *is of the nonlinear limit-point type.*

If we restrict our attention to equations of the form

$$y^{(n)} + r(t)f(y) = 0,$$

then (C_1) becomes

$$\frac{1}{u} \leq f(u) \leq M_1(1 + u)$$

for $u \geq M > 0$, which is certainly true, for example, if f is an increasing function with

$$|f(u)| \leq A + B|u| \qquad \text{for large } u,$$

or if $f(u) = u^{\gamma}$ where $0 < \gamma \leq 1$ is the ratio of odd positive integers.

Remark 4. The left hand inequality in (C_1) is not unreasonable. For example, for third order equations, Bartušek and Došlá (see Theorem 3.3 and Remark 3.4 in [1]) proved that if $r(t) \leq -K < 0$ and there exists $\beta > \frac{3}{2}$ such that

$$|f(x_1, x_2, x_3)| \leq \frac{1}{|x_1|^{\beta}} \qquad \text{for } |x_1| \geq M > 0,$$

then (E) is of the nonlinear limit-circle type. Whether their result is true for $n > 3$ remains an open question.

The proof of Theorem 3, as well as the other theorems in this section, are somewhat long and technical in nature. They make use of an energy type function, some integral inequalities, and knowledge of the behavior of oscillatory solutions of (E). Hence, we will omit the proofs, and concentrate on the nature of the results.

3.2 The Case $r \geq 0$

In studying the asymptotic behavior of solutions of higher order equations, the order itself sometimes plays an important role. Observe that the set of positive integers can be divided into the three disjoint sets, $\{n : n = 4k,\ k = 1, 2, \dots\}$, $\{n : n = 2k+1,\ k = 1, 2, \dots\}$, and $\{n : n = 4k+2,\ k = 1, 2, \dots\}$.

Theorem 5. *If $n = 4k$, (2) holds, $r(t) \geq 0$ on $[t_0, \infty)$, and there exist constants $M_1 > 0$, $M_2 > 0$, and $\lambda \in (0, 1]$ such that*

$$M_1 |x_1|^\lambda \leq |f(x_1, x_2, \dots, x_n)| \leq M_2(1 + |x_1|) \qquad \text{on } \mathbf{R}^n, \tag{C$_2$}$$

then (E) is of the nonlinear limit-point type.

Observe once again that if $f(x_1, x_2, \dots, x_n) = f(x_1) = x_1^\gamma$ with $0 < \gamma \leq 1$ the ratio of odd positive integers, then condition (C_2) is clearly satisfied.

Theorem 6. *If $n \geq 3$, (2) holds, and there exist constants $M > 0$, $M_1 > 0$, $M_2 > 0$, and $\lambda \in (0, 1]$ such that*

$$0 \leq r(t) \leq M,$$

and

$$M_1 |x_1|^\lambda \leq |f(x_1, x_2, \dots, x_n)| \leq M_2 |x_1|^\lambda \qquad \text{on } \mathbf{R}^n,$$

then (E) is of the nonlinear limit-point type.

Remark 7. The case $n = 3$ is contained in [1, Theorem 3.7] under a slightly weaker nonlinearity condition on f; the proof for $n \geq 4$ appears in [4, Theorem 3].

The following two theorems generalize the nonlinearity condition imposed on f in Theorem 6, but at the same time, restrict the values of n allowed.

Theorem 8. *Suppose $n = 2k + 1$, there exist constants $M_1 > 0$ and $M_2 > 0$ such that*

$$M_1 \leq r(t) \leq M_2,$$

and there is a positive constant M and a continuous function $g : \mathbf{R}_+ \to \mathbf{R}$ such that $g(0) = 0$, $g(x) > 0$ for $x > 0$, $\liminf_{x\to\infty} g(x) > 0$, and

$$g(|x_1|) \leq |f(x_1, \dots, x_n)| \leq M(1 + |x_1|) \qquad \text{on } \mathbf{R}^n.$$

Then (E) is of the nonlinear limit-point type.

Theorem 9. *Suppose that* $n = 4k$, (2) *holds, and that there exist constants* K_i, $i = 0, 1, 2, 3, 4$, *and* x^* *such that*

$$0 \le r(t) \le K_0\, t^\delta \text{ on } (t_0, \infty),$$
$$g_1(|x_1|) \le |f(x_1, \dots, x_n)| \le g_2(|x_1|) \text{ on } \mathbf{R}^n,$$

where $\delta = \frac{n+1}{n-2}$ *and*

$$g_1(x) = \begin{cases} K_1 x & \text{for } x \in [0, x^*] \\ K_2 & \text{for } x \in (x^*, \infty) \end{cases}$$

$$g_2(x) = \begin{cases} K_3 & \text{for } x \in [0, x^*] \\ K_4 x & \text{for } x \in (x^*, \infty). \end{cases}$$

Then (E) *is of the nonlinear limit-point type.*

Observe that in Theorems 6 and 8, $r(t)$ is bounded above, while in Theorem 9, $r(t)$ is allowed to grow with t.

4 Applications of Main Results

Our first corollary concerns equation (E) and is an immediate consequence of Theorems 3 and 5.

Corollary 10. *If* $n = 4k$, *and* (1)–(2) *and* (C_2) *hold, then* (E) *is of the nonlinear limit-point type.*

Next, we apply our results to the equation

$$y^{(4k)} + r(t)y = 0 \tag{L_{4k}}$$

and obtain a positive answer to the conjecture raised in Section 2.

Corollary 11. *If* $r(t)$ *satisfies* (1) *or is an oscillatory function that is either bounded from above or bounded from below, then* (L_{4k}) *is not limit-circle.*

Proof. If r satisfies (1), then the conclusion follows immediately from Corollary 10. Suppose that r is an oscillatory function that is bounded from below. Then there exists a constant $K > 0$ such that $r(t) \ge -K$. By Corollary 10,

$$y^{(4k)} + (r(t) + K)y = 0$$

is not limit-circle. By a result of Naimark [16, §23, Theorem 1, p.192], it follows that the equation

$$y^{(4k)} + (r(t) + K + q(t))y = 0$$

is not limit-circle whenever q is a measurable and essentially bounded function. Thus, letting $q = -K$ we obtain that (L_{4k}) is also not of the limit-circle type. A similar argument holds if $r(t)$ is bounded from above.

Remark 12. Corollary 11 does not follow from Fedorjuk [9, Theorem 5.1] because additional assumptions on the integrability of the derivatives of r would be needed.

As another application of our results, we consider the Emden-Fowler equation

$$y^{(n)} + r(t)|y|^{\lambda} \operatorname{sgn} y = 0, \quad \lambda \in (0, 1]. \tag{E-F}$$

From Theorems 3–6, we have the following corollary (see [4]).

Corollary 13. (a) *If $n = 4k$ and (1) holds, then (E-F) always has a solution $y \notin L^{1+\lambda}[0,\infty)$.*

(b) *Suppose $n = 2k+1$ or $n = 4k+2$. If either $r(t) \le 0$ or $0 \le r(t) \le M$, then (E-F) always has a solution $y \notin L^{1+\lambda}[0,\infty)$.*

5 More on Fourth Order Equations

Now that we have seen that equation (L_4) is not a limit-circle equation (the only possibile exception being if r is an oscillatory function that is unbounded from above and below), it seems appropriate to ask if there are other fourth order equations that are of the limit-circle type. This leads us to the study of fourth order equations in self-adjoint form, namely,

$$y^{(4)} - (p(t)y')' + r(t)f(y) = 0, \tag{SA}$$

where p, $r : [0,\infty) \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ are continuous, and $uf(u) \ge 0$ on $\mathbf{R}$ (see [3]). For equation (SA), the definitions of nonlinear limit-point and limit-circle take the following form.

Definition 14. Equation (SA) is of the *nonlinear limit-circle type* if every continuable solution y satisfies

$$\int_0^\infty y(t)f(y(t))\,dt < \infty,$$

and if there is at least one continuable solution y such that

$$\int_0^\infty y(t)f(y(t))\,dt = \infty,$$

then equation (SA) is said to be of the *nonlinear limit-point type.*

We have the following result in the case where f is sublinear, that is, there exists $K > 0$ such that

$$\frac{1}{|y|} \le |f(y)| \le 1 + |y| \quad \text{for } |y| \ge K. \tag{C_3}$$

Theorem 15. *Let* (C_3) *hold.*

(a) *If* $r(t) \le 0$ *and either*
  (i) $p(t) \ge 0$, *or*
  (ii) $p(t) \le 0$ *and* $I(p) = \int_0^\infty s|p(s)|ds < \infty$,
  then (SA) *is of the nonlinear limit-point type.*

(b) *If* $r(t) \ge 0$ *is bounded,* $p(t) \ne 0$, *and* $I(p) < \infty$, *then* (SA) *is of the nonlinear limit-point type.*

A special case of equation (SA), namely, the self-adjoint linear equation

$$My \equiv y^{(4)} - (p(t)y')' + r(t)y = 0 \tag{SAL}$$

plays an important role in the spectral theory of singular differential operators (see, for example, [5,6,7,8,9,16]) in which the so called deficiency index is defined as follows.

Definition 16. The equation

$$y^{(4)} - (p(t)y')' + r(t)y = \lambda y, \quad \text{Im}\,\lambda \ne 0, \tag{SAL$_\lambda$}$$

is said to be *limit-ν* if it has ν linearly independent solutions in $L^2(0,\infty)$. The differential expression M has the *deficiency index* (ν,ν) if (SAL$_\lambda$) is limit-ν.

It is known from the spectral theory of linear operators that $\nu \in \{2,3,4\}$ for equation (SAL$_\lambda$). When $\nu = 2$, (SAL$_\lambda$) is said to be *limit-point*; when $\nu = 4$, (SAL$_\lambda$) is said to be *limit-circle*. Note that this agrees with our Definition 14 above.

We will make use of the following two results from spectral theory. The first describes the relationship between equations (SAL) and (SAL$_\lambda$), and enables us to give criteria under which (SAL) is not limit-circle.

Lemma 17. (Naimark [16, Theorem 4, p.93]) *Equation* (SAL$_\lambda$) *is limit-4 if and only if equation* (SAL) *has all its solutions belonging to* $L^2(0,\infty)$.

Lemma 18. (Naimark [16, §23, Theorem 1, p.192]) *Let* q *be a real, measurable, essentially bounded function on* $\mathbf{R}_+$. *Then the deficiency index of the expression* M *is not changed by adding the function* q *to* r.

The following conjecture is still open (see, e.g., Paris and Wood [17] or Schultz [18]).

Conjecture 19. *Real formally self-adjoint expressions with nonnegative coefficients are not limit-circle.*

Kauffman [15] proved this conjecture in the case where the coefficients are finite sums of real multiples of real powers satisfying certain other conditions. We can provide additional information about this conjecture with our next result.

Theorem 20. *The equation* (SAL) *is not limit-circle, equivalently,* (SAL_λ) *is either limit-2 or limit-3, or equivalently, the deficiency index of M is either (2,2) or (3,3), if any one of the following conditions is satisfied:*

(i) $r(t) \le 0$ *and* $p(t) \ge 0$*,*

(ii) $r(t) \le 0$*,* $p(t) \le 0$*, and* $I(p) = \int_0^\infty s|p(s)|ds < \infty$*, or*

(iii) r *is bounded.*

Proof. Parts (i) and (ii) follow immediately from Theorem 15 and Lemma 17. To prove (iii), first observe that the equation

$$y^{(4)} - (p(t)y')' = 0$$

is never of the limit-circle type since $y(t) \equiv 1 \notin L^2$ is a solution. Hence,

$$y^{(4)} - (p(t)y')' + r(t)y = 0$$

is not limit-circle by Lemma 18.

Note 21. Results analogous to Theorems 15 and 20 for self-adjoint equations of order $n > 4$ are not yet known.

We conclude this section with the following open problem.

Problem 22. Under what conditions, such as $|r(t)| \le |R(t)|$ for all $t > t_0$, is the following statement true.
If

$$y^{(4)} - (p(t)y')' + R(t)y = 0$$

is not limit-circle, then

$$y^{(4)} - (p(t)y')' + r(t)y = 0$$

is not limit-circle.

To be of interest, it should be assumed that $r(t)$ is an unbounded function (see Theorem 20). Moreover, if $p(t) \equiv 0$, then $r(t)$ should be assumed to be oscillatory as well (see Corollary 11).

6 Concluding Remarks

We conclude this paper by noting the implication of the above results on the study of the nonlinear limit-point/limit-circle problem. Nonlinear equations of the form

$$y^{(n)} + r(t)f(y) = 0 \qquad \text{(NL)}$$

have always been popular objects of study; this has been especially true for second order equations. As a consequence of Corollary 11, unless r is an unbounded oscillatory function, it would not be possible to find sufficient conditions for equation

(NL) to be of the nonlinear limit-circle type if the conditions on the nonlinear function f include linear functions as a special case. This is not the case for second order equations as can be seen from the work of Graef et al. [10,11,12,13,19,20]. Finally, it would be interesting to examine the relationships, if any, between the nonlinear limit-point/limit-circle property and the boundedness, oscillation, or convergence to zero of solutions. These interconnections for second order equations were studied in [10,11,12,13], but for higher order equations, it remains an open question.

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A New Approach to the Existence of Almost Everywhere Solutions of Nonlinear PDEs

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Abstract. We discuss the existence of almost everywhere solutions of nonlinear PDE's of first (in the scalar and vectorial cases) and second order.

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Keywords. A. e. solutions of nonlinear PDE's, Baire category theorem, quasiconvex hull

1 Introduction

This article presents some recent results obtained jointly with P. Marcellini (see [10], [11] and [12]). We propose a new approach for existence of almost everywhere solutions of nonlinear partial differential equations of the first and second order. This approach does not use the notion of viscosity solution since it is mainly intended for handling vectorial problems of non elliptic type. We also give an example (c.f. Theorem 3 and for more general results see [3]) where our method contrasts with the viscosity approach.

Our results establish only existence of solutions; it remains open, in general, to find a criterion of selection among the many solutions which are provided by our existence theorems. Of course when a Lipschitz viscosity solution exists and is unique, then this is, in general, the best criterion.

Our original motivation to study such problems comes from the calculus of variations and its applications to nonlinear elasticity and optimal design (see [9]).

2 First order PDE, the scalar case

Consider the Dirichlet problem

$$\begin{cases} F\left(Du\left(x\right)\right)=0, \text{ a.e. } x\in\Omega \\ u\left(x\right)=\varphi\left(x\right), \, x\in\partial\Omega \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded (or unbounded) open set, $F:\mathbb{R}^n\to\mathbb{R}$ and $\varphi\in W^{1,\infty}\left(\Omega\right)$. We then have

Theorem 1 (c.f. [10]). *Let $E=\{\xi\in\mathbb{R}^n:F\left(\xi\right)=0\}$, if*

$$D\varphi\left(x\right) \text{ is compactly contained in } intcoE, \text{ a.e. in } \Omega \tag{2}$$

where $intcoE$ stands for the interior of the convex hull of E, then there exists (a dense set of) $u\in W^{1,\infty}\left(\Omega\right)$ that satisfies (1). If in addition $\varphi\in C^1\left(\Omega\right)$ and if E is closed then (2) can be replaced by

$$D\varphi\left(x\right)\in E\cup intcoE \text{ in } \Omega. \tag{3}$$

Remark 2. (i) One should note that no hypotheses of convexity or coercivity on F are made. The condition is close to the necessary condition which, in some sense, is

$$D\varphi\left(x\right)\in coE \text{ in } \Omega.$$

(ii) The condition (3) excludes, as it should do, the linear case since then $intcoE=\emptyset$.

(iii) The above theorem can be generalized to the case where $F=F(x,u,Du)$, c.f. [11], c.f. also [1] and [15].

(iv) It is interesting to compare the above result with the classical hypotheses (c.f. [17], [7], [18]) ensuring existence of Lipschitz viscosity solution to (1) i.e. F is convex, coercive ($\lim F\left(\xi\right)=+\infty$ if $|\xi|\to\infty$) then

$$E\cup intcoE=\{\xi\in\mathbb{R}^n:F\left(\xi\right)\leq 0\}$$

and we recover the usual compatibility condition $F\left(D\varphi\right)\leq 0$.

Proof. We very roughly outline the idea of the proof in the classical case i.e. when F is convex, coercive and $F\left(D\varphi\right)\leq 0$. We set

$$V=\left\{u\in\varphi+W_0^{1,\infty}\left(\Omega\right):F\left(Du\right)\leq 0\right\}.$$

Then $\varphi\in V$ and when endowed with the C^0 metric it becomes a complete metric space (this results from the convexity and coercivity of F). We then define

$$V^k=\left\{u\in V:\int_\Omega F\left(Du\right)>-\frac{1}{k}\right\}.$$

Then V^k is open and dense in V, the first property follows from the convexity of F while the second one is more difficult and is some kind of relaxation theorem used in the calculus of variations.

We then use Baire category theorem which ensures that

$$\bigcap V^k = \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) : F(Du) = 0 \right\}$$

is dense (and hence non empty) in V. This achieves the outline of the proof.

The idea to use Baire theorem for Cauchy problem for ordinary differential inclusion is due to Cellina [5], c.f. also [14]. □

A natural question is then to ask if under the general assumption of the theorem one can always find among the many solutions a viscosity one (when F is convex and coercive this is the case). The answer is in general negative unless strong *geometric restrictions* are assumed. A necessary and sufficient condition is given in [3]. We give below such a result only in a particular example which sheds some light on the nature of these *geometric restrictions*. We will denote for $u = u(x, y)$ its partial derivatives by u_x, u_y.

Theorem 3 ([3]). *Let $\Omega \subset \mathbb{R}^2$ be convex. Then*

$$\begin{cases} F(Du) = \left(u_x^2 - 1\right)^2 + \left(u_y^2 - 1\right)^2 = 0, \ \text{a.e. in } \Omega \\ u = 0, \ \text{on } \partial\Omega \end{cases} \tag{4}$$

has a $W^{1,\infty}$ viscosity solution if and only if Ω is a rectangle whose faces are orthogonal to the vectors $(1,1)$ and $(1,-1)$.

Remark 4. Note that by Theorem 1 the problem (4) has a $W^{1,\infty}$ solution since

$$0 \in intcoE = \left\{ \xi \in \mathbb{R}^2 : |\xi_1|, |\xi_2| < 1 \right\}.$$

3 First order PDE, the vectorial case

We now want to discuss the analogue of Theorem 1 in the vectorial case. The problem is then

$$\begin{cases} F_1(Du) = \cdots = F_N(Du) = 0, \ \text{a.e. in } \Omega \\ u = \varphi, \ \text{on } \partial\Omega \end{cases} \tag{5}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n, m > 1$, and $F_i : \mathbb{R}^{m \times n} \to \mathbb{R}$, $i = 1, ..., N$.

We then let

$$E = \left\{ \xi \in \mathbb{R}^{m \times n} : F_i(\xi) = 0, i = 1, ..., N \right\}.$$

A natural conjecture (c.f. [11]) is then

Conjecture 5. *The system* (5) *has a* $W^{1,\infty}$ *solution provided* $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^m\right)$ *is such that*

$$D\varphi\left(x\right) \in E \cup intQcoE,\ \ in\ \Omega$$

where $QcoE$ *denotes the quasiconvex (in the sense of Morrey) hull of* E*.*

This conjecture is a theorem under some extra technical conditions which are discussed in [11]. In the scalar case the notions of convexity and quasiconvexity are equivalent, therefore $QcoE = coE$. As in the scalar case the conjecture is close to the necessary condition which is, in some sense,

$$D\varphi\left(x\right) \in QcoE,\ \text{ in } \Omega.$$

These types of problems are important in the calculus of variations (see [9]) and in nonlinear elasticity (phase transitions, problem of potential wells, c.f. also in this case [20]) or in optimal design.

We now give one typical case that can be handled by our method (c.f. [11] and [13], c.f. also [4]).

Let $\xi \in \mathbb{R}^{n\times n}$ and denote by $0 \le \lambda_1\left(\xi\right) \le ... \le \lambda_n\left(\xi\right)$ the singular values of the matrix ξ (i.e. the eigenvalues of $\left(\xi^t\xi\right)^{1/2}$). This implies in particular that

$$\left|\xi\right|^2 = \sum_{i,j=1}^{n} \xi_{ij}^2 = \sum_{i=1}^{n}\left(\lambda_i\left(\xi\right)\right)^2,\ \left|\det\xi\right| = \prod_{i=1}^{n}\lambda_i\left(\xi\right).$$

Theorem 6. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $a_i : \overline{\Omega}\times\mathbb{R}^n \to \mathbb{R}$*,* $i = 1, ..., n$ *be continuous functions satisfying*

$$0 < c \le a_1\left(x,s\right) \le ... \le a_n\left(x,s\right)$$

for some constant c *and for every* $\left(x,s\right) \in \overline{\Omega}\times\mathbb{R}^n$*. Let* $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^n\right)$ *satisfy*

$$\prod_{i=\nu}^{n}\lambda_i\left(D\varphi\left(x\right)\right) < \prod_{i=\nu}^{n}a_i\left(x,\varphi\left(x\right)\right),\ x \in \Omega,\ \nu = 1,...,n \tag{6}$$

(in particular $\varphi \equiv 0$*), then there exists (a dense set of)* $u \in W^{1,\infty}\left(\Omega;\mathbb{R}^n\right)$ *such that*

$$\begin{cases} \lambda_i\left(Du\left(x\right)\right) = a_i\left(x,u\left(x\right)\right),\ a.e.\ x \in \Omega,\ i = 1,...,n \\ u\left(x\right) = \varphi\left(x\right),\ x \in \partial\Omega. \end{cases} \tag{7}$$

Remark 7. If $a_i \equiv 1$, for every $i = 1, ..., n$, then (6) becomes

$$\lambda_n\left(D\varphi\left(x\right)\right) < 1,\ x \in \Omega.$$

The problem (7) can then equivalently be rewritten as

$$Du\left(x\right) \in O\left(n\right),\ \text{a.e. in } \Omega.$$

The case $n = 3$, $a_i \equiv 1$ and $\varphi \equiv 0$ has also been studied in [6].

4 Second order case

Since second order equations can be rewritten as first order systems, this section seems to fall in the preceding one; however some of the equations are then linear and hence this corresponds to the case where

$$intQcoE = \emptyset.$$

We here present two types of results of more general ones, see [12].

The first one deals with one single equation. For this purpose we introduce the following notations and terminology

$$\mathbb{R}_s^{n\times n} = \left\{\xi \in \mathbb{R}^{n\times n} : \xi = \xi^t\right\}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n} \to \mathbb{R}$, $F = F(x, s, p, \xi)$, we say that F is *coercive* with respect to the last variable ξ *in the rank one direction* λ, if $\lambda \in \mathbb{R}_s^{n\times n}$ with $rank\{\lambda\} = 1$, and for every bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n}$ there exist constants $m, q > 0$, such that

$$F(x, s, p, \xi + t\lambda) \geq m|t| - q$$

for every $t \in \mathbb{R}$ and every (x, s, p, ξ) that vary in the bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n}$. Examples of such functions are

$$F(\xi) = |\xi|^2 - 1 = \sum_{i,j=1}^{n} \left(\xi_{ij}^2\right) - 1 \text{ or } F(\xi) = |\text{trace } \xi| - 1.$$

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n} \to \mathbb{R}$ be a continuous function, convex with respect to the last variable and coercive in a rank one direction λ. Let $\varphi \in C^2(\mathbb{R}^n)$ satisfy*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq 0,\ x \in \overline{\Omega}. \tag{8}$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F\left(x, u(x), Du(x), D^2u(x)\right) = 0,\ a.e.\ x \in \Omega \\ u(x) = \varphi(x),\ Du(x) = D\varphi(x),\ x \in \partial\Omega. \end{cases}$$

Remark 9. (i) The theorem remains valid if convexity is replaced by quasiconvexity in the sense of Morrey (for this notion see [19] or [8]).

(ii) The coercivity condition in a rank one direction excludes from our analysis linear equations as well as the so called *fully non linear elliptic equations* (in the sense of [2], [7], [16] or [21]).

(iii) Note that if u and φ are smooth functions and $\partial\Omega$ is smooth, then to write $u = \varphi$, $Du = D\varphi$, on $\partial\Omega$ is equivalent as simultaneously prescribing the normal and tangential derivatives. Therefore the boundary conditions are at the same time of Dirichlet and Neumann type.

Examples of applications of this result are

Example 10. (i) The following Dirichlet-Neumann problem admits a $W^{2,\infty}$ solution

$$\begin{cases} |\Delta u| = a\left(x, u\left(x\right), Du\left(x\right)\right), \text{ a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, \text{ on } \partial\Omega \end{cases}$$

provided the compatibility condition is satisfied, namely

$$|\Delta\varphi| \leq a\left(x, \varphi\left(x\right), D\varphi\left(x\right)\right).$$

(ii) Similarly the problem

$$\begin{cases} \left|D^2 u\right| = a\left(x, u\left(x\right), Du\left(x\right)\right), \text{ a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, \text{ on } \partial\Omega \end{cases}$$

has a $W^{2,\infty}$ solution provided

$$\left|D^2\varphi\right| \leq a\left(x, \varphi\left(x\right), D\varphi\left(x\right)\right).$$

Similar results can be established for systems of equations (c.f. [12]). We only quote here the following second order version of Theorem 6 that we get by our method.

Theorem 11. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\varphi \in C^2\left(\mathbb{R}^n\right)$ satisfy*

$$\lambda_n\left(D^2\varphi\left(x\right)\right) < 1,\ x \in \Omega \tag{9}$$

(in particular $\varphi \equiv 0$), then there exists (a dense set of) $u \in W^{2,\infty}\left(\Omega\right)$ such that

$$\begin{cases} \lambda_i\left(D^2 u\left(x\right)\right) = 1,\ a.e.\ x \in \Omega,\ i = 1, ..., n \\ u\left(x\right) = \varphi\left(x\right),\ Du\left(x\right) = D\varphi\left(x\right),\ x \in \partial\Omega. \end{cases} \tag{10}$$

Remark 12. (i) Observe that since in this theorem the matrices are symmetric then the singular values are the absolute values of the eigenvalues of the matrices.

(ii) Note that as a consequence of the above theorem we have that if (9) holds, then the following Dirichlet-Neumann problem admits a solution

$$\begin{cases} \left|\det D^2 u\right| = \prod_{i=1}^{n} \lambda_i\left(D^2 u\right) = 1, \text{ a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, \text{ on } \partial\Omega. \end{cases}$$

Observe that because of the Dirichlet-Neumann boundary data the above problem cannot be handled as a corollary of the results on Monge-Ampère equation.

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Behaviour of Solutions of Linear Differential Equations with Delay

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Abstract. This contribution is devoted to the problem of asymptotic behaviour of solutions of scalar linear differential equation with variable bounded delay of the form

$$\dot{x}(t) = -c(t)x(t-\tau(t)) \tag{*}$$

with positive function $c(t)$. Results concerning the structure of its solutions are obtained with the aid of properties of solutions of auxiliary homogeneous equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t-\tau(t))]$$

where the function $\beta(t)$ is positive. A result concerning the behaviour of solutions of Eq. (*) in critical case is given and, moreover, an analogy with behaviour of solutions of the second order ordinary differential equation

$$x''(t) + a(t)x(t) = 0$$

for positive function $a(t)$ in critical case is considered.

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Keywords. Positive solution, oscillating solution, convergent solution, linear differential equation with delay, topological principle of Ważewski (Rybakowski's approach).

1 Introduction

This contribution is devoted to the problem of asymptotic behaviour of solutions of scalar linear differential equation with variable bounded delay of the form

$$\dot{x}(t) = -c(t)x(t - \tau(t)) \tag{1}$$

with positive function $c(t)$. Results concerning the structure of its solutions are obtained with the aid of properties of solutions of auxiliary homogeneous equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))] \tag{2}$$

where the function $\beta(t)$ is positive. It is known that, supposing existence of a positive solution $x = \omega(t)$ of Eq. (1), the substitution $y(t) = x(t)/\omega(t)$ gives an equation of the type (2) where $\beta(t) \equiv c(t)\omega(t - \tau(t))/\omega(t)$. On the other hand equation of the type (1) can be obtained from Eq. (2) by means of transformation $y(t) = x(t)\exp\left(\int_{t_0}^{t}\beta(s)\,ds\right)$. This means that both equations (1) and (2) are equivalent in this sense. Eq. (2) has very suitable form for investigations since an obvious property (see Lemma 1 below), that any monotone initial function generates monotone solution, implies many further properties concerning behaviour of all solutions.

A result concerning the behaviour of solutions of Eq. (1) in critical case (when $\tau(t) \equiv \tau = \text{const}$ and $\lim_{t\to\infty} c(t) = 1/\tau e$) is given and, moreover, an analogy with behaviour of solutions of the second order ordinary differential equation

$$x''(t) + a(t)x(t) = 0 \tag{3}$$

when positive continuous function $a(t)$ satisfies the condition $\lim_{t\to\infty} t^2 a(t) = 1/4$ is showed. Comparisons with known results are given.

2 Convergence of solutions of Eq. (2)

Let us consider Eq. (2)

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$$

where $\tau \in C(I_{-1}, \mathbb{R}^+)$, $I_{-1} = [t_{-1}, \infty)$, $t_{-1} \in \mathbb{R}$, $\mathbb{R}^+ = (0, \infty)$, $t - \tau(t)$ is an increasing function on I_{-1}, $\tau(t) \le r, t \in I_{-1}, 0 < r = \text{const}$ and $\beta \in C(I_{-1}, \mathbb{R}^+)$. Let us denote $I = [t_0, \infty)$, $I_1 = [t_1, \infty)$ where $t_0 = t_{-1} + \tau(t_0)$ and $t_1 = t_0 + \tau(t_1)$. The symbol "$\cdot$" represents the right-hand derivative.

A function y is called a *solution of Eq.* (2) *corresponding to initial point* $t^* \in I$ if y is defined and is continuous on $[t^* - \tau(t^*), \infty)$, differentiable on $[t^*, \infty)$ and satisfies (2) for $t \ge t^*$. By a *solution* of (2) we mean a solution corresponding to some initial point $t^* \in I$. We denote $y(t^*, \varphi)(t)$ a solution of Eq. (2) corresponding to initial point $t^* \in I$ which is generated by continuous *initial* function $\varphi : [t^* -$

$\tau(t^*), t^*] \mapsto \mathbb{R}$. In the case of linear Eq. (2) solution $y(t^*, \varphi)(t)$ is unique on its maximal existence interval $D_{t^*,\varphi} = [t^*, \infty)$ ([20]).

By analogy we define these notions for Eq. (1) or for other classes of differential equations with delay. If in the text of the paper an initial point is not indicated, we suppose it equals t_0.

We say that a solution of Eq. (2) corresponding to initial point t^* is *convergent* or *asymptotically convergent* if it has a finite limit at $+\infty$.

Let us start with the following trivial lemma:

Lemma 1. (J. Diblík [8]) *Let the initial function $\varphi(t)$ be defined and continuous on $[t_{-1}, t_0]$ and*

$$\varphi(t) < \varphi(t_0) \tag{4}$$

or

$$\varphi(t) > \varphi(t_0)\,, \tag{5}$$

where $t \in [t_{-1}, t_0)$. Then the corresponding solution $y(t, \varphi)$ of Eq. (2) is on I increasing in the case of inequality (4) or decreasing in the case of inequality (5).

This lemma establishes an obvious fact concerning monotony of solutions of Eq. (2). Immediately there arise the questions concerning the conditions for convergence and divergence of such solutions. In this section and in the next one we shall try some of these questions answered.

Theorem 2. (Convergence Criterion) (J. Diblík [6]) *For the convergence of all solutions of Eq. (2), corresponding to initial point t_0, a necessary and sufficient condition is that there exists function $k \in C(I_{-1}, \mathbb{R}^+)$ satisfying the integral inequality*

$$1 + k(t) \geq \exp\left[\int_{t-\tau(t)}^{t} \beta(s)k(s)\,ds\right] \tag{6}$$

on interval I.

The following corollary gives known sufficient condition for convergence of solutions of Eq. (2) which can be obtained as a consequence of Theorem 2 if $k(t) \equiv k = const$ where k is a sufficiently small positive number.

Corollary 3. *All solutions of Eq. (2) are convergent if*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} \beta(s)\,ds < 1.$$

As further consequences we can obtain more accurate sufficient conditions for convergence if $\tau(t) \equiv \tau$ and $k(t) \equiv (t \ln^{\varepsilon} t)^{-1}$ where $\varepsilon > 1$ or $k(t) \equiv \varepsilon t^{-1}(1/\tau - L/t)^{-1}$ where ε is a small positive constant. These sufficient conditions were at first obtained by F. V. Atkinson and J. R. Haddock [2].

Corollary 4. *All solutions of Eq.* (2) *are convergent if*

$$\int_t^{t+\tau} \beta(s)\,ds \le 1 - \frac{\tau}{t} - \frac{L}{t\ln t}$$

for some $L > \tau$ *and all sufficiently large* t *or if*

$$\beta(t) \le \frac{1}{\tau} - \frac{L}{t}$$

where $L > 1/2$ *and* t *is sufficiently large.*

3 Divergence of solutions of Eq. (2)

It is easy to see that the nonexistence of the function $k \in C(I, \mathbb{R}^+)$ in Theorem 2 implies existence of divergent solutions of Eq. (2) and vice versa.

Theorem 5. (Divergence Criterion) (J. Diblík [6]) *Sufficient and necessary condition for existence of solution of Eq.* (2), *corresponding to initial point* t_0, *with property* $y(\infty) = \infty$ *is nonexistence of function* $k \in C(I_{-1}, \mathbb{R}^+)$ *satisfying the integral inequality* (6) *on interval* I.

A consequence of this criterion (if an additional property of $k(t)$ in (6) is taken into account (see [6])) is:

Corollary 6. (J. Diblík [6]) *For existence of solution of Eq.* (2), *corresponding to initial point* t_0, *with property* $y(\infty) = \infty$ *it is sufficient that*

$$\int_{t-\tau(t)}^{t} \beta(s)\,ds \ge 1, \quad t \in I. \tag{7}$$

Consider Eq. (2) where $\tau(t) \equiv 1$. Such type of equation was considered in the paper by S. N. Zhang [41] with connection of investigation of structure and asymptotic behaviour of solutions in *divergent* case (see an unpublished manuscript by F. V. Atkinson and S. N. Zhang of the identical title too). His main conditions (except condition $\beta(t) > 0$) are:

$$\int_t^{t+1} \beta(s)\,ds \ge 1, \quad \int_t^{t+1} \beta(s)\,ds \not\equiv 1, \quad t > t_0. \tag{8}$$

As we can see, these conditions are a special case of (7).

A more detailed sufficient condition for divergence which is sometimes suitable in the case when $\lim_{t\to\infty} \beta(t)\tau(t) = 1$ is given in the next theorem. (This case can be called *critical* in view of Corollary 3 and Corollary 6.)

Theorem 7. (J. Diblík [8]) *Eq.* (2) *has on* I_{-1} *a solution* $y = y(t)$ *with property* $y(\infty) = \infty$ *if*

$$\frac{1}{\tau(t)\beta(t)} - 1 \leq \int_{t-\tau(t)}^{t} \left[\frac{1}{\tau(s)} - \beta(s)\right] ds, \ t \in I$$

and, moreover,

$$\int^{+\infty} \left[\frac{1}{\tau(s)} - \beta(s)\right] ds = +\infty.$$

4 Structure of solutions of Eq. (2) in convergent case

In the convergent case each solution has a finite limit. In this case we can give the estimate of the rate of convergence to this limit.

Theorem 8. (J. Diblík [6]) *Let there be a function* $k \in C(I_{-1}, \mathbb{R}^+)$ *which satisfies the integral inequality* (6) *on* I. *Then for each solution* $y(t)$ *of Eq.* (2), *corresponding to initial point* t_0, *representation*

$$y(t) = K + \zeta(t) \tag{9}$$

holds on I_{-1}, *where* $K = y(\infty)$, *and* $\zeta(t)$ *is a vanishing function. Moreover,*

$$|\zeta(t)| < \psi(t), \ t \in I_1,$$

where $\psi(\infty) = 0$,

$$\psi(t) \equiv \delta e^{-\int\limits_{t_0-r}^{t} \beta(s)k(s)\,ds} - \delta e^{-\int\limits_{t_0-r}^{\infty} \beta(s)k(s)\,ds}$$

and δ *is a fixed positive number such that*

$$\delta > M \left\{ \min_{[t_0, t_0+r]} \left[\beta(t)k(t)e^{-\int_{t_0-r}^{t} \beta(s)k(s)\,ds}\right] \right\}^{-1},$$

where

$$M = \max_{[t_0, t_0+r]} |\dot{y}(t)|.$$

On the other hand, to each $K \in \mathbb{R}$ *there corresponds a solution* $y(t)$ *of Eq.* (2) *and a function of the type* $\zeta(t)$ *such that representation* (9) *holds and for* $K = 0$ *there is an indicated representation with positive function* $\zeta(t)$.

5 Structure of solutions of Eq. (2) in divergent case

Existence of a solution, tending to ∞, plays the main role in the characterization of the family of solutions of Eq. (2) in nonconvergent case. Let us state the following result concerning the structure formula for the solutions of Eq. (2). The unique assumption of it is the existence of a solution $y(t) = Y(t)$ of Eq. (2) with property $Y(\infty) = \infty$. This result generalizes the result by S. N. Zhang [41] (which is contained in the above mentioned manuscript of F. V. Atkinson and S. N. Zhang too) where the main assumptions are: $\beta(t) > 0$ and (8).

Theorem 9. (J. Diblík [8]) *Let $Y(t)$ be a solution of Eq. (2) on I_{-1} with property $Y(\infty) = \infty$. Then for each solution $y(t)$ of Eq. (2), corresponding to initial point t_0, representation*

$$y(t) = K \cdot Y(t) + \delta(t) \tag{10}$$

holds on I_{-1}, where $K \in \mathbb{R}$ is a constant, dependent on $y(t)$, and $\delta(t)$ is a bounded solution of (2) on I_{-1} dependent on $y(t)$. This representation is unique (with respect to K and $\delta(t)$). On the other hand, to each $K \in \mathbb{R}$ there corresponds a solution $y(t)$ of Eq. (2) and a function of the type $\delta(t)$ such that representation (10) holds and for any real K, L, M the expression $K \cdot Y(t) + L + M\delta(t)$ gives a solution of Eq. (2).

Remark 10. In the paper by J. Diblík [8] it is proved that (under certain conditions) bounded nonconstant and nonmonotone solutions of Eq. (2) exist.

6 Concluding remarks concerning the solutions of Eq. (2)

As an analysis of properties of solutions of Eq. (2) shows, the affirmations of the following theorems are equivalent. The indicated conjectures are included as some open problems.

Theorem 11. (Convergent case) *The following assertions are equivalent:*

1) *All solutions of Eq. (2) are convergent.*
2) *There is a function $k \in C(I_{-1}, \mathbb{R}^+)$ which satisfies the integral inequality (6) on I.*
3) *There is a convergent nonconstant and monotone solution of Eq. (2).*
4) *Solution of Eq. (2) with infinite limit does not exist.*
5) (***Conjecture***) *There is a convergent nonconstant and nonmonotone solution of Eq. (2).*
6) (***Conjecture***) *Divergent bounded solution of Eq. (2) does not exist.*

Theorem 12. (Divergent case) *The following assertions are equivalent:*

1) *There is a solution of Eq. (2) with an infinite limit.*
2) *A function $k \in C(I_{-1}, \mathbb{R}^+)$, which satisfies the integral inequality (6) on interval I, does not exist.*

3) *Each nonconstant monotone solution of Eq. (2) has infinite limit.*
4) (***Conjecture***) *A convergent nonconstant solution of Eq. (2) does not exist.*
5) (***Conjecture***) *There is divergent bounded solution of Eq. (2).*

7 Properties of solutions of Eq. (1)

Let us suppose that $c \in C(I_{-1}, \mathbb{R}^+)$. All assumptions with respect to the delay $\tau(t)$ remain the same as above.

As usual, a solution of Eq. (1) is called *oscillatory* if it has arbitrary large zeros. Otherwise it is called *non-oscillatory* (*positive* or *negative*).

At first we prove theorem concerning existence of positive solutions of Eq. (1)

$$\dot{x}(t) = -c(t)x(t - \tau(t))$$

with nonzero limit. In this theorem we shall suppose $\int^{\infty} c(s)\, ds < \infty$ and the point t_0 so large that $\int_{t_0-r}^{\infty} c(s)\, ds < 1$.

Theorem 13. *Eq. (1) has a positive solution with nonzero limit if and only if*

$$\int^{\infty} c(t)\, dt < \infty. \tag{11}$$

Proof. Without loss of generality we shall suppose that $\int_{t_0-r}^{\infty} c(t)\, dt = m < 1$. Let us define $\omega(t)$, where $t \in I$, as the set of functions $\lambda \in C([t-r, t], \mathbb{R})$ such that

$$\varphi_1(t+\theta) < \lambda(t+\theta) < \varphi_2(t+\theta)$$

for all $\theta \in [-r, 0)$ where

$$\varphi_1(t) \equiv 1 + \delta_1 \int_t^{\infty} c(s)\, ds, \quad \varphi_2(t) \equiv 1 + \delta_2 \int_t^{\infty} c(s)\, ds, \quad t \in I_{-1},$$

$\delta_1, \delta_2 = \text{const}, 0 < \delta_1 < 1; 1/(1-m) < \delta_2$ and either $\lambda(t) = \varphi_1(t)$ or $\lambda(t) = \varphi_2(t)$. Let us define function

$$W(t, x) \equiv (x - \varphi_1(t)) \cdot (x - \varphi_2(t)), \quad t \in I_{-1}$$

and find the sign of derivative of this function along the solutions of Eq. (1) on the set $\omega(t)$ for each $t \in I$. We obtain

$$\frac{dW(t,x)}{dt} =$$

$$-(c(t)x(t-\tau(t)) + \varphi_1'(t)) \cdot (x - \varphi_2(t)) - (x - \varphi_1(t)) \cdot (c(t)x(t-\tau(t)) + \varphi_2'(t)).$$

For each $\lambda \in \omega$, such that $\lambda(t) = \varphi_1(t)$, $t \in I$, we have

$$\left.\frac{dW(t,x)}{dt}\right|_{x=\lambda} = -(c(t)\lambda(t-\tau(t)) + \varphi_1'(t))(\varphi_1(t) - \varphi_2(t)) >$$

$$> (\delta_2 - \delta_1)c(t)\left(1 - \delta_1 + \delta_1 \int_{t-\tau(t)}^{\infty} c(s)\,ds\right)\int_t^{\infty} c(s)\,ds > 0$$

and for each $\lambda \in \omega$, such that $\lambda(t) = \varphi_2(t)$, $t \in I$, we get

$$\left.\frac{dW(t,x)}{dt}\right|_{x=\lambda} = -(\varphi_2(t) - \varphi_1(t))(c(t)\lambda(t-\tau(t)) + \varphi_2'(t)) >$$

$$> (\delta_2 - \delta_1)c(t)\left(\delta_2 - 1 - \delta_2 \int_{t-\tau(t)}^{\infty} c(s)\,ds\right)\int_t^{\infty} c(s)\,ds > 0.$$

Therefore in both cases, for $t \in I$, the following is true:

$$\left.\frac{dW(t,x)}{dt}\right|_{x=\lambda} > 0.$$

Now, by the topological method of T. Ważewski (see, for instance, [38]) in the adaptation which is suitable for the retarded functional differential equations (given by K. P. Rybakowski [36]), there is a solution of Eq. (1) $x = \tilde{x}(t), t \in I$ such that $\tilde{x}(t) \in \omega(t)$ for each $t \in I$. From the form of the set $\omega(t)$ it follows that $\varphi_1(t) < \tilde{x}(t) < \varphi_2(t)$ on I_{-1} and, moreover, $\lim_{t\to\infty} \tilde{x}(t) = 1$ since $\lim_{t\to\infty} \varphi_1(t) = \lim_{t\to\infty} \varphi_2(t) = 1$. The details of the application of the topological principle are omitted because they can be found e.g. in [8,9,10], [36].

Now, let us suppose that $\int^{\infty} c(t)\,dt = \infty$. If there is a positive solution $x = \tilde{x}(t), t \in I_{-1}$, of Eq. (1) with nonzero limit $\tilde{x}(\infty) = L > 0$, then integration of this equation with limits t_0 and ∞ gives

$$L - \tilde{x}(t_0) = -\int_{t_0}^{\infty} c(s)\tilde{x}(s - \tau(t))\,ds. \tag{12}$$

We obtain a contradiction since the left hand side of (12) is equal to a negative number although the right hand side is equal to $-\infty$. The theorem is proved.

Corollary 14. *As it follows from the proof of Theorem 13 if (11) holds then there is a solution $x = x(t)$ of Eq. (1) on I_{-1} such that*

$$1 + \delta_1 \int_t^{\infty} c(s)\,ds < x(t) < 1 + \delta_2 \int_t^{\infty} c(s)\,ds,$$

where $t \in I_{-1}, \delta_1, \delta_2 = \text{const}, \delta_1 \in (0,1), \delta_2 \in (1/(1-m), \infty)$ and $m = \int_{t_0-r}^{\infty} c(s)\,ds$.

Remark 15. As it follows from Theorem 13, each positive solution of Eq. (1) tends to zero if $\int^{\infty} c(t)\,dt = \infty$.

8 Structure formulas for solutions of Eq. (1)

Theorem 16. *Let us suppose the existence of a positive solution* $x = \tilde{x}(t)$ *of Eq.* (1) *on* I_{-1}. *Then every solution* $x = x(t)$ *of Eq.* (1) *is by a unique way represented either by the formula*

$$x(t) = \tilde{x}(t)(K + \zeta(t)), \tag{13}$$

where $K \in \mathbb{R}$ *is a constant, dependent on* $x(t)$, *and* $\zeta(t)$, $\zeta(\infty) = 0$ *is a continuous function defined on* I_{-1} *dependent on* $x(t)$, *or by the formula*

$$x(t) = \tilde{x}(t)(KY(t) + \delta(t)) \tag{14}$$

where $Y(t)$ *is a continuous increasing function which is the same for each* $x(t)$, $Y(\infty) = \infty$, $K \in \mathbb{R}$ *is a constant, dependent on* $x(t)$, *and* $\delta(t)$ *is a bounded continuous function defined on* I_{-1} *dependent on* $x(t)$. *On the other hand, to each* $K \in \mathbb{R}$ *there corresponds a solution of* $x(t)$ *Eq.* (1) *and a function of the type* $\zeta(t)$ *(if in* (13) $K = 0$, *then there is a representation of a solution* $x(t)$ *with positive function* $\zeta(t)$) *or of the type* $\delta(t)$ *such that either formula* (13) *holds or formula* (14) *is valid. Moreover, in this case the representation* (14) *gives a solution of Eq.* (1) *if* $\delta(t)$ *is shifted by any constant or is equal to any constant.*

Proof. Let us introduce a new variable $y(t)$ by means of formula

$$y(t) = x(t)/\tilde{x}(t)$$

where $x(t)$ is any solution of Eq. (1). Then $y(t)$ satisfies the equation of the type of Eq. (2), i.e. the equation

$$\dot{y}(t) = \frac{c(t)\tilde{x}(t-\tau(t))}{\tilde{x}(t)}\,[y(t) - y(t-\tau(t))]. \tag{15}$$

We can conclude that either there is a positive function $k(t)$ on I_{-1} which satisfies the integral inequality (6) on I if

$$\beta(t) \equiv \frac{c(t)\tilde{x}(t-\tau(t))}{\tilde{x}(t)}$$

or such function does not exist. This means: either the convergence criterion (Theorem 2) holds or the divergence criterion (Theorem 5) is valid. If the first case occurs, then formula (13) immediately follows from Theorem 8 (formula (9)). If we deal with the second possibility, then Theorem 9 is true and the representation (14) follows immediately from formula (10). The theorem is proved.

Remark 17. Let us suppose that Theorem 16 holds. Then there are two linearly independent positive solutions of Eq. (1) $x_1(t), x_2(t)$ on I_{-1}, defined in the case (13) as

$$x_1(t) = \tilde{x}(t), \;\; x_2(t) = \tilde{x}(t)\zeta(t)$$

(the existence of a positive function $\zeta(t)$ follows from Lemma 1) and in the case (14) as

$$x_1(t) = \tilde{x}(t)Y(t), \ \ x_2(t) = \tilde{x}(t).$$

Obviously $\lim_{t\to\infty} x_2(t)/x_1(t) = 0$. Then formula (14) turns into $x(t) = Kx_1(t) + O(x_2(t))$. In the next theorem it is shown that this formula covers both representations (13), (14).

Theorem 18. *Let there be a positive solution $x = \tilde{x}(t)$, $t \in I_{-1}$, of Eq. (1). Then there are two positive solutions $x_1(t)$, $x_2(t)$, $t \in I_{-1}$, of Eq. (1) such that $\lim_{t\to\infty} x_2(t)/x_1(t) = 0$. Moreover, every solution $x = x(t)$, $t \in I_{-1}$, of Eq. (1) is represented by the formula*

$$x(t) = Kx_1(t) + O(x_2(t)), \ \ t \in I_{-1}, \tag{16}$$

where $K \in \mathbb{R}$ depends on $x(t)$.

Proof. In view of Theorems 9, 16, Lemma 1 and Remark 17 it is sufficient to prove formula (16) if representation (13) holds. Let us introduce a new variable $y(t)$ by means of formula

$$y(t) = x(t)/(\tilde{x}(t)\zeta(t))$$

where $x(t)$ is any solution of Eq. (1) and $\zeta(t) > 0$. Proceeding as above, we conclude that for corresponding equation of the type (15) the structure formula (10) holds. This means

$$y = \tilde{K}\ \tilde{Y}(t) + \tilde{\delta}(t)$$

where the sense of $\tilde{K}, \tilde{Y}(t)$ and $\tilde{\delta}(t)$ is the same as the sense of $K, Y(t)$ and $\delta(t)$ in formula (10). The representation (13) can be written in the form

$$x(t) = \tilde{x}(t)\zeta(t)(\tilde{K}\ \tilde{Y}(t) + \tilde{\delta}(t)).$$

This representation is simultaneously the representation of the type (14) for which the affirmation was proved in Remark 17. The theorem is proved.

Remark 19. For previous results in this direction we refer to the papers by E. Kozakiewicz [28,29,30] and the book of A. D. Myshkis [32]. Note, except this, that (if Theorem 18 holds) any oscillating solution $x = x(t)$ of (1) satisfies relation $x(t) = O(x_2(t))$ and, consequently, tends to zero if $t \to \infty$.

Example 20. Let us consider the equation of the type Eq. (1)

$$\dot{x}(t) = -(1/t)x(t-1). \tag{17}$$

In the papers by J. Diblík [9], [10] it was shown that asymptotic behavior of two linearly independent positive solutions $x_1(t), x_2(t)$ of Eq. (17) is given by relations

$$|x_1(t) - (t-1)^{-1}| < (t-1)^{-2}$$

and

$$\exp[-3(t+1/4)\ln(t+1/4)] < x_2(t) < \exp[-(t/2+1/8)\ln(t-1/4)].$$

Then, by Theorem 18, the representation (16) holds.

Example 21. For the equation of the type Eq. (1)

$$\dot{x}(t) = -(1/e\tau)x(t-\tau) \tag{18}$$

where $\tau = \text{const}$ it is known that there are two asymptotically different positive solutions, namely $x_1(t) = t\exp(-t/\tau)$, $x_2(t) = \exp(-t/\tau)$. In accordance with Theorem 18 the representation (16) holds and each solution is representable in the form

$$x(t) = te^{-t/\tau}\left(K + O\left(1/t\right)\right).$$

9 Existence of positive solution of Eq. (1)

In D. Zhou [42], L. H. Erbe, Q. Kong, B. G. Zhang [16] or J. Diblík [7], [11] some criterions for existence of positive solution of Eq. (1) are given. Let us give one of them which will be used in the sequel.

Theorem 22. (L. H. Erbe, Q. Kong, B. G. Zhang [16], p. 29) *Eq.* (1) *has a positive solution with respect to t_0 if and only if there exists a continuous function $\lambda(t)$ on I_{-1} such that $\lambda(t) > 0$ on I and*

$$\lambda(t) \geq c(t)e^{\int_{t-\tau(t)}^{t}\lambda(s)\,ds}, \ \ t \in I. \tag{19}$$

A very well known sufficient condition, given (under various slightly different assumption for (1) or for modified classes of this equation) by many authors (see, e.g., L. H. Erbe, Q. Kong, B. G. Zhang [16], K. Gopalsamy [17], I. Györi, G. Ladas [18], I. Györi, M. Pituk [19], R. G. Koplatadze, T. A. Chanturija [25], M. Pituk [35]) is a consequence of this criterion:

Corollary 23. (L. H. Erbe, Q. Kong, B. G. Zhang [16], p. 29) *If*

$$\int_{t-\tau(t)}^{t} c(s)\,ds \leq 1/e, \ \ t \in I \tag{20}$$

then Eq. (1) *has a positive solution with respect to t_0.*

This consequence gives that, in the case $\tau(t) \equiv \text{const}$ for existence of a positive solution with respect to t_0 of Eq. (1), the inequality

$$c(t) \leq 1/e\tau, \ \ t \in I_{-1} \tag{21}$$

is sufficient. In the next section the case

$$\lim_{t\to\infty} c(t) = \frac{1}{\tau e}$$

is considered.

10 Behaviour of solutions of Eq. (1) in critical case

Y. Domshlak [13], [14] was the first who noticed that among the equations of the form (1) with $\lim_{t\to\infty} c(t) = 1/\tau e$ there exist equations such that all their solutions are oscillatory in spite of the fact that the corresponding limiting equation (18) admits a non-oscillatory solution (see Example 21). This situation is called critical.

Let us give an improvement of the last sufficient condition (21) together with the sufficient condition for oscillation of all solutions of Eq. (1).

Let us denote

$$\ln_p t = \underbrace{\ln\ln\ldots\ln}_{p} t,\ \ p \geq 1,$$

if $t > \exp_{p-2} 1$, where

$$\exp_p t \equiv (\underbrace{\exp(\exp(\ldots\exp}_{p} t))),\ p \geq 1,$$

$\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$. Moreover, let us define $\ln_0 t \equiv t$. Instead of expressions $\ln_0 t, \ln_1 t$, we will write only t and $\ln t$ in the sequel. The following holds:

Theorem 24. (J. Diblík [11])
A) Let us assume that $\tau(t) \equiv \tau = \mathrm{const}$,

$$c(t) \leq c_p(t) \tag{22}$$

for $t \to \infty$ *and an integer* $p \geq 0$, *where*

$$c_p(t) \equiv \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_p t)^2}\ .$$

Then there is a positive solution $x = x(t)$ *of Eq.* (1)*. Moreover,*

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\ldots\ln_p t}$$

as $t \to \infty$.
B) Let us assume that $\tau(t) \equiv \tau = \mathrm{const}$,

$$c(t) \geq c_{p-1}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\ldots\ln_p t)^2} \tag{23}$$

for $t \to \infty$, *an integer* $p \geq 1$ *and a constant* $\theta > 1$*. Then all solutions of Eq.* (1) *oscillate.*

The proof of the part A) of this theorem can be made with the aid of Theorem 22. Indeed, it is easy to see that the inequality (19), where $c(t) \equiv c_p(t)$, has (for sufficiently large t) a solution

$$\lambda(t) = \frac{1}{\tau} - \frac{1}{2t} - \frac{1}{2t\ln t} - \frac{1}{2t\ln t\ln_2 t} - \cdots - \frac{1}{2t\ln t\ln_2 t\ldots\ln_p t}\ .$$

In process of verification it is necessary to find an asymptotic representation of the right hand side of inequality (19). After this, as usual, we compare the coefficients of identical functional terms on left hand side and on right hand side. The following equalities for determination of coefficients of the functional terms indicated below are valid:

$$1 \quad : \qquad 1/\tau = 1/\tau,$$

$$1/(t \ln t \dots \ln_j t),\ 0 \le j \le p \quad : \qquad -1/2 = -1/2,$$

$$1/(t \ln t \dots \ln_j t)^2,\ 0 \le j \le p \quad : \qquad 0 = \tau/8 - \tau/8,$$

$$1/[(t \ln t \dots \ln_s t)^2 (\ln_{s+1} t \dots \ln_j t)],\ 0 < s < j < p \quad : \qquad 0 = \tau/8 - \tau/8\,.$$

For the next asymptotic smaller terms we have

$$0 \ge -\frac{\tau^2}{16t^3} - \frac{\tau^2}{16t^3} + o\left(\frac{1}{t^3}\right) = -\frac{\tau^2}{8t^3} + o\left(\frac{1}{t^3}\right).$$

This inequality holds for $t \to \infty$. The verification is ended.

In the paper by J. Diblík [11] this part is proved by another equivalent way.

The proof of the part B) is made in cited paper by using the method of Y. Domshlak. In this part, Theorem 24 generalizes Theorem 3 of the recent paper by Y. Domshlak and I. P. Stavroulakis [14]).

Remark 25. The behaviour of solutions in the critical case was investigated by many authors. For example, the papers (except the above mentioned ones) by Li Bingtuan [3], [4], by Á. Elbert and I. P. Stavroulakis [15], by J. Jaroš and I. P. Stavroulakis [23], by E. Kozakiewicz [26], [27] and by J. Werbowski [39] are devoted to this case. We refer to these papers for further bibliography (and history) concerning this question.

Problem 26. An analogy of Theorem 24 is not yet given if inequalities (22), (23) are substituted by inequalities (or by a slightly modified inequalities) obtained from (22), (23) by integrating with limits $t - \tau$ and t, i.e. an analogy is not given if corresponding inequalities are given in terms of the integral average $\int_{t-\tau}^{t} c(s)\, ds$ of the function $c(t)$ instead in terms of values of the function $c(t)$ itself. The first step in this direction is inequality (20). This can serve as a motivation for further investigations in this direction. (As far as this question in the oscillation case is concerned, we refer to the paper [14].) See this situation with an analogous one in Corollary 4.

Remark 27. Let us observe that if inequality (23) holds, then integral inequality (19) has not a positive solution, satisfying conditions indicated in Theorem 22. Note, moreover, that in the papers by F. Neuman (e.g. [33], [34]) a theoretical possibility is given for transformation of an equation with *variable* delays to an equation of the same class with *constant* delays. This perhaps can serve as a possibility of generalization of Theorem 24 if the delay is not constant.

11 Comparison with behaviour of solutions of Eq. (3) in critical case

Let us define functions

$$\mu_p(t) \equiv t \ln t \ln_2 t \dots \ln_p t,$$

$$a_p(t) \equiv \frac{1}{4}\left(\frac{1}{t^2} + \frac{1}{(t\ln t)^2} + \dots + \frac{1}{(t\ln t \dots \ln_{p-1} t)^2} + \frac{1+A}{(t\ln t \dots \ln_p t)^2}\right),$$

where $p \geq 0$, $A \in \mathbb{R}$ and t is sufficiently large.

Lemma 28. *The equation of the type of* (3)

$$x''(t) + a_p(t)x(t) = 0, \ p \geq 0, \tag{24}$$

has following linearly independent solutions:
A)

$$x_1(t) = \sqrt{\mu_p(t)}\, \sin\left(\frac{a}{2}\ln_{p+1} t\right), \ \ x_2(t) = \sqrt{\mu_p(t)}\, \cos\left(\frac{a}{2}\ln_{p+1} t\right),$$

if $A = a^2, a > 0, p \geq 0$;
B)

$$x_1(t) = \sqrt{\mu_p(t)}\,, \ \ x_2(t) = \sqrt{\mu_p(t)}\, \ln_{p+1} t,$$

if $A = 0, p \geq 0$;
C)

$$x_1(t) = \sqrt{\mu_{p-1}(t)}\, (\ln_p t)^{\lambda_1}\,, \ \ x_2(t) = \sqrt{\mu_{p-1}(t)}\, (\ln_p t)^{\lambda_2} \ \text{ for } \ p \geq 1,$$

and

$$x_1(t) = t^{\lambda_1}, \ \ x_2(t) = t^{\lambda_2} \ \text{ for } \ p = 0$$

if $A < 0$ *and* λ_1, λ_2 *are roots of the quadratic equation*

$$\lambda^2 - \lambda + (1+A)/4 = 0, \ \text{ i.e. } \ \lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{-A}\right).$$

Proof. It is easy to verify this affirmation by means of substitution of the expressions $x_1(t), x_2(t)$ into Eq. (24).

Let us formulate the known result concerning oscillatory and nonoscillatory properties of all solutions of Eq. (3) which can be proved by standard arguments with the aid of Lemma 28 and Sturmian Comparison Method (see e.g. [21]).

Theorem 29. *Let* $a \in C(I, \mathbb{R}^+)$. *All solutions of Eq.* (3) *oscillate on* I *if* $a(t) \geq a_p(t)$, $t \in I$ *for some integer* $p \geq 0$ *and* $A > 0$. *If* $a(t) \leq a_p(t)$, $t \in I$ *for some* $p \geq 0$ *and for* $A = 0$ *then Eq.* (3) *is nonoscillatory on* I.

Remark 30. Theorem 24 is an analogy of Theorem 29 since there is a parallel between oscillatory and nonoscillatory properties of solutions of Eq. (1) and Eq. (3). Previous analogues in the case of equations with delay (for $p = 0$ and for $p = 1$) with classical Kneser's theorem [24], [37] and with result due to Hille [22], [37] were given in the cited paper by Y. Domshlak and I. P. Stavroulakis [14]. Note, except this, that conditions concerning functions $a(t)$ and $c(t)$ are very similar. Comparison functions $a_p(t)$ and $c_p(t)$ consist of the same functional terms and differ only in their multipliers and in additive constant.

Remark 31. Some close problems for similar classes of equations and systems of equations (with respect to Eq. (1) and Eq. (2)) are considered e.g. by O. Arino, M. Pituk [1], by J. Čermák [5], by T. Krisztin [31] and for equations with impulses by A. Domoshnitsky, M. Drakhlin [12] and by Yu Jiang, Yan Jurang [40].

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Additive groups connected with asymptotic stability of some differential equations*

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Abstract. The asymptotic behaviour of a Sturm-Liouville differential equation with coefficient $\lambda^2 q(s),\ s \in [s_0, \infty)$ is investigated, where $\lambda \in \mathbb{R}$ and $q(s)$ is a nondecreasing step function tending to ∞ as $s \to \infty$. Let S denote the set of those λ's for which the corresponding differential equation has a solution not tending to 0. It is proved that S is an additive group. Four examples are given with $S = \{0\}$, $S = \mathbb{Z}$, $S = \mathbb{D}$ (i.e. the set of dyadic numbers), and $\mathbb{Q} \subset S \subsetneq \mathbb{R}$.

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1 Introduction and new results

In [1] F. V. Atkinson investigated the differential equations of the form

$$y''(s) + \left(\lambda^2 q(s) + \lambda\sqrt{q(s)}g(s)\right) y(s) = 0 \quad \lambda \in \mathbb{R},\ s \in (s_0, \infty)$$

with a coefficient $q(s) > 0$, which is continuous, nondecreasing and $\lim_{s\to\infty} q(s) = \infty$, and $\int^\infty |g(s)|\, ds < \infty$. He defined the set S of those λ's for which there exist a $g(s)$ and a solution $y(s)$ of this differential equation such that the relation

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$\lim_{s\to\infty} y(s) = 0$ does not hold. He found that S is an additive group and he gave examples when $S = \{0\}$, $S = \mathbb{Z}$.

Here we consider the cases when $q(s)$ is a step function, i.e.

$$q(s) = k_i^2 \quad \text{for } s_i \le s < s_{i+1}, \quad i = 0, 1, \dots, \tag{1}$$

where $0 < k_0 < k_1 < \dots$, $\lim_{i\to\infty} k_i = \infty$ and we consider the differential equation

$$y''(s) + \lambda^2 q(s)\, y(s) = 0 \quad s \ge s_0,\ \lambda \in \mathbb{R}. \tag{2}$$

The function $y(s)$ is a solution of this differential equation if $y(s)$ is continuously differentiable, $y'(s)$ is piecewise continuously differentiable and it satisfies (2) on that pieces of interval.

In [3] we have shown that (2) has at least one solution for which $\lim_{s\to\infty} y(s) = 0$ holds provided $\lambda \ne 0$. It is a question whether all solutions of (2) tend to zero or there are some which do not do this. This property may depend heavily on the actual value of λ. Here we extend the Atkinson's result in the following way.

Theorem. *Let S denote the set of those λ's for which (2) has a solution $y_\lambda(s)$ such that the limit $\lim_{s\to\infty} y_\lambda(s) = 0$ does not hold. Then S is an additive group.*

The set S is never empty because $0 \in S$: for $\lambda = 0$ in (2) we have the solution $y_0(s) \equiv 1$ which does not tend to 0. On the other hand, if $\lambda \ne 0$ and $\lambda \in S$, then $-\lambda \in S$ because in (2) only the value λ^2 counts.

In [3] we have investigated similar problems and we have seen that the stability properties of differential equation (2) are equivalent to the stability of the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \mathcal{D}(d_i)\mathcal{E}(\lambda\omega_i) \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \quad i = 0, 1, \dots, \tag{3}$$

where

$$d_i = \frac{k_i}{k_{i+1}}, \quad \omega_i = k_i(s_{i+1} - s_i), \quad \mathcal{D}(d) = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}, \quad \mathcal{E}(\omega) = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}. \tag{4}$$

Clearly, the sequences $\{d_i\}_{i=0}^\infty$, $\{\omega_i\}_{i=0}^\infty$ are subject to the restrictions

$$0 < d_i < 1, \quad \prod_{i=0}^\infty d_i = 0, \quad \sum_{i=0}^\infty \omega_i d_0 \dots d_{i-1} = \infty. \tag{5}$$

It is evident that if the sequences $\{d_i\}_{i=0}^\infty$, $\{\omega_i\}_{i=0}^\infty$ are given, satisfying (5), and knowing the initial data k_0 and s_0, we can reconstruct the function $q(s)$ of the form (1). Hence the correspondence between the differential equation (2) and the difference equation (3) is one to one.

We shall give examples for different additive groups S.

Example 1. Let $d_i < d_{i+1} < 1$ $(i = 0, 1, \dots)$ and $\lim_{i\to\infty} \omega_i = 0$ such that (5) is satisfied and

$$\sum_{i=0}^{\infty} (1 - d_{i+1})\omega_i^2 = \infty.$$

Then $S = \{0\}$.

Particularly, for $d_i = \frac{i+1}{i+2}$, $\omega_i = \frac{1}{\sqrt{\log(i+2)}}$ all the requirements of Example 1 are satisfied.

Example 2. Let $\omega_i = \pi$ and $d_i < d_{i+1} < 1$ with $\prod_{i=0}^{\infty} d_i = 0$. Then $S = \mathbb{Z}$.

Let $\mathbb{D}$ denote the set of dyadic numbers, i.e. the rational numbers of the form $n/2^m$ for all $n, m \in \mathbb{Z}$. Clearly, this set is an additive group.

Example 3. Let $\omega_i = 2^i\pi$ and $d_i = d \in [\frac{1}{2}, 1)$ be fixed. Then $S = \mathbb{D}$.

Example 4. Let $\omega_i = i!\,\pi$ and $d_i = d \in (0, 1)$. Then $\frac{1}{2}e \notin S$, where $e = 2.718\dots$ is the Euler number and $\mathbb{Q} \subset S \subsetneqq \mathbb{R}$.

Open problem. For the case $S = \mathbb{R}$ we have no other example than the trivial one (see also in [1]) when $q(s)$ tends to a positive constant or $q(s) \equiv const > 0$. We guess that there is no example for $S = \mathbb{R}$ and $\lim_{s\to\infty} q(s) = \infty$.

In the next section we prepare the tools for the proof of the above theorem and examples and the proof itself will be carried out in Section 3.

2 Preliminaries

In [1] the proof goes on the Prüfer transformation technique. Also here we shall follow this way. First we consider the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \mathcal{D}(d_i)\mathcal{E}(\omega_i) \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \quad i = 0, 1, \dots, \tag{6}$$

with parameters d_i, ω_i as in (5). According to the results in [2], we know that the limit $\lim_{i\to\infty}(a_i^2 + b_i^2)$ exists for all solutions $\{[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}], [\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}], \dots\}$. We say that the difference equation (6) is asymptotically stable if for all solutions $\lim_{i\to\infty}(a_i^2 + b_i^2) = 0$, otherwise we say that (6) is not asymptotically stable. Clearly, $\lambda \in S$ if and only if (3) is not asymptotically stable. Therefore we look for criteria to decide when a difference equation is asymptotically stable or not asymptotically stable.

Let r_i, φ_i be defined by

$$a_i = r_i \cos\varphi_i, \; b_i = -r_i \sin\varphi_i, \quad (r_i > 0). \tag{7}$$

Then $\{r_i\}_{i=0}^{\infty}$ is defined uniquely by $r_i = \sqrt{a_i^2 + b_i^2}$. Also φ_0 is unique if we make the restriction $0 \le \varphi_0 < 2\pi$. The desirable uniqueness of the values $\varphi_1, \varphi_2, \ldots$ will be guaranteed by a continuity consideration given later. By (6) we have

$$\begin{aligned} a_{i+1} &= r_{i+1}\cos\varphi_{i+1} = r_i\cos(\omega_i + \varphi_i), \\ b_{i+1} &= -r_{i+1}\sin\varphi_{i+1} = -d_i\, r_i\sin(\omega_i + \varphi_i), \end{aligned} \qquad i = 0, 1, \ldots. \tag{8}$$

Hence

$$r_{i+1}^2 = r_i^2[1 - (1 - d_i^2)\sin^2(\omega_i + \varphi_i)], \quad i =, 0, 1, \ldots,$$

consequently

$$r_{i+1}^2 = r_0^2 \prod_{j=0}^{i} [1 - (1 - d_j^2)\sin^2(\omega_j + \varphi_j)].$$

Clearly, (6) is not asymptotically stable if and only if there exists an initial value φ_0 (and $r_0 = 1$), such that the sequences $\{\varphi_i\}_{i=0}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ satisfy (8) and

$$\prod_{i=0}^{\infty} [1 - (1 - d_i^2)\sin^2(\omega_i + \varphi_i)] > 0,$$

or equivalently,

$$\sum_{i=0}^{\infty} (1 - d_i^2)\sin^2(\omega_i + \varphi_i) < \infty. \tag{9}$$

In this criterion only the knowledge of the sequence $\varphi_0, \varphi_1, \ldots$ is important and we do not have to calculate the sequence $\{r_1, r_2, \ldots\}$ to decide the asymptotic stability of the difference equation (6).

Let us introduce the continuous function $\Phi(d, \alpha)$: $(0, \infty) \times \mathbb{R} \to \mathbb{R}$ by the relations:

$$\begin{aligned} \Phi(1, \alpha) &= \alpha, \\ \Phi(d, k\frac{\pi}{2}) &= k\frac{\pi}{2}, \quad d > 0,\ k \in \mathbb{Z}, \\ \tan\Phi(d, \alpha) &= d\tan\alpha, \quad d > 0, \alpha \ne (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}. \end{aligned} \tag{10}$$

Clearly, $\Phi(d, \alpha)$ is strictly increasing function of α when d is fixed. Hence there exists its inverse $\Phi^{-1}(d, \alpha)$, too. Making use of the function $\Phi(d, \alpha)$, we have by (8)

$$\varphi_{i+1} = \Phi(d_i, \omega_i + \varphi_i), \quad i = 0, 1, \ldots, \tag{11}$$

which defines uniquely the values of $\varphi_1, \varphi_2, \ldots$.

Let the function $\sigma(d, \alpha, \beta)$ be defined on $(0, \infty) \times \mathbb{R}^2$ by one of the following (equivalent) relations:

$$\begin{aligned} &\sigma(d, \alpha, \beta) = \Phi^{-1}(d, \Phi(d, \alpha) + \Phi(d, \beta)) - \alpha - \beta, \\ &\Phi(d, \alpha + \beta + \sigma(d, \alpha, \beta)) = \Phi(d, \alpha) + \Phi(d, \beta). \end{aligned} \tag{12}$$

Clearly, we have $\sigma(1, \alpha, \beta) \equiv 0$. The most important property of this function is formulated as follows.

Lemma. *Let* $0 < d < 1$*, then*

$$|\sigma(d,\alpha,\beta)| \le \frac{\pi}{2}\,(1-d^2)\,|\sin\alpha|\,|\sin\beta|,$$

where the equality holds if and only if either $\sin\alpha = 0$ *or else* $\sin\beta = 0$*.*

The proof of this lemma will be given in the next section.

On asymptotic stability or non stability we can find sufficient conditions in [2] or in [3]. We recall them as follows.

Theorem A. *The difference equation* (6) *is asymptotically stable if*

$$\sum_{i=0}^{\infty} \min\{1-d_i, 1-d_{i+1}\}\sin^2\omega_i = \infty.$$

Theorem B. *If the sum* $\sum_{i=0}^{\infty} |\sin\omega_i| < \infty$*, then the difference equation* (6) *is not asymptotically stable.*

Let $\mathcal{M}$ be a 2×2 (real) matrix and let $\mathbf{x} = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ with the norm $|\mathbf{x}| = \sqrt{a^2+b^2}$. Define the spectral norm $\|\mathcal{M}\|$ of the matrix $\mathcal{M}$ by

$$\|\mathcal{M}\| = \max_{|\mathbf{x}|=1} |\mathcal{M}\mathbf{x}|.$$

Consider the difference equation

$$\begin{bmatrix} \hat{a}_{i+1} \\ \hat{b}_{i+1} \end{bmatrix} = \mathcal{M}_i \begin{bmatrix} \hat{a}_i \\ \hat{b}_i \end{bmatrix} \qquad i = 0, 1, \dots, \tag{13}$$

where $\mathcal{M}_i$ is nonsingular 2×2 matrix for $i = 0, 1, \dots$. We say that (13) is an ℓ_1-perturbation of (6) if

$$\sum_{i=0}^{\infty} \|\mathcal{M}_i - \mathcal{D}(d_i)\mathcal{E}(\omega_i)\| < \infty \tag{14}$$

holds. Here we recall another result from [2, Theorem 6 and Remark 1, Proposition 3]:

Theorem C. *Suppose* (13) *is an* ℓ_1*-perturbation of* (6)*. Then these difference equations are either both asymptotically stable or both not asymptotically stable.*

3 Proofs

We start with the proof of Lemma because we have to apply it to the proof of Theorem.

Proof of the Lemma. Suppose that $\tan\alpha$ and $\tan\beta$ are defined (i.e. $\alpha \not\equiv \frac{\pi}{2} \pmod{\pi}$ and $\beta \not\equiv \frac{\pi}{2} \pmod{\pi}$). Let $\alpha_1 = \Phi(d,\alpha)$, $\beta_1 = \Phi(d,\beta)$. Again we suppose that $\alpha_1+\beta_1 \not\equiv \frac{\pi}{2} \pmod{\pi}$. Then by (10), (12) we have

$$\tan(\alpha_1+\beta_1) = d\tan(\alpha+\beta+\sigma) = d\frac{\tan(\alpha+\beta)+\tan\sigma}{1-\tan(\alpha+\beta)\tan\sigma} = \\ = \frac{\tan\alpha_1+\tan\beta_1}{1-\tan\alpha_1\,\tan\beta_1} = d\frac{\tan\alpha+\tan\beta}{1-d^2\tan\alpha\tan\beta}\,,$$

therefore

$$\tan\sigma = \tan\sigma(d,\alpha,\beta) = -(1-d^2)\frac{\sin\alpha\,\sin\beta\,\sin(\alpha+\beta)}{1+(1-d^2)\sin\alpha\,\sin\beta\,\cos(\alpha+\beta)}. \tag{15}$$

Also by this formula it is clear that $\sigma(1,\alpha,\beta)\equiv 0$ and $\sigma(d,\alpha,\beta)$ is defined for all $(\alpha,\beta)\in\mathbb{R}^2$ if $d\in(0,1]$, i.e. $|\sigma(d,\alpha,\beta)|<\frac{\pi}{2}$.

By (15) it follows that

$$\sigma(d,\alpha,\beta)=\sigma(d,\beta,\alpha),\quad \sigma(d,\alpha+\pi,\beta)=\sigma(d,\alpha,\beta),\quad \sigma(d,-\alpha,-\beta)=-\sigma(d,\alpha,\beta).$$

Thus it is sufficient to prove our Lemma for $0\le|\beta|\le\alpha\le\frac{\pi}{2}$. If $\beta=0$, the statement is trivial. Let $0<\beta\le\alpha\le\frac{\pi}{2}$. First we show that $|\sigma(s,\alpha,-\beta)|\le|\sigma(d,\alpha,\beta)|$ or

$$(1-d^2)\frac{\sin\alpha\,\sin\beta\,\sin(\alpha-\beta)}{1-(1-d^2)\sin\alpha\,\sin\beta\,\cos(\alpha-\beta)} \le (1-d^2)\frac{\sin\alpha\,\sin\beta\,\sin(\alpha+\beta)}{1+(1-d^2)\sin\alpha\,\sin\beta\,\cos(\alpha+\beta)}$$

or simplifying by $(1-d^2)\sin\alpha\,\sin\beta$:

$$(1-d^2)\sin\alpha\,\sin\beta\,\sin 2\alpha \le 2\cos\alpha\,\sin\beta$$

whence the equality holds if $\alpha=\frac{\pi}{2}$, and the sharp inequality $(1-d^2)\sin^2\alpha<1$ in other cases.

Introducing the quantity $x=\frac{\pi}{2}(1-d^2)\sin\alpha\,\sin\beta$, we have to show by (15) that

$$|\tan\sigma| = \frac{\frac{2}{\pi}x\sin(\alpha+\beta)}{1+\frac{2}{\pi}x\cos(\alpha+\beta)} < \tan x = \frac{\sin x}{\cos x},\qquad 0<x<\frac{\pi}{2}$$

or equivalently

$$\sin(\alpha+\beta-x) < \frac{\pi}{2}\frac{\sin x}{x}.$$

The function on the right hand side is strictly decreasing and only at $x=\frac{\pi}{2}$ would attain the value 1, and this fact proves our Lemma. □

Proof of the Theorem. We have to show that if $\lambda,\mu\in S$ (and $\lambda+\mu\neq 0$), then $\lambda+\mu\in S$. According to (3) and (9) there exist φ_0 and ψ_0 such that for the sequences $\{\varphi_i\}_{i=0}^{\infty}$, $\{\psi_i\}_{i=0}^{\infty}$ defined by (11):

$$\varphi_{i+1}=\Phi(d_i,\lambda\omega_i+\varphi_i),\qquad \psi_{i+1}=\Phi(d_i,\mu\omega_i+\psi_i)$$

satisfy the relations

$$\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\omega_i+\varphi_i)<\infty,$$
$$\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\mu_i+\psi_i)<\infty. \tag{16}$$

Let $\sigma_i = \sigma(d_i, \lambda\omega_i+\varphi_i, \mu\omega_i+\psi_i)$ be defined by (12) and consider the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & d_i \end{bmatrix} \begin{bmatrix} \cos\bar{\omega}_i & \sin\bar{\omega}_i \\ -\sin\bar{\omega}_i & \cos\bar{\omega}_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} \qquad i=0,1,\dots, \tag{17}$$

where $\bar{\omega}_i = (\lambda+\mu)\omega_i+\sigma_i$. Let $\bar{\varphi}_i = \varphi_i+\psi_i$. Then by definition of $\bar{\omega}_i$ and by (12) we obtain

$$\begin{aligned}\bar{\varphi}_{i+1} &= \varphi_{i+1}+\psi_{i+1} = \Phi(d_i,\lambda\omega_i+\varphi_i)+\Phi(d_i,\mu\omega_i+\psi_i) = \\ &= \Phi(d_i,\lambda\omega_i+\varphi_i+\mu\omega_i+\psi_i+\sigma_i) = \Phi(d_i,(\lambda+\mu)\omega_i+\sigma_i+\bar{\varphi}_i) = \\ &= \Phi(d_i,\bar{\omega}_i+\bar{\varphi}_i).\end{aligned}$$

Now the difference equation (17) is not asymptotically stable because it has a solution not tending to 0. To see this we apply relation (9). We find

$$\begin{aligned}\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\bar{\omega}_i+\bar{\varphi}_i) &= \sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\omega_i+\varphi_i+\mu\omega_i+\psi_i+\sigma_i) \le \\ &\le 3\sum_{i=0}^{\infty}(1-d_i^2)\left[\sin^2(\lambda\omega_i+\varphi_i)+\sin^2(\mu\omega_i+\psi_i)+\sin^2\sigma_i\right] = \\ &= 3\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\omega_i+\varphi_i)+3\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\mu\omega_i+\psi_i)+ \\ &\quad +3\sum_{i=0}^{\infty}(1-d_i^2)\sin^2\sigma_i.\end{aligned}$$

The first two terms are convergent because of (16). By Lemma we have

$$\sin^2\sigma_i \le \sigma_i^2 \le \frac{\pi^2}{4}(1-d_i^2)^2\sin^2(\lambda\omega_i+\varphi_i)\sin^2(\mu\omega_i+\psi_i),$$

hence also the third term is convergent. Thus we have got

$$\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\bar{\omega}_i+\bar{\varphi}_i)<\infty,$$

which implies the existence of a solution of (17) not tending to 0.

To complete the proof, we show that (17) is an ℓ_1-perturbation of the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \mathcal{D}(d_i)\mathcal{E}((\lambda+\mu)\omega_i) \begin{bmatrix} a_i \\ b_i \end{bmatrix} \quad i = 0, 1, \dots . \tag{18}$$

By Theorem C we have to estimate the spectral norm of the difference of the coefficient matrices:

$$\|\mathcal{D}(d_i)\,[\mathcal{E}(\bar{\omega}_i) - \mathcal{E}((\lambda+\mu)\omega_i)]\| \le \|\mathcal{D}(d_i)\|\,\|\mathcal{E}((\lambda+\mu)\omega_i)\|\,\|\mathcal{E}(\sigma_i) - \mathcal{E}(0)\| \le$$
$$\le 1 \cdot 1 \cdot \sqrt{\sin^2\sigma_i + (1-\cos\sigma_i)^2} \le |\sigma_i|$$

because $\bar{\omega}_i = (\lambda+\mu)\omega_i + \sigma_i$ and $\mathcal{E}(\alpha+\beta) = \mathcal{E}(\alpha)\mathcal{E}(\beta)$. By Lemma and by (16) we conclude that

$$\sum_i^{\infty} |\sigma_i| \le \frac{\pi}{2} \sum_{i=0}^{\infty} (1-d_i^2)\left(\sin^2(\lambda\omega_i + \varphi) + \sin^2(\mu\omega_i + \psi_i)\right) < \infty,$$

i.e. the difference equation (18) is not asymptotically stable. Finally we observe that this difference equation corresponds to the differential equation

$$y''(s) + (\lambda+\mu)^2 q(s) y(s) = 0,$$

hence $\lambda + \mu \in S$. □

Proof of Example 1. Let $\lambda \ne 0$, then we have $\lim_{i\to\infty} \lambda\omega_i = 0$. Let i_0 be sufficiently large integer such that $|\lambda\omega_i| < \frac{\pi}{2}$ for $i \ge i_0$. Applying the inequality $\sin x/x > 1/\sqrt{2}$ for $|x| < \frac{\pi}{2}$, we obtain

$$\sum_{i=0}^{\infty} (1-d_{i+1}) \sin^2 \lambda\omega_i \ge \frac{\lambda^2}{2} \sum_{i=i_0}^{\infty} (1-d_{i+1})\omega_i^2 = \infty,$$

hence by Theorem A we conclude that $\lambda \notin S$, which proves that $S = \{0\}$. □

Proof of Example 2. Let $\lambda = k \in \mathbb{Z}$, then

$$\sum_{i=0}^{\infty} |\sin k\omega_i| = \sum_{i=0}^{\infty} |\sin k\pi| = 0,$$

and by Theorem B $k \in S$, i.e. $\mathbb{Z} \subset S$.

If $\lambda \notin \mathbb{Z}$, then $\sin\lambda\pi \ne 0$ and

$$\sum_{i=0}^{\infty} (1-d_{i+1}) \sin^2 \lambda\pi = \sin^2\lambda\pi \sum_{i=1}^{\infty} (1-d_i) = \infty$$

because by (5) the restriction $\prod_{i=0}^{\infty} d_i = 0$ is equivalent to $\sum_{i=0}^{\infty}(1-d_i) = \infty$. By Theorem A all solutions of (3) tend to 0 if $\lambda \notin \mathbb{Z}$, consequently for these λ's we have $\lambda \notin S$, which proves this example. □

Proof of Example 3. The restriction $d \in [\frac{1}{2}, 1)$ is justified by the requirement in (5): $\sum_{i=0}^{\infty} 2^i \pi d^i = \pi \sum_{i=0}^{\infty} (2d)^i = \infty$. Let $\lambda = \frac{1}{2^n}$, $n \in \mathbb{N}$. Then

$$\sum_{i=0}^{\infty} |\sin \lambda\omega_i| = \sum_{i=0}^{\infty} |\sin \frac{2^i}{2^n}\pi| = \sum_{i=0}^{n-1} |\sin 2^{i-n}\pi| < \infty$$

and by Theorem B $\frac{1}{2^n} \in S$, consequently $\mathbb{D} \subset S$.

Since $1 \in S$ and S is an additive group, it is sufficient to show that if $\lambda \notin \mathbb{D}$, $\lambda \in (0, 1)$, then $\lambda \notin S$. A real number λ in $(0, 1)$ can be represented in the form

$$\lambda = \sum_{n=1}^{\infty} \frac{e_n}{2^n}, \qquad \text{where } e_n \in \{0, 1\}.$$

Then the condition $\lambda \notin \mathbb{D}$ is equivalent to the restriction that in the sequence e_1, e_2, e_3, ... there are infinitely many 0's and 1's. We claim that

$$\sum_{i=0}^{\infty} \sin^2 2^i \lambda\pi = \infty. \tag{19}$$

We prove this in indirect way. If this sum is convergent, then $\lim_{i\to\infty} \sin^2 2^i \lambda\pi = 0$ and there exists index $k \geq 1$ such that $\sin^2 2^i \lambda\pi < \frac{1}{4}$ or $|\sin 2^i \lambda\pi| < \frac{1}{2}$ for $i = k, k+1, \ldots$. Since

$$\sin 2^i \lambda\pi = \sin\left(\sum_{n=1}^{\infty} \frac{e_n}{2^n} 2^i \pi\right) = \pm \sin\left(\sum_{n=i+1}^{\infty} \frac{e_n}{2^{n-i}}\right)\pi. \tag{20}$$

Taking into account the bound $|\sin 2^i \lambda\pi| < \frac{1}{2} = \sin \frac{\pi}{6}$ for $i \geq k$, we have two possibilites: (1): $e_{k+1} = 0$, (2): $e_{k+1} = 1$.

(1) We claim that $e_{k+1} = 0$ implies $e_{k+2} = 0$. Suppose the contrary, i.e. $e_{k+2} = 1$, then $\frac{1}{4} \leq \sum_{n=k+1}^{\infty} \frac{e_n}{2^{n-k}} < \frac{1}{2}$ and by (20) $\sin \frac{\pi}{4} \leq |\sin 2^k \lambda\pi| < \sin \frac{\pi}{2}$ which contradicts the restriction $|\sin 2^i \lambda\pi| < \frac{1}{2}$ for $i = k, k+1, \ldots$. Repeating this argumentation, we find that $e_i = 0$ for $i = k+1, k+2, \ldots$, hence $\lambda \in \mathbb{D}$, which was excluded.

(2) Similarly, we claim that $e_{k+1} = 1$ implies $e_{k+2} = 1$. Again, we suppose the contrary, i.e. let $e_{k+2} = 0$. Then $\frac{1}{2} \leq \sum_{n=k+1}^{\infty} \frac{e_n}{2^{n-k}} < \frac{1}{2} + \sum_{n=k+3}^{\infty} \frac{1}{2^{n-k}} = \frac{5}{4}$ and by (20) we find $|\sin 2^k \lambda\pi| > \sin \frac{5\pi}{4} > \frac{1}{2}$ contradicting our assumption on k. Consequently, we must have $e_i = 1$ for all $i \geq k+1$, which again contradicts the assumption $\lambda \notin \mathbb{D}$.

Thus we have proved that the sum in (19) is indeed, divergent. Then Theorem A implies the asymptotic stability of (3), hence $\lambda \notin \mathbb{D}$ implies $\lambda \notin S$, which completes the proof of the relation $S = \mathbb{D}$. □

Proof of Example 4. Let $n \in \mathbb{N}$, $n \neq 0$. Let $\lambda = \frac{1}{n}$. Since

$$\sum_{i=0}^{\infty} \left|\sin \frac{i!\,\pi}{n}\right| = \sum_{i=0}^{n-1} \left|\sin \frac{i!\,\pi}{n}\right| < \infty$$

by Theorem B we conclude $\frac{1}{n} \in S$, hence $\mathbb{Q} \subset S$ because $\mathbb{Q}$ is the smallest additive group which contains all the reciprocals $\frac{1}{n}$, $n = 1, 2, \dots$.

We are going to show that $\frac{1}{2}e \notin S$. Consider the sum $\sum_{i=0}^{\infty} \sin^2(i!\, e\frac{\pi}{2})$! We have for $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{i!} + \frac{1}{(i+1)!} + \frac{1}{(i+2)!} + \dots$

$$i!\, e = i!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(i-2)!}\right) + i + 1 + \frac{1}{i+1} + \frac{1}{(i+1)(i+2)} + \cdots =$$
$$= 2k_i + i + 1 + \frac{1}{i+\theta_i}, \quad 0 < \theta_i < 1,\ k_i \in \mathbb{N},\ i \geq 2,$$

therefore

$$\sum_{i=0}^{\infty} \sin^2\left(i!\, e\frac{\pi}{2}\right) = \sum_{i=0}^{\infty} \sin^2\left(\frac{i+1}{2}\pi + \frac{\pi}{2(i+\theta_i)}\right) \geq$$
$$\geq \sum_{i=0}^{\infty} \sin^2\left(\frac{2i+1}{2}\pi + \frac{\pi}{2(2i+\theta_{2i})}\right) = \sum_{i=0}^{\infty} \cos^2 \frac{\pi}{2(2i+\theta_{2i})} = \infty,$$

hence by Theorem A $\frac{e}{2} \notin S$, and $S \neq \mathbb{R}$, i.e. S is a proper subset of $\mathbb{R}$. However, it is still an open problem whether the relation $\mathbb{Q} = S$ holds. □

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Singular Eigenvalue Problems for Second Order Linear Ordinary Differential Equations

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Abstract. We consider linear differential equations of the form

$$(p(t)x')' + \lambda q(t)x = 0 \quad (p(t) > 0,\ q(t) > 0) \tag{A}$$

on an infinite interval $[a, \infty)$ and study the problem of finding those values of λ for which (A) has principal solutions $x_0(t;\lambda)$ vanishing at $t = a$. This problem may well be called a singular eigenvalue problem, since requiring $x_0(t;\lambda)$ to be a principal solution can be considered as a boundary condition at $t = \infty$. Similarly to the regular eigenvalue problems for (A) on compact intervals, we can prove a theorem asserting that there exists a sequence $\{\lambda_n\}$ of eigenvalues such that $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$, and the eigenfunction $x_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ has exactly n zeros in (a, ∞), $n = 0, 1, 2, \ldots$. We also show that a similar situation holds for nonprincipal solutions of (A) under stronger assumptions on $p(t)$ and $q(t)$.

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1 Introduction

We consider the second order linear differential equation

$$(p(t)x')' + \lambda q(t)x = 0, \qquad t \geq a, \tag{A}$$

where $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, $a \geq 0$, and λ is a real parameter. We assume that (A) is nonoscillatory at $t = \infty$ for all $\lambda > 0$ (and hence for all $\lambda \in \mathbb{R}$). It is known [1, Theorem 6.4, p. 355] that there exists a solution $x_0(t; \lambda)$ of (A) which is uniquely determined up to a constant factor by the condition

$$\int^{\infty} \frac{dt}{p(t)(x_0(t; \lambda))^2} = \infty, \tag{1}$$

and that any solution $x_1(t; \lambda)$ of (A) linearly independent of $x_0(t; \lambda)$ has the property that

$$\int^{\infty} \frac{dt}{p(t)(x_1(t; \lambda))^2} < \infty. \tag{2}$$

A solution $x_0(t; \lambda)$ satisfying (1) is called a *principal solution* of (A) (at $t = \infty$), and a solution $x_1(t; \lambda)$ satisfying (2) is called a *nonprincipal solution* of (A) (at $t = \infty$).

We are concerned with the problem of finding principal solutions $x_0(t; \lambda)$ of (A) which satisfy the boundary condition

$$x_0(a; \lambda) = 0. \tag{3}$$

This problem falls within the category of general eigenvalue problems formulated by Hartman [2]. A solution $x_0(t; \lambda)$ of this problem will be said to be a *principal eigenfunction* and the corresponding value of λ a *principal eigenvalue*. Our task is to establish the existence of principal eigenvalues and count the number of zeros of the corresponding principal eigenfunctions.

We begin by introducing the notation needed in stating the main results. With regard to the function $p(t)$ the following two cases are possible:
either

$$\int_a^{\infty} \frac{dt}{p(t)} = \infty \tag{4}$$

or

$$\int_a^{\infty} \frac{dt}{p(t)} < \infty. \tag{5}$$

We define the functions $P(t)$ and $\pi(t)$ as follows:

$$P(t) = \int_a^t \frac{ds}{p(s)}, \qquad t \geq a, \text{ in case (4) holds;} \tag{6}$$

$$\pi(t) = \int_t^{\infty} \frac{ds}{p(s)}, \qquad t \geq a, \text{ in case (5) holds.} \tag{7}$$

It is clear that $P(t) \to \infty$ and $\pi(t) \to 0$ as $t \to \infty$. Our fundamental hypotheses on (A) are:

$$\int_a^\infty P(t)q(t)dt < \infty \qquad \text{in case (4) holds;} \tag{8}$$

$$\int_a^\infty \pi(t)q(t)dt < \infty \qquad \text{in case (5) holds.} \tag{9}$$

It is well known that (8) [resp. (9)] implies the existence of solutions $x_0(t;\lambda)$ of (A) satisfying the following boundary condition (10) [resp. (11)] at $t = \infty$:

$$\lim_{t\to\infty} x_0(t;\lambda) = 1, \qquad \lim_{t\to\infty} P(t)p(t)x_0'(t;\lambda) = 0 \qquad \text{in case (4) holds;} \tag{10}$$

$$\lim_{t\to\infty} \frac{x_0(t;\lambda)}{\pi(t)} = 1, \qquad \lim_{t\to\infty} p(t)x_0'(t;\lambda) = -1 \qquad \text{in case (5) holds.} \tag{11}$$

Since this solution $x_0(t;\lambda)$ satisfies (1), we easily find that, under the condition (8) or (9), the requirement that $x_0(t;\lambda)$ be a principal solution of (A) is equivalent to the requirement that $x_0(t;\lambda)$ be a solution of (A) satisfying (10) or (11).

One of the main results of this paper now follows.

Theorem I. *Suppose that* (8) *or* (9) *holds. Then, there exists a sequence of principal eigenvalues* $\{\lambda_n\}$*:*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty$$

such that the corresponding principal eigenfunction $x_0(t;\lambda_n)$ *satisfying* (10) *or* (11) *has exactly* n *zeros in* (a,∞), $n = 0, 1, 2, \ldots$.

The proof of this theorem will be given in Section 1. It will be shown that Theorem I follows from the corresponding result for the particular equation $x'' + \lambda q(t)x = 0$.

Let us now turn to the consideration of nonprincipal solutions of the nonoscillatory equation (A). A nonprincipal solution of (A) is by no means unique. It may happen, however, that certain additional restrictions on the functions $p(t), q(t)$ and/or the asymptotic behavior of the solution determine a unique nonprincipal solution $x_1(t;\lambda)$ of (A) for each fixed λ. If this is the case one could speak of a *nonprincipal eigenvalue problem* for (A) which consists in finding its nonprincipal solutions $x_1(t;\lambda)$ satisfying the boundary condition (3); such a solution $x_1(t;\lambda)$ is termed a *nonprincipal eigenfunction* and the corresponding value of λ a *nonprincipal eigenvalue*. This kind of problem has not been studied in the literature.

For example, if we assume (8) or (9), then (A) has a nonprincipal solution, non-unique, $x_1(t;\lambda)$ such that

$$\lim_{t\to\infty} \frac{x_1(t;\lambda)}{P(t)} = 1, \qquad \lim_{t\to\infty} p(t)x_1'(t;\lambda) = 1 \qquad \text{in case (4) holds;} \tag{12}$$

$$\lim_{t\to\infty} x_1(t;\lambda) = 1, \qquad \lim_{t\to\infty} \pi(t)p(t)x_1'(t;\lambda) = 0 \qquad \text{in case (5) holds.} \tag{13}$$

However, if we require that $p(t)$ and $q(t)$ satisfy the more stringent condition

$$\int_a^\infty (P(t))^2 q(t)dt < \infty \qquad \text{in case (4) holds} \tag{14}$$

or

$$\int_a^\infty q(t)dt < \infty \qquad \text{in case (5) holds,} \tag{15}$$

then there exists, for each λ, a unique nonprincipal solution $x_1(t;\lambda)$ of (A) such that

$$\lim_{t\to\infty} [x_1(t;\lambda) - P(t)] = 0, \qquad \lim_{t\to\infty} P(t)[p(t)x_1'(t;\lambda) - 1] = 0 \qquad \text{in case (4) holds} \tag{16}$$

or

$$\lim_{t\to\infty} \frac{x_1(t;\lambda) - 1}{\pi(t)} = 0, \qquad \lim_{t\to\infty} p(t)x_1'(t;\lambda) = 0 \qquad \text{in case (5) holds.} \tag{17}$$

From these solutions $x_1(t;\lambda)$ one can extract a sequence of nonprincipal eigenfunctions having the prescribed numbers of zeros as is shown by the following theorem which is another main result of this paper.

Theorem II. *Suppose that* (14) *or* (15) *holds. Then, there exists a sequence of nonprincipal eigenvalues* $\{\lambda_n\}$*:*

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty$$

such that the corresponding nonprincipal eigenfunction $x_1(t;\lambda_n)$ *satisfying* (16) *or* (17) *has exactly* n *zeros in* (a,∞), $n = 0, 1, 2, \ldots$.

We will prove this theorem in Section 2 by reducing the problem under study to the corresponding problem for the simpler equation $x'' + \lambda q(t)x = 0$. We remark that since (14) and (15) are stronger than (8) and (9), respectively, the hypotheses of Theorem II guarantee the existence of both principal and nonprincipal eigenvalues for the equation (A). Section 3 will be devoted to a discussion of applicability of Theorems I and II to the qualitative study of a certain class of linear elliptic partial differential equations in exterior domains.

2 Principal eigenvalue problem

A) *A reduced problem.* Consider the particular equation

$$x'' + \lambda q(t)x = 0, \qquad t \ge a, \tag{B}$$

where $q(t)$ is a positive continuous function on $[a,\infty)$ and λ is a real parameter. Theorem I specialized to (B) reads as follows.

Theorem 1. *Suppose that*

$$\int_a^\infty tq(t)dt < \infty. \tag{18}$$

Then, there exists a sequence of positive constants $\{\lambda_n\}$*:*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \tag{19}$$

such that, for each $\lambda = \lambda_n$*, the equation* (B) *possesses a solution* $x_0(t;\lambda)$ *satisfying the boundary conditions*

$$x_0(a;\lambda_n) = 0, \qquad \lim_{t\to\infty} x_0(t;\lambda_n) = 1, \qquad \lim_{t\to\infty} tx_0'(t;\lambda_n) = 0 \tag{20}$$

and having exactly n *zeros in* (a,∞)*,* $n = 0, 1, 2, \ldots$*.*

We will show that Theorem I follows from its specialized version: Theorem 1.

First consider the case where $p(t)$ and $q(t)$ satisfy (4) and (8). In this case the change of variables $(t,x) \to (\tau,\xi)$ defined by

$$\tau = P(t), \qquad \xi(\tau;\lambda) = x(t;\lambda) \tag{21}$$

transforms (A) into

$$\ddot{\xi} + \lambda Q(\tau)\xi = 0, \qquad \tau \geq 0, \tag{22}$$

where a dot denotes differentiation with respect to τ and $Q(\tau) = p(t)q(t)$. Since (22) is of the type (B) and since

$$\int_0^\infty \tau Q(\tau)d\tau = \int_a^\infty P(t)q(t)dt < \infty$$

by (8), it follows from Theorem 1 that there exist a sequence of positive constants $\{\lambda_n\}_{n=0}^\infty$ satisfying (19) and the corresponding sequence of solutions $\{\xi_0(\tau;\lambda_n)\}_{n=0}^\infty$ of (22) such that

$$\xi_0(0;\lambda_n) = 0, \qquad \lim_{\tau\to\infty} \xi_0(\tau;\lambda_n) = 1, \qquad \lim_{\tau\to\infty} \tau\dot{\xi}_0(\tau;\lambda_n) = 0. \tag{23}$$

Define $x_0(t;\lambda_n) = \xi_0(P(t);\lambda_n)$. Then, $x_0(t;\lambda_n)$ is clearly a solution of (A) on $[a,\infty)$, and in view of (21), (23) implies that it satisfies the boundary conditions (3) and (10).

Next suppose that $p(t)$ and $q(t)$ satisfy (5) and (9). Perform the change of variables $(t,x) \to (\tau,\eta)$ given by

$$\tau = \frac{1}{\pi(t)}, \qquad \eta(\tau;\lambda) = \tau x(t;\lambda). \tag{24}$$

The equation (A) then transforms into

$$\ddot{\eta} + \lambda R(\tau)\eta = 0, \qquad \tau \geq \frac{1}{\pi(a)}, \tag{25}$$

where a dot denotes differentiation with respect to τ and $R(\tau) = p(t)q(t)/\tau^4$. In view of (9) we have

$$\int_{1/\pi(a)}^{\infty} \tau R(\tau)d\tau = \int_a^{\infty} \pi(t)q(t)dt < \infty,$$

and so applying Theorem 1 to (25) we see that there exists a sequence of positive constants $\{\lambda_n\}_{n=0}^{\infty}$ satisfying (19) and the corresponding solutions $\{\eta_0(\tau;\lambda_n)\}_{n=0}^{\infty}$ of (25) such that

$$\eta_0(\frac{1}{\pi(a)};\lambda_n) = 0, \qquad \lim_{\tau\to\infty} \eta_0(\tau;\lambda_n) = 1, \qquad \lim_{\tau\to\infty} \tau\dot{\eta}_0(\tau;\lambda_n) = 0. \tag{26}$$

Define $x_0(t;\lambda_n) = \pi(t)\eta_0(1/\pi(t);\lambda_n)$. As it is easily seen, $x_0(t;\lambda_n)$ is a solution of (A) on $[a,\infty)$ and satisfies the boundary conditions (3) and (11). Thus the proof of Theorem I is reduced to that of Theorem 1.

B) *Proof of Theorem* 1. The condition (18) ensures the existence of a unique principal solution $x_0(t;\lambda)$ of (B) such that

$$\lim_{t\to\infty} x_0(t;\lambda) = 1, \qquad \lim_{t\to\infty} tx_0'(t;\lambda) = 0. \tag{27}$$

This $x_0(t;\lambda)$ is characterized as the solution to the integral equation

$$x_0(t;\lambda) = 1 - \lambda \int_t^{\infty} (s-t)q(s)x_0(s;\lambda)ds, \qquad t \geq a, \tag{28}$$

and is subject to the estimate

$$|x_0(t;\lambda)| \leq \exp\left[|\lambda| \int_a^{\infty} sq(s)ds\right] \equiv K(\lambda), \qquad t \geq a. \tag{29}$$

For this fact, see e.g. Hille [3, Theorem 9.1.1 and its proof].

We need only to examine positive values of λ, since the boundary condition $x_0(a;\lambda) = 0$ is not satisfied for $\lambda \leq 0$.

A simple consequence of (28) and (29) is that $x_0(t;\lambda)$ is positive on $[a,\infty)$ if $\lambda > 0$ is so small that

$$\lambda K(\lambda) \int_a^{\infty} sq(s)ds < 1$$

that is, $x_0(t;\lambda)$ has no zero in $[a,\infty)$ for such small values of λ.

It can be shown that $x_0(t;\lambda)$ has a zero in (a,∞) if $\lambda > 0$ is sufficiently large and that the number of zeros of $x_0(t;\lambda)$ in $[a,\infty)$, denoted by $N[x_0(\lambda)]$, tends to

∞ as $\lambda \to \infty$. In fact, let $k \in \mathbb{N}$ be given. Put $q_* = \min\{q(t) : a \le t \le a + \pi\}$ and define $\lambda_k = k^2/q_*$. Then, $\lambda > \lambda_k$ implies $\lambda q(t) > k^2$ on $[a, a + \pi]$. We now compare (B) with the harmonic oscillator $y'' + k^2 y = 0$. Noting that a solution $y(t) = \sin k(t - a)$ of the latter equation has $k + 1$ zeros in $[a, a + \pi]$, we conclude from Sturm's comparison theorem that every solution of (B), and hence $x_0(t; \lambda)$, has at least k zeros in $(a, a + \pi)$ provided $\lambda > \lambda_k$. Since k is arbitrary, it follows that $N[x_0(\lambda)] \to \infty$ as $\lambda \to \infty$.

We now make use of the Prüfer transformation:

$$x_0(t; \lambda) = \rho(t; \lambda) \sin \varphi(t; \lambda), \qquad x_0'(t; \lambda) = \rho(t; \lambda) \cos \varphi(t; \lambda), \tag{30}$$

or equivalently,

$$\begin{aligned} \rho(t; \lambda) &= \left[(x_0(t; \lambda))^2 + (x_0'(t; \lambda))^2\right]^{\frac{1}{2}} > 0, \\ \varphi(t; \lambda) &= \arctan \frac{x_0(t; \lambda)}{x_0'(t; \lambda)}. \end{aligned} \tag{31}$$

As it is well-known, $\rho(t; \lambda)$ and $\varphi(t; \lambda)$ are continuously differentiable functions of t on $[a, \infty)$, and $\varphi(t; \lambda)$ satisfies the differential equation

$$\varphi'(t; \lambda) = \cos^2 \varphi(t; \lambda) + \lambda q(t) \sin^2 \varphi(t; \lambda), \qquad t \ge a. \tag{32}$$

Note that the boundary condition (27) imposed on $x_0(t; \lambda)$ at $t = \infty$ corresponds via (31) to the "terminal" condition for $\varphi(t; \lambda)$:

$$\lim_{t \to \infty} \varphi(t; \lambda) \equiv \frac{\pi}{2} \pmod{\pi}.$$

There is no loss of generality in requiring at the outset that

$$\lim_{t \to \infty} \varphi(t; \lambda) = \frac{\pi}{2}. \tag{33}$$

We will prove that, for each fixed $t \ge a$, $\varphi(t; \lambda)$ is a continuous decreasing function of $\lambda > 0$. From the equation

$$\begin{aligned} x_0(t; \lambda) - x_0(t; \mu) = -\lambda \int_t^\infty (s - t) q(s) [x_0(s; \lambda) - x_0(s; \mu)] ds \\ -(\lambda - \mu) \int_t^\infty (s - t) q(s) x_0(s; \mu) ds \,, \end{aligned}$$

which follows from (28), we see with the use of (29) that $u(t) = |x_0(t; \lambda) - x_0(t; \mu)|$ satisfies

$$u(t) \le |\lambda - \mu| K(\mu) \int_a^\infty s q(s) ds + \lambda \int_t^\infty s q(s) u(s) ds, \qquad t \ge a.$$

Using the Gronwall inequality and an easy calculation one may conclude that

$$u(t) \le |\lambda - \mu| K(\lambda) K(\mu) \int_a^\infty s q(s) ds, \qquad t \ge a,$$

which shows that $x_0(t;\lambda)$ is continuous with respect to λ. The continuity of $x_0'(t;\lambda)$ with respect to λ follows from the relation

$$x_0'(t;\lambda) = \lambda \int_t^\infty q(s)x_0(s;\lambda)ds, \qquad t \geq a.$$

Then (31) implies that $\varphi(t;\lambda)$ is continuous with respect to λ.

The decreasing property of $\varphi(t;\lambda)$ with respect to λ is verified by contradiction. Suppose that

$$\varphi(b;\lambda) \geq \varphi(b;\mu) \tag{34}$$

for some $b \in [a,\infty)$ and λ and μ with $\lambda > \mu > 0$. Since the right-hand side of (32) written as $\cos^2\varphi + \lambda q(t)\sin^2\varphi$ is increasing with respect to λ, the initial inequality (34) implies that

$$\varphi(t;\lambda) > \varphi(t;\mu) \qquad \text{for } t > b,$$

or

$$\arctan\frac{x_0(t;\lambda)}{x_0'(t;\lambda)} > \arctan\frac{x_0(t;\mu)}{x_0'(t;\mu)}, \qquad t > b.$$

Consequently, there exists $c > b$ such that

$$\frac{x_0(t;\lambda)}{x_0'(t;\lambda)} > \frac{x_0(t;\mu)}{x_0'(t;\mu)}, \qquad t \geq c. \tag{35}$$

Put

$$X(t;\lambda,\mu) = x_0(t;\lambda)x_0'(t;\mu) - x_0'(t;\lambda)x_0(t;\mu).$$

Then $X(t;\lambda,\mu) > 0, t \geq c$, by (35), and since

$$X'(t;\lambda,\mu) = (\lambda-\mu)q(t)x_0(t;\lambda)x_0(t;\mu) > 0, \qquad t \geq c,$$

provided c is taken sufficiently large, $X(t;\lambda,\mu)$ tends to a positive constant as $t \to \infty$. But this is impossible, since the boundary condition (27) implies that $X(t;\lambda,\mu) \to 0$ as $t \to \infty$. Therefore, $\varphi(t;\lambda)$ must be a decreasing function of $\lambda > 0$ for each fixed $t \geq a$.

Finally consider the values $\varphi(a;\lambda)$ for $\lambda > 0$. Since $\varphi(t;\lambda)$ is an increasing function of t for fixed $\lambda > 0$, we have $\varphi(a;\lambda) < \pi/2$ (cf. (33)). If $\lambda > 0$ is sufficiently small, then $\varphi(a;\lambda) > 0$, because $x_0(t;\lambda)$ has no zero in $[a,\infty)$ as proven above. On the other hand, the fact that $N[x_0(\lambda)] \to \infty$ as $\lambda \to \infty$ shows that $\varphi(a;\lambda) \to -\infty$ as $\lambda \to \infty$. Since $\varphi(a;\lambda)$ is decreasing with respect to $\lambda > 0$, for every $n \in \mathbb{N} \cup \{0\}$, there exists $\lambda_n > 0$ such that

$$\varphi(a;\lambda_n) = -n\pi, \tag{36}$$

which means that the principal solution $x_0(t;\lambda_n)$ of (B) satisfies the boundary condition $x_0(a;\lambda_n) = 0$ and has exactly n zeros in (a,∞). It is almost trivial to see that (19) holds for the sequence of principal eigenvalues $\{\lambda_n\}$. This completes the proof of Theorem 1.

Remark. It is well-known [6] that the equation (A) is nonoscillatory for all $\lambda > 0$ if and only if

$$\lim_{t\to\infty} P(t) \int_t^\infty q(s)ds = 0 \qquad \text{in case (4) holds;} \tag{37}$$

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty (\pi(s))^2 q(s)ds = 0 \qquad \text{in case (5) holds.} \tag{38}$$

The condition (8) or (9) required in Theorem I is stronger than (37) or (38), respectively. We conjecture that an analogue of Theorem I will hold under the most general condition (37) or (38).

3 Nonprincipal eigenvalue problem

Let us turn to the nonprincipal eigenvalue problem for (A) mentioned in the Introduction. As in the preceding section it can be shown that our main result, Theorem II, follows from the corresponding result for the particular equation (B).

Theorem 2. *Suppose that*

$$\int_a^\infty t^2 q(t)dt < \infty. \tag{39}$$

Then, there exists a sequence of numbers $\{\lambda_n\}$*:*

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \tag{40}$$

such that, for each $\lambda = \lambda_n$*, the equation* (B) *possesses a solution* $x_1(t;\lambda_n)$ *satisfying the boundary conditions*

$$x_1(a;\lambda_n) = 0, \qquad \lim_{t\to\infty} [x_1(t;\lambda_n) - t] = 0, \qquad \lim_{t\to\infty} t[x_1'(t;\lambda_n) - 1] = 0 \tag{41}$$

and having exactly n *zeros in* (a,∞)*,* $n = 0, 1, 2, \ldots$*.*

We will give a proof of this theorem, leaving the reduction of Theorem II to Theorem 2 to the reader.

Because of (39) there exists, for each λ, a unique nonprincipal solution $x_1(t;\lambda)$ of (B) such that

$$\lim_{t\to\infty} [x_1(t;\lambda) - t] = 0, \qquad \lim_{t\to\infty} t[x_1'(t;\lambda) - 1] = 0. \tag{42}$$

This solution is characterized as the solution to the integral equation

$$x_1(t;\lambda) = t - \lambda \int_t^\infty (s-t)q(s)x_1(s;\lambda)ds, \qquad t \ge a, \tag{43}$$

and this enables us to obtain the estimate

$$|x_1(t;\lambda) - t| \le |\lambda| K(\lambda) \int_a^\infty s^2 q(s) ds \equiv L(\lambda), \qquad t \ge a, \tag{44}$$

where $K(\lambda)$ is the constant defined in (29). For the details the reader is referred to Hille [3, Theorem 9.1.1].

Since

$$x_1(t;\lambda) - x_1(t;\mu) = -\lambda \int_t^\infty (s-t) q(s)[x_1(s;\lambda) - x_1(s;\mu)] ds$$
$$-(\lambda-\mu) \int_t^\infty (s-t) q(s) x_1(s;\mu) ds, \qquad t \ge a,$$

using (43), we see that the function $u(t) = |x_1(t;\lambda) - x_1(t;\mu)|$ satisfies

$$u(t) \le |\lambda - \mu| \int_a^\infty s q(s)[s + L(\mu)] ds + |\lambda| \int_t^\infty s q(s) u(s) ds, \qquad t \ge a,$$

and hence we have

$$u(t) \le |\lambda - \mu| M(\mu) \exp\left[|\lambda| \int_a^\infty s q(s) ds \right], \qquad t \ge a,$$

where

$$M(\mu) = \int_a^\infty s q(s)[s + L(\mu)] ds.$$

This shows that $x_1(t;\lambda)$ is a continuous function of λ for each fixed $t \ge a$. The continuity of $x_1'(t;\lambda)$ with respect to λ follows from the equation

$$x_1'(t;\lambda) = 1 + \lambda \int_t^\infty q(s) x_1(s;\lambda) ds, \qquad t \ge a.$$

Nonnegative values of λ [or negative values of λ] may be excluded from our consideration in the case $a > 0$ [or in the case $a = 0$], since it follows from (A) and (43) that $x_1(t;\lambda)$ is unable to satisfy the boundary condition $x_1(a;\lambda) = 0$ for such values of λ.

Now we perform the Prüfer transformation:

$$x_1(t;\lambda) = \rho(t;\lambda) \sin \varphi(t;\lambda), \qquad x_1'(t;\lambda) = \rho(t;\lambda) \cos \varphi(t;\lambda), \tag{45}$$

or equivalently

$$\begin{gathered} \rho(t;\lambda) = \left[(x_1(t;\lambda))^2 + (x_1'(t;\lambda))^2 \right]^{\frac{1}{2}} > 0, \\ \varphi(t;\lambda) = \arctan \frac{x_1(t;\lambda)}{x_1'(t;\lambda)}. \end{gathered} \tag{46}$$

The function $\varphi(t;\lambda)$ satisfies (32), and so it is an increasing function of t for $\lambda > 0$. Also, $\varphi(t;\lambda)$ is continuous with respect to λ, since so are $x_1(t;\lambda)$ and $x_1'(t;\lambda)$ as stated above.

From (45) and (42) we have $\rho(t;\lambda)/t \to 1$, $\sin\varphi(t;\lambda) \to 1$ and $\cos\varphi(t;\lambda) \to 0$ as $t \to \infty$, which implies that

$$\lim_{t\to\infty} \varphi(t;\lambda) \equiv \frac{\pi}{2} \pmod{\pi}.$$

To fix the idea we suppose that

$$\lim_{t\to\infty} \varphi(t;\lambda) = \frac{\pi}{2}. \tag{47}$$

Proceeding exactly as in the proof of Theorem 1 we can show that the number of zeros of $x_1(t;\lambda)$ in $[a,\infty)$ can be made as large as possible if $\lambda > 0$ is chosen sufficiently large. It follows that $\varphi(a;\lambda) \to -\infty$ as $\lambda \to \infty$.

To examine the values $\varphi(a;\lambda)$ for small $\lambda > 0$, we distinguish the two cases: either $a = 0$ or $a > 0$. Consider the case where $a = 0$. Let $\lambda = 0$. Then, $x_1(t;0) = t$ by inspection. This solution has no zero in $(0,\infty)$. It should be noted that $x_1(t;0)$ itself is a nonprincipal eigenfunction for (B) corresponding to a nonprincipal eigenvalue $\lambda = 0$. Next consider the case where $a > 0$ and claim that $x_1(t;\lambda) > 0$ on $[a,\infty)$ for all sufficiently small $\lambda > 0$. In fact, let $\lambda > 0$ be small enough so that

$$\lambda \int_a^\infty sq(s)[s + L(\lambda)]ds < a,$$

where $L(\lambda)$ is defined in (44). Then, from (43) and (44) we obtain

$$\begin{aligned} x_1(t;\lambda) &\geq a - \lambda \int_a^\infty sq(s)|x_1(s;\lambda)|ds \\ &\geq a - \lambda \int_a^\infty sq(s)[s + L(\lambda)]ds > 0, \qquad t \geq a. \end{aligned}$$

It remains to establish the decreasing property of $\varphi(t;\lambda)$ with respect to $\lambda > 0$. Suppose to the contrary that $\varphi(b;\lambda) \geq \varphi(b;\mu)$ for some $b \in [a,\infty)$ and λ and μ with $\lambda > \mu > 0$. Applying the argument which derived (35) from (34), we see that the function

$$X(t;\lambda,\mu) = x_1(t;\lambda)x_1'(t;\mu) - x_1'(t;\lambda)x_1(t;\mu)$$

and its derivative $X'(t;\lambda,\mu)$ are positive for all sufficiently large t. It follows that $X(t;\lambda,\mu)$ tends to a positive constant or ∞ as $t \to \infty$. On the other hand, using the relation

$$\begin{aligned} X(t;\lambda,\mu) = {}& [x_1(t;\lambda) - t]x_1'(t;\mu) - [x_1(t;\mu) - t]x_1'(t;\lambda) \\ & - t[x_1'(t;\lambda) - 1] + t[x_1'(t;\mu) - 1] \end{aligned}$$

and (42), we find that $X(t;\lambda,\mu) \to 0$ as $t \to \infty$. This contradiction proves that $\varphi(t;\lambda)$ is a decreasing function of $\lambda > 0$ for each fixed $t \geq a$.

From the above observations we conclude that, for every $n \in \mathbb{N} \cup \{0\}$, there exists λ_n such that $\varphi(a; \lambda_n) = -n\pi$, so that $x_1(t; \lambda_n)$ is a desired nonprincipal eigenfunction for (B). It is clear that the sequence of eigenvalues $\{\lambda_n\}$ satisfies (40). Note that $\lambda_0 = 0$ if $a = 0$ and $\lambda_0 > 0$ if $a > 0$. This completes the proof of Theorem 2.

Example. As an example of equations to which Theorems 1 and 2 apply we give Halm's equation ([3, p. 357])

$$x'' + \lambda(1 + t^2)^{-2} x = 0, \qquad t \geq 0.$$

4 Application to elliptic equations

Our purpose here is to show that Theorems I and II can be applied to a qualitative study of elliptic partial differential equations of the type

$$\Delta u + \lambda c(|x|)u = 0, \qquad x \in E_a, \tag{C}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $N \geq 2$, $|x| = \left(\Sigma_{i=1}^N x_i^2\right)^{1/2}$, Δ is the N-dimensional Laplace operator, $E_a = \{x \in \mathbb{R}^N : |x| \geq a\}$, $a > 0$, $c(t)$ is a positive continuous function on $[a, \infty)$, and λ is a real parameter. We are interested in the existence of radially symmetric solutions $u(x)$ which satisfy the Dirichlet condition

$$u(x) = 0, \qquad x \in \partial E_a = \{x \in \mathbb{R}^N : |x| = a\}. \tag{48}$$

Radial symmetry of a solution means that it depends only on $|x|$, that is, it is of the form $u(x) = y(|x|)$.

It is easy to see that a radially symmetric function $u(x) = y(|x|)$ is a solution of the exterior Dirichlet problem (C)–(48) if and only if the function $y(t)$ is a solution of the ordinary differential equation

$$(t^{N-1} y')' + \lambda t^{N-1} c(t) y = 0, \qquad t \geq a \tag{49}$$

satisfying

$$y(a) = 0. \tag{50}$$

The equation (49) is a special case of (A) in which

$$p(t) = t^{N-1} \qquad \text{and} \qquad q(t) = t^{N-1} c(t). \tag{51}$$

We note that:

(i) (4) holds if and only if $N = 2$, in which case the function $P(t)$ defined by (6) is

$$P(t) = \log \frac{t}{a}\,, \qquad t \geq a; \tag{52}$$

(ii) (5) holds if and only if $N \geq 3$, in which case the function $\pi(t)$ defined by (7) is

$$\pi(t) = \frac{t^{2-N}}{N-2}, \qquad t \geq a. \tag{53}$$

Therefore, the conditions (8), (9) reduce to

$$\int^{\infty} t(\log t)c(t)dt < \infty, \tag{54}$$

$$\int^{\infty} tc(t)dt < \infty, \tag{55}$$

and the conditions (14), (15) to

$$\int^{\infty} t(\log t)^2 c(t)dt < \infty, \tag{56}$$

$$\int^{\infty} t^{N-1}c(t)dt < \infty. \tag{57}$$

The next result follows from Theorem I applied to (49)–(50).

Theorem 3. (i) *Let $N = 2$ and suppose that (54) holds. Then, there exists a sequence of positive numbers $\{\lambda_n\}$:*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \tag{58}$$

such that, for each $\lambda = \lambda_n$, the exterior Dirichlet problem (C)–(48) possesses a radially symmetric solution $u(x;\lambda_n)$ satisfying

$$\lim_{|x|\to\infty} u(x;\lambda_n) = 1 \tag{59}$$

and having exactly n spherical nodes in the interior of E_a, $n = 0, 1, 2, \ldots$.

(ii) *Let $N \geq 3$ and suppose that (55) holds. Then, there exists a sequence of positive numbers $\{\lambda_n\}$ with the property (58) such that, for each $\lambda = \lambda_n$, the problem (C)–(48) possesses a radially symmetric solution $u(x;\lambda_n)$ satisfying*

$$\lim_{|x|\to\infty} |x|^{N-2}u(x;\lambda_n) = 1 \tag{60}$$

and having exactly n spherical nodes in the interior of E_a, $n = 0, 1, 2, \ldots$.

Theorem II specialized to (49)–(50) yields another result for the exterior Dirichlet problem under consideration.

Theorem 4. (i) *Let $N = 2$ and suppose that (56) holds. Then, there exists a sequence of positive numbers $\{\lambda_n\}$ with the property (58) such that, for each $\lambda = \lambda_n$, the problem (C)–(48) possesses a radially symmetric solution $u(x;\lambda_n)$ satisfying*

$$\lim_{|x|\to\infty} \frac{u(x;\lambda_n)}{\log|x|} = 1 \tag{61}$$

and having exactly n *spherical nodes in the interior of* E_a, $n = 0, 1, 2, \ldots$.

(ii) *Let* $N \geq 3$ *and suppose that* (57) *holds. Then, there exists a sequence of positive numbers* $\{\lambda_n\}$ *with the property* (58) *such that, for each* $\lambda = \lambda_n$*, the problem* (C)–(48) *possesses a radially symmetric solution* $u(x; \lambda_n)$ *satisfying*

$$\lim_{|x| \to \infty} u(x; \lambda_n) = 1 \tag{62}$$

and having exactly n *spherical nodes in the interior of* E_a, $n = 0, 1, 2, \ldots$.

Remark. A related problem for (C) in the entire space $\mathbb{R}^N$ has been studied by Naito [5] and Kabeya [4].

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Bifurcation of Periodic and Chaotic Solutions in Discontinuous Systems

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Abstract. Chaos generated by the existence of Smale horseshoe is the well-known phenomenon in the theory of dynamical systems. The Poincaré-Andronov-Melnikov periodic and subharmonic bifurcations are also classical results in this theory. The purpose of this note is to extend those results to ordinary differential equations with multivalued perturbations. We present several examples based on our recent achievements in this direction. Singularly perturbed problems are studied as well. Applications are given to ordinary differential equations with both dry friction and relay hysteresis terms.

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1 Introduction

There are many concrete problems in mechanics with non-smooth nonlinearities. Such discontinuities arise in the context of modelling Coulomb friction [1,3,9,19]. One of the simplest examples for such problems is provided by the pendulum with dry friction given by

$$\ddot{x} + x + \mu \operatorname{sgn} \dot{x} = \psi(t)\,. \tag{1}$$

Here $\operatorname{sgn} r = r/|r|$ for $r \in \mathbb{R} \setminus \{0\}$. Equation (1) is studied for the periodic case in [7] and for the almost periodic case in [8]. Also a rather complete picture of the asymptotic behaviour of (1) is derived. The range of μ is found such that nontrivial almost periodic and periodic motions exist. The question of uniqueness of such motions is studied as well.

The numerical analysis is presented in the papers [26,27,28] for a mechanical model of a friction-oscillator with simultaneous self- and external excitation given by the equation of motion

$$\ddot{x} + x = F_R(v_r) + u_0 \cos \Omega t\,, \tag{2}$$

where u_0, Ω are positive constants, $v_r = v_0 - \dot{x}$ is a relative velocity and F_R is the friction force defined by

$$\begin{aligned} F_R(v_r) &= \mu(v_r) F_N \operatorname{sgn}(v_r) \quad \text{for the slip mode} \quad v_r \neq 0 \\ F_R(v_r) &= x(t) - u_0 \cos \Omega t \quad \text{for the stick mode} \quad v_r = 0\,. \end{aligned}$$

Here $\mu(v_r)$ is a friction coefficient and F_N is the normal force. Three different types of friction coefficients $\mu(v_r)$ are studied in [26,27,28] including the Coulomb one $\mu(v_r) = 1$, $v_r \neq 0$. The bifurcation behaviour and the routes to chaos of (2) are investigated for a wide range of parameters. The influence of these three types of friction coefficients is described and the admissibility of smoothing procedures is examined by comparing results gained for non-smooth and smoothed friction coefficients. These papers [26,27,28] present a nice introduction to the phenomenon of dry friction problem as well.

The boundedness of solutions of the equation

$$\ddot{x} + x + \mu \operatorname{sgn} x = \psi(t) \tag{3}$$

is studied in [23] as well as the existence of infinitely many periodic and quasiperiodic solutions of (3) is established for all $\mu > 0$ sufficiently large.

By using Lyapunov exponents, the qualitative analysis for (1) and for a similar friction-oscillator is given in [20] and [21], respectively. A numerical analysis of the same friction-oscillator is presented in [22].

Finally, let us note that equations of the type (1) also appear in electrical engineering (see [1, Chap. III]), related problems are studied in control systems (see [31]) as well, and dry friction problems were investigated already in [29], [30].

This note is based on recent results derived in the papers [10,11,12,13,14,15]. We focus on concrete examples rather than presenting theoretical results. In Section 2, we give examples with chaotic solutions. Section 3 deals with bifurcation of periodic solutions for a friction-oscillator. A problem with small relay hysteresis is studied in Section 4.

In this paper, the dry friction is modelled by the Coulomb law [9], [19] which includes a static coefficient of friction μ_s and a dynamic coefficient of friction μ_d. If $\mu_s = \mu_d = \mu$, then the friction law may be written as $\dot{x} \to \mu \operatorname{sgn} \dot{x}$. On the other hand, since usually $\mu_s > \mu_d$, the smooth approximation of $\operatorname{sgn} r$ given, for instance, by

$$\Phi(r) = \frac{1}{\pi}\big(7 \arctan 8sr - 5 \arctan 4sr\big), \quad s \gg 1$$

seems to be physically more relevant than the mathematically convenient approximation of the form

$$r \to \frac{2}{\pi} \arctan sr, \quad s \gg 1\,.$$

The function Φ has two symmetric spikes at $r = \pm\frac{\sqrt{6}}{8s}$ of the values

$$\pm\frac{1}{\pi}\Big(7 \arctan \sqrt{6} - 5 \arctan \frac{\sqrt{6}}{2}\Big) \doteq \pm 1{,}2261344\,.$$

Moreover, $\Phi(r)$ is close to 1 or -1 when $r > 0$ or $r < 0$, respectively, tends off 0. Summarizing, we can take for any $\eta \geq 0$, $\zeta \geq 1$, $0 < \kappa \leq 1$ the multivalued function $\operatorname{Sgn}_{\eta,\zeta,\kappa} r$ defined by

$$\operatorname{Sgn}_{\eta,\zeta,\kappa} r = \begin{cases} -1 & \text{for } r < -\eta\,, \\ [-\zeta, -\kappa] & \text{for } -\eta \leq r < 0\,, \\ [-\zeta, \zeta] & \text{for } r = 0\,, \\ [\kappa, \zeta] & \text{for } 0 < r \leq \eta\,, \\ 1 & \text{for } r > \eta\,. \end{cases}$$

The term $\operatorname{Sgn}_{\eta,\zeta,\kappa} \dot{x}$ can be viewed as an extension for modelling dry friction including static and dynamic frictions as well.

2 Chaos in Dry Friction Problems

Dry friction forces acting on a moving particle due to its contact to walls have in certain situations the form $\mu(x)(g_0(\dot{x}) + \operatorname{sgn} \dot{x})$, where x is displacement from the rest state, $\dot{x}$ is velocity, μ and g_0 are non-negative bounded continuous, and $\operatorname{sgn} r = r/|r|$ for $r \in \mathbb{R} \setminus \{0\}$, see [1,5,19]. If there is also damping , restoring and external forces, the following equation is studied

$$\ddot{x} + g(x) + \mu_1 \operatorname{sgn} \dot{x} + \mu_2 \dot{x} = \mu_3 \psi(t)\,, \tag{4}$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are small parameters, $g \in C^2(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, $g'(0) < 0$, $\psi \in C^1(\mathbb{R}, \mathbb{R})$ and ψ is periodic.

If a smooth small perturbation is included in (4), then by using a method developed in dynamical systems (see [18]), it would be possible to show the existence of chaos for such ordinary differential equations.

By introducing the multivalued mapping

$$\operatorname{Sgn} r = \begin{cases} \operatorname{sgn}\, r & \text{for } r \neq 0\,, \\ [-1, 1] & \text{for } r = 0\,, \end{cases}$$

(4) is rewritten as follows

$$\ddot{x} + g(x) + \mu_2 \dot{x} - \mu_3 \psi(t) \in -\mu_1 \operatorname{Sgn} \dot{x}\,. \tag{5}$$

By a solution of any first order differential inclusion we mean a function which is absolute continuous and satisfying differential inclusion almost everywhere.

We assume the existence of a homoclinic solution ω of $\ddot{x} + g(x) = 0$ such that $\lim_{t\to\pm\infty} \omega(t) = 0$ and $\omega(0) > 0$.

Lemma 1. ([14]) *There is a unique $t_0 \in \mathbb{R}$ satisfying $\dot{\omega}(t_0) = 0$. Consequently, $\dot{\omega}(t) > 0$, $\forall\, t < t_0$ and $\dot{\omega}(t) < 0$, $\forall\, t > t_0$.*

We consider a mapping M_μ, $\mu = (\mu_1, \mu_2, \mu_3)$, of the form

$$M_\mu(\alpha) = -2\omega(t_0)\mu_1 - \mu_2 \int\limits_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds + \mu_3 \int_{-\infty}^{\infty} \dot{\omega}(s)\psi(s+\alpha)\, ds\,.$$

Since $\omega(0) > 0$, Lemma 1 implies $\omega(t_0) > 0$. By putting

$$A(\alpha) = \int_{-\infty}^{\infty} \dot{\omega}(s)\psi(s+\alpha)\, ds\,,$$

we arrive at

$$M_\mu(\alpha) = A(\alpha)\mu_3 - 2\omega(t_0)\mu_1 - \mu_2 \int\limits_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds\,.$$

We note that A is periodic and C^1-smooth. We put $\bar{m} = \min A$, $\bar{M} = \max A$. By applying results of [12], we obtain the following theorem.

Theorem 2. *Assume that A has critical points only at maximums and minimums. Then there is an open subset $\mathcal{R}$ of $\mathbb{R}^3$ of all sufficiently small (μ_1, μ_2, μ_3) satisfying $\mu_3 \neq 0$ together with*

$$\bar{m} < \frac{2\omega(t_0)\mu_1 + \mu_2 \int\limits_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds}{\mu_3} < \bar{M}\,,$$

on which equation (4) has chaotic solutions in the following sense:

If $J\colon \mathcal{E} = \big\{E :\ E \in \{0,1\}^{\mathbb{Z}}\big\} \to \mathcal{E}$ is the Bernoulli shift defined by $J\big(\{e_j\}_{j\in\mathbb{Z}}\big) = \{\tilde{e}_j\}_{j\in\mathbb{Z}}$, $\tilde{e}_j = e_{j+1}$, then for any $\mu \in \mathcal{R}$ and $m \in \mathbb{N}$ sufficiently large, (4) possesses a family of solutions $\big\{x_{m,E}\big\}_{E\in\mathcal{E}}$ such that

- (i) *$E \to x_{m,E}$ is injective;*
- (ii) *$x_{m,J(E)}(t)$ is orbitally close to $x_{m,E}(t + \Omega m)$, where $\Omega > 0$ is the period of ψ.*

To be more precise, we consider Duffing-type equation (4) with $g(x) = -x + 2x^3$, $\psi(t) = \cos t$. Hence (4) has the form

$$\ddot{x} - x + 2x^3 + \mu_1 \operatorname{sgn} \dot{x} + \mu_2 \dot{x} = \mu_3 \cos t\,. \tag{6}$$

Then (see [18]), $\omega(t) = \operatorname{sech} t$. So we have

$$A(\alpha) = \int_{-\infty}^{\infty} \dot{\operatorname{sech}}\, s \, \cos(s+\alpha)\, ds = \pi \operatorname{sech} \frac{\pi}{2} \, \sin \alpha \,,$$

and $\bar{M} = -\bar{m} = \pi \operatorname{sech} \frac{\pi}{2}$, $t_0 = 0$, $\omega(t_0) = 1$, $\int\limits_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds = 2/3$.

Corollary 3. *Equation (6) has chaotic solutions in the sense of Theorem 2 provided that the parameters* μ_1, μ_2, μ_3 *are sufficiently small satisfying*

$$0 < 3\pi|\mu_3| \, sech \frac{\pi}{2} < |6\mu_1 + 2\mu_2| \,.$$

The next example is a modification of (4)

$$\ddot{x} + \delta g(x) + \frac{\eta}{\sqrt{\delta}} \dot{x} + \psi(t) \operatorname{sgn} \dot{x} = 0 \,, \tag{7}$$

where $g \in C^2(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, $g'(0) < 0$, $\delta > 0$ is a large parameter, $\psi \in C^1(\mathbb{R}, (0,\infty))$ is periodic and η is a constant. We assume the existence of a homoclinic solution ω of $\ddot{x} + g(x) = 0$ such that $\lim\limits_{t \to \pm\infty} \omega(t) = 0$ and $\omega(0) < 0$. Then again there is a unique $t_0 \in \mathbb{R}$ satisfying $\dot{\omega}(t_0) = 0$. Consequently, $\dot{\omega}(t) < 0$, $\forall\, t < t_0$, $\dot{\omega}(t) > 0$, $\forall\, t > t_0$ and $\omega(t_0) < 0$.

The equation (7) is rewritten in the form

$$\begin{aligned} \varepsilon \dot{x} &= y, \quad \varepsilon = \sqrt{1/\delta} \\ \varepsilon \dot{y} &= -g(x) - \varepsilon^2 \big(\eta y + \psi(t) \operatorname{sgn} y\big) \,. \end{aligned} \tag{8}$$

Hence (8) is a singularly perturbed discontinuous problem. Results of [12] imply the next theorem.

Theorem 4. *If the function*

$$M(\alpha) = 2\psi(\alpha)\omega(t_0) - \eta \int_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds$$

has a simple root, then for any $\delta > 0$ *sufficiently large, equation (7) has chaotic solutions in the sense of Theorem 2.*

We consider again the Duffing-type equation (7) of the form

$$\ddot{x} + \delta(-x + 2x^3) + \frac{\eta}{\sqrt{\delta}} \dot{x} + (2 + \cos t) \operatorname{sgn} \dot{x} = 0 \,. \tag{9}$$

Hence $g(x) = -x + 2x^3$, $\psi(t) = 2 + \cos t$, $\omega(t) = -\operatorname{sech} t$, $t_0 = 0$, $\omega(t_0) = -1$. Consequently, Theorem 4 gives the next corollary.

Corollary 5. *Equation (9) has chaotic solutions in the sense of Theorem 2 provided that* $\delta > 0$ *is sufficiently large and* $\eta \in (-9, -3)$.

3 Bifurcation of Periodic Solutions

Consider a mass attached to a mechanical device on a moving ribbon with a speed $v_0 > 0$. If there is also an external force and damping then the resulting differential equation [1,3,9] has the form

$$\ddot{x} + q(x) + \mu_1 \operatorname{sgn}(\dot{x} - v_0) + \mu_2 \dot{x} = \mu_3 \sin \omega t\,, \tag{10}$$

where $\operatorname{sgn} r$ corresponds to the dry friction between the mass and ribbon, $q \in C^2(\mathbb{R}, \mathbb{R})$ represents the force of the mechanical device and $\mu_1, \mu_2, \mu_3,\ \omega > 0$ are constants. Since $\operatorname{sgn} r$ is discontinuous in $r = 0$, (10) is considered as a perturbed differential inclusion of the form

$$\dot{x} = y, \quad \dot{y} \in -q(x) - \mu_1 \operatorname{Sgn}(y - v_0) - \mu_2 y + \mu_3 \sin \omega t\,. \tag{11}$$

Moreover, we assume

(i) There are numbers $0 < c < e$ and a C^2-mapping $\gamma\colon (c, e) \times \mathbb{R} \to \mathbb{R}$ such that $\gamma(\theta, t)$ has the minimum period θ in t, $\dot{\gamma}(\theta, 0) = 0$ and $\gamma(\theta, \cdot)$ is a solution of $\ddot{x} + q(x) = 0$.

If $c < 2\pi/\omega < e$ then we take

$$B(\alpha) = \int_0^{2\pi/\omega} \sin \omega(t + \alpha) \dot{\gamma}(2\pi/\omega, t)\, dt\,.$$

Since B is periodic, we put $\tilde{m} = \min B$, $\tilde{M} = \max B$.

Theorem 6. *Let $v_0 > 0$ be sufficiently small and let B have only critical points at minimums and maximums. If (i) holds and $c < 2\pi/\omega < e$, then for any sufficiently small $\mu = (\mu_1, \mu_2, \mu_3)$, $\mu_3 \neq 0$ satisfying*

$$\tilde{m} < \frac{1}{\mu_3}\Big(\mu_2 \int_0^{2\pi/\omega} \dot{\gamma}^2(2\pi/\omega, t)\, dt + 2\mu_1 \big|\gamma(2\pi/\omega, \pi/\omega) - \gamma(2\pi/\omega, 0)\big|\Big) < \tilde{M}\,,$$

equation (10) has a $2\pi/\omega$-periodic solution in a neighbourhood of the family $\gamma(\theta, t)$, $\theta \in (c, e)$ from (i).

Proof. We apply Corollary 3.2 of [10]. The formula (3.6) of [10] has the form

$$\begin{aligned} M_{\mu, v_0}(\alpha) = \Big\{ &\int_0^{2\pi/\omega} h(s) \dot{\gamma}(2\pi/\omega, s)\, ds \;\mid\; h \in L^2(0, 2\pi/\omega)\,, \\ &h(t) \in \mu_1 \operatorname{Sgn}(\dot{\gamma}(2\pi/\omega, t) - v_0) + \mu_2 \dot{\gamma}(2\pi/\omega, t) - \mu_3 \sin \omega(t + \alpha) \\ &\quad \text{a.e. on} \quad [0, 2\pi/\omega] \Big\}\,. \end{aligned}$$

Lemma 5.5. of [10] gives that $\dot\gamma(2\pi/\omega,0)=0$, $\dot\gamma(2\pi/\omega,\pi/\omega)=0$ and $\dot\gamma(2\pi/\omega,t)\neq 0$ $\forall t\in(0,2\pi/\omega)\setminus\{\pi/\omega\}$. Since $\ddot\gamma+g(\gamma)=0$, we see that $\ddot\gamma(2\pi/\omega,0)\neq 0$, $\ddot\gamma(2\pi/\omega,\pi/\omega)\neq 0$. Consequently for $v_0>0$ sufficiently small, $\dot\gamma(2\pi/\omega,t)=v_0$ has the only solutions $t_1(v_0)+k2\pi/\omega$, $t_2(v_0)+k2\pi/\omega$, $t_1(v_0)<t_2(v_0)$, where $k\in\mathbb{Z}$. Moreover, either $t_1(0)=0$, $t_2(0)=\pi/\omega$ or $t_1(0)=\pi/\omega$, $t_2(0)=2\pi/\omega$, and $t_{1,2}$ are smooth and $\dot\gamma(2\pi/\omega,t)>v_0$ on $(t_1(v_0),t_2(v_0))$; $\dot\gamma(2\pi/\omega,t)<v_0$ on $(t_2(v_0),t_1(v_0)+2\pi/\omega)$. Hence we obtain

$$\begin{aligned}M_{\mu,v_0}(\alpha)=&\mu_2\int_0^{2\pi/\omega}\dot\gamma^2(2\pi/\omega,t)\,dt+2\mu_1(\gamma(2\pi/\omega,t_2(v_0))-\gamma(2\pi/\omega,t_1(v_0))\\&-\mu_3\int_0^{2\pi/\omega}\sin\omega(t+\alpha)\dot\gamma(2\pi/\omega,t)\,dt\,.\end{aligned}$$

We note

$$M_{\mu,0}(\alpha)=\mu_2\int_0^{2\pi/\omega}\dot\gamma^2(2\pi/\omega,t)\,dt+2\mu_1\big|\gamma(2\pi/\omega,\pi/\omega)-\gamma(2\pi/\omega,0)\big|-\mu_3B(\alpha)\,.$$

The assumptions of our theorem imply that M_{μ,v_0} changes the sign on $\mathbb{R}$ for v_0 sufficiently small and for μ given in the theorem. Consequently, Corollary 3.2 of [10] can be applied to (10). The proof is finished. □

We refer the reader for more examples to [10].

4 Systems with Small Relay Hysteresis

In this section, we deal with relay hysteresis [2,24,25]. So there is given a pair of real numbers $\alpha<\beta$ (thresholds) and a pair of real-valued continuous functions $h_o\in C([\alpha,\infty),\mathbb{R})$, $h_c\in C((-\infty,\beta],\mathbb{R})$ such that $h_o(u)\geq h_c(u)\,\forall\,u\in[\alpha,\beta]$. Moreover, we suppose that h_o, h_c are bounded on $[\alpha,\infty)$, $(-\infty,\beta]$, respectively.

For a given continuous input $u(t)$, $t\geq t_0$, one defines the output $v(t)=f(u)(t)$ of the relay hysteresis operator as follows

$$f(u)(t)=\begin{cases}h_o(u(t)) & \text{if}\quad u(t)\geq\beta\,,\\ h_c(u(t)) & \text{if}\quad u(t)\leq\alpha\,,\\ h_o(u(t)) & \text{if}\quad u(t)\in(\alpha,\beta)\quad\text{and}\quad u(\tau(t))=\beta\,,\\ h_c(u(t)) & \text{if}\quad u(t)\in(\alpha,\beta)\quad\text{and}\quad u(\tau(t))=\alpha\,,\end{cases}$$

where $\tau(t)=\sup\{s:\ s\in[t_0,t],\ u(s)=\alpha \text{ or } u(s)=\beta\}$. If $\tau(t)$ does not exist (i.e. $u(\sigma)\in(\alpha,\beta)$ for $\sigma\in[t_0,t]$), then $f(u)(\sigma)$ is undefined and we have to initially set the relay open or closed when $u(t_0)\in(\alpha,\beta)$. Of course, when either $h_o(\beta)>h_c(\beta)$ or $h_o(\alpha)>h_c(\alpha)$ then $f(u)$ is generally discontinuous.

Let us consider the problem

$$\dddot{y} - \ddot{y} + \dot{y} - y = \mu f(y)\,, \tag{12}$$

where $\mu \in \mathbb{R}$ and f is of the form

$$\alpha = -\delta, \quad \beta = \delta, \quad \delta > 0, \quad h_o = g + p, \quad h_c = g - p$$

with $p > 0$ constant and $g \in C^1(\mathbb{R}, \mathbb{R})$.

Theorem 7. *If $\theta_0 > \delta$ is a simple root of the function*

$$4p\Big(\sqrt{1 - \frac{\delta^2}{\theta^2}} - \frac{\delta}{\theta}\Big) + \int_0^{2\pi} g(\theta \sin t) \sin t\, dt\,,$$

then there is a constant $K > 0$ such that for any μ sufficiently small there are θ_μ, ω_μ, $|\theta_0 - \theta_\mu| \le K|\mu|$, $|\omega_0 - \omega_\mu| \le K|\mu|$, $\omega_0 = -\frac{2\delta p}{\pi\theta_0^2}$ and a $2\pi(1 + \mu\omega_\mu)$-periodic solution y_μ of (12) satisfying

$$\sup_{t\in\mathbb{R}} \Big| y_\mu(t) - \theta_\mu \sin \frac{t}{1 + \mu\omega_\mu} \Big| \le K|\mu|\,.$$

Proof. We apply Theorem 2.2 of [15], by taking

$$\mathcal{O} = (\delta, \infty), \quad \phi_1(t) = \psi_1(t) = \sin t, \quad \theta > \delta\,,$$
$$\phi_2(t) = \psi_2(t) = \cos t, \quad \eta(\theta, t) = \theta \sin t, \quad t_0 = \arcsin\frac{\delta}{\theta}\,.$$

The formula (2.8) of [15] has the form

$$M(\omega, \theta) = \big(M_1(\omega, \theta), M_2(\omega, \theta)\big)\,,$$

where

$$M_1(\omega, \theta) = \int_0^{2\pi} \omega(\theta\cos t + 2\theta \sin t - 3\theta\cos t)\sin t\, dt + \int_{t_0}^{t_0+\pi} (g(\theta \sin t) + p)\sin t\, dt$$
$$+ \int_{t_0+\pi}^{t_0+2\pi} (g(\theta\sin t) - p)\sin t\, dt = 2\pi\theta\omega + 4p\sqrt{1 - \frac{\delta^2}{\theta^2}} + \int_0^{2\pi} g(\theta\sin t)\sin t\, dt\,,$$
$$M_2(\omega, \theta) = \int_0^{2\pi} \omega(\theta\cos t + 2\theta \sin t - 3\theta\cos t)\cos t\, dt + \int_{t_0}^{t_0+\pi} (g(\theta \sin t) + p)\cos t\, dt$$
$$+ \int_{t_0+\pi}^{t_0+2\pi} (g(\theta\sin t) - p)\cos t\, dt = -2\pi\theta\omega - 4\frac{\delta p}{\theta}\,.$$

Clearly (ω_0, θ_0) is a simple zero of M. Consequently, [15, Theorem 2.2] implies the result. The proof is finished. □

More examples are given in [15].

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A Note on Asymptotic Expansion for a Periodic Boundary Condition

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Abstract. The aim of this contribution is to present a new result concerning asymptotic expansion of solutions of the heat equation with periodic Dirichlet–Neuman boundary conditions with the period going to zero in 3D.

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Keywords. Heat equation, asymptotic expansion, homogenization

1 Introduction

In the recent paper [3, Filo–Luckhaus] we have determined the first two terms in the asymptotic expansion (with respect to a small parameter ε) of the solution $u_\varepsilon = u_\varepsilon(x,t)$ to the following problem:

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} &= \Delta u_\varepsilon + f(x,t) && \text{in } \Omega \times (0,T), \\ \frac{\partial u_\varepsilon}{\partial \nu} &= \vartheta(x,t) - \sigma(x,t) u_\varepsilon && \text{on } n^\varepsilon \times (0,T), \\ u_\varepsilon &= 0 && \text{on } d^\varepsilon \times (0,T), \\ u_\varepsilon &= \varphi && \text{on } \Omega \times \{t=0\}. \end{aligned} \tag{1}$$

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Here $\Omega \subset \mathbb{R}^2$ is a bounded domain whose boundary is given by a C^3 simple closed curve Γ,

$$\Gamma = \{(p(\tau), q(\tau));\ 0 \le \tau \le \pi\}, \qquad (p'(\tau))^2 + (q'(\tau))^2 = 1,$$

a is 2π periodic function such that

$$a(\sigma) = \begin{cases} 0 & : \quad \sigma \in [\pi - \delta, \pi + \delta] \\ 1 & : \quad \sigma \in [0, \pi - \delta) \cup (\pi + \delta, 2\pi] \end{cases}$$

for some $\delta \in (0, \pi)$,

$$n^\varepsilon = \left\{ x \in \Gamma;\ x = (p(\tau), q(\tau)),\ a\left(\frac{\tau}{\varepsilon}\right) = 1,\ 0 \le \tau \le \pi \right\},$$

$$d^\varepsilon = \left\{ x \in \Gamma;\ x = (p(\tau), q(\tau)),\ a\left(\frac{\tau}{\varepsilon}\right) = 0,\ 0 \le \tau \le \pi \right\}$$

and

$$\varepsilon^{-1} \qquad \text{is an even integer .}$$

We have shown, under certain smoothness assumptions on the data f, σ, ϑ and φ, that

$$u_\varepsilon = u + \varepsilon u^1 + \varepsilon \mathcal{O}(\varepsilon)\ , \tag{2}$$

where

$$\mathcal{O}(\varepsilon) \longrightarrow 0 \qquad \text{strongly in } L_p(\Omega \times (0,T)) \qquad \text{if} \qquad \varepsilon \to 0$$

for any $p,\ 1 \le p < 4$ and

$$\frac{u_\varepsilon - u}{\varepsilon} \rightharpoonup \omega_0(\vartheta - \partial_\nu u) \qquad \text{weakly in } L_2(\Gamma \times (0,T))\ . \tag{3}$$

The functions u and u^1 are solutions of the problems

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(x,t) && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \Gamma \times (0,T), \\ u &= \varphi && \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{\partial u^1}{\partial t} &= \Delta u^1 && \text{in } \Omega \times (0,T), \\ u^1 &= \omega_0 \left(\vartheta - \tfrac{\partial u}{\partial \nu}\right) && \text{on } \Gamma \times (0,T), \\ u^1 &= 0 && \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{5}$$

respectively. Here

$$\omega_0 = \frac{1}{\pi}\int_0^\pi \omega(x_1,0)\,dx_1 \ ,$$

where $\omega = \omega(x_1,x_2)$ is the unique nonnegative 2π periodic (in the x_1 variable) solution of the following boundary value problem

$$\begin{aligned} \Delta\omega &= 0 && \text{in } \mathbb{R}^2_+, \\ a(x_1)\left(\frac{\partial\omega}{\partial x_2}(x_1,0)+1\right) &+ (1-a(x_1))\omega(x_1,0) = 0 && \text{for } x_1 \in \mathbb{R}, \end{aligned}$$

satisfying

$$\|\omega\|_{L^\infty(\mathbb{R}^2_+)} + \int_0^\infty\int_0^\pi |\nabla\omega|^2(x_1,x_2)\,dx_1dx_2 < \infty \ .$$

Moreover, we have demonstrated, that

$$\left\|\frac{u_\varepsilon - u}{\varepsilon} - w_\varepsilon(\vartheta - \partial_\nu u)\right\|_{L_2(\Gamma\times(0,T))} \le C\ \sqrt{\varepsilon}$$

for

$$w_\varepsilon(x) \equiv \omega\left(\frac{\tau(x)}{\varepsilon}, \frac{\delta(x)}{\varepsilon}\right)$$

where the functions τ, δ are defined for $x \in \overline{\Omega}$ sufficiently close to Γ such that $\delta(x) = dist(x,\Gamma)$ and

$$p'(\tau(x))(x_1 - p(\tau(x))) + q'(\tau(x))(x_2 - q(\tau(x))) = 0 \ .$$

In addition,

$$\frac{u_\varepsilon - u}{\varepsilon} - w_\varepsilon\mathcal{G} \rightharpoonup u^1 - \omega_0\mathcal{G}$$

weakly in $V_2^{1,0}(\Omega\times(0,T))$, where

$$\mathcal{G}(x,t) \equiv \vartheta(x,t) - \xi(x)\partial_\nu u(p(\tau(x)), q(\tau(x)), t)$$

and ξ is a cutoff function that equals 1 in a neighbourhood of Γ and $\xi(x) = 0$ for any $x \in \Omega$, $dist(x,\Gamma) \ge \delta_0$ for some positive δ_0.
For definitions of function spaces we refer to [5, Ladyzenskaja at al.].

It is the aim of this contribution to present a generalization of the previous result to the case of more space dimensions developed in [4, Luckhaus–Filo].

2 Motivation

Our original goal was to study flow problems in porous media with a part of the boundary covered by a fluid. For one incompressible fluid in porous medium one has to solve the equation

$$\frac{\partial \theta(p)}{\partial t} = \nabla \cdot (k(\theta(p))(\nabla p + e)), \tag{6}$$

where p is the unknown pressure, θ the water content, k the conductivity of the porous medium, and $-e$ the direction of gravity (see [1, Bear], for mathematical treatment of (6) [2, Alt - Luckhaus], for example).

The part of the boundary, which is covered by the fluid and where the infiltration takes place is supposed to behave like a impervious layer with many small holes. It is assumed that the holes are distributed uniformly and create a periodic structure with period ε. The pressure is supposed to be 0 on the holes, where the fluid infiltrates into the porous medium, and the condition $(k(\theta(p))(\nabla p + e)) \cdot \nu = 0$ is assumed to be satisfied on the impervious part of the boundary. As the period and the diameter of the hole is of order ε and the domain occupied by the porous medium is large, it is natural to let $\varepsilon \to 0$ and to ask on the behaviour of solutions to (6).

However, since this nonlinear problem was not yet treatable, we have studied the heat equation, i.e. equation (6) with

$$\theta(p) \equiv p, \quad k(\theta(p)) \equiv 1 \quad \text{and} \quad e = 0 \ .$$

3 Model Problem in $\mathbb{R}^3$

Let Λ be the square in $\mathbb{R}^2$, i.e. $\Lambda \equiv (0, 2\ell) \times (0, 2\ell)$ for some positive ℓ and $\theta \colon \mathbb{R}^2 \to \mathbb{R}_+, \mathbb{R}_+ \equiv (0, \infty)$ be a smooth function, say, $C^3(\mathbb{R}^2)$, even and 2ℓ-periodic in each of its variable. Points in $\mathbb{R}^3$ are denoted by $x = (\bar{x}, x_3)$ $\bar{x} = (x_1, x_2)$ and we define

$$\Omega \equiv \{x \in \mathbb{R}^3 \mid \ \bar{x} \in \Lambda, \ \theta(\bar{x}) < x_3 < d\}$$

for some positive d greater than the maximum of the function θ and define

$$\Gamma \equiv \{x \in \partial\Omega \mid \ x_3 = \theta(\bar{x}), \ \bar{x} \in \Lambda\}.$$

Now let $\mathcal{F} = \{\bar{x} \in \Lambda \mid \ |\bar{x} - \bar{\ell}| \le \delta\}$, $\bar{\ell} = (\ell, \ell)$ for some $0 < \delta < \ell$ and set

$$\overline{a}(\bar{x}) = \begin{cases} 0 & : \quad \bar{x} \in \mathcal{F} \\ 1 & : \quad \bar{x} \in \Lambda \setminus \mathcal{F}. \end{cases}$$

Denote by $a(\bar{x})$ for $\bar{x} \in \mathbb{R}^2$ the 2ℓ-periodic extension of the function $\overline{a}$ on the whole $\mathbb{R}^2$. Let $\varepsilon^{-1} = 2^k$ for $k \in \{0, 1, 2, \cdots\}$, define

$$\begin{aligned} \mathcal{D}^\varepsilon &\equiv \{x \in \Gamma \mid \ a(\varepsilon^{-1}\bar{x}) = 0\}, & \mathcal{D}^\varepsilon_T &\equiv \mathcal{D}^\varepsilon \times (0, T), \\ \mathcal{N}^\varepsilon &\equiv \{x \in \Gamma \mid \ a(\varepsilon^{-1}\bar{x}) = 1\}, & \mathcal{N}^\varepsilon_T &\equiv \mathcal{N}^\varepsilon \times (0, T), \\ D &\equiv \{\bar{x} \in \mathbb{R}^2 \mid \ a(\bar{x}) = 0\}, & N &\equiv \{\bar{x} \in \mathbb{R}^2 \mid \ a(\bar{x}) = 1\}. \end{aligned}$$

and for simplicity of notation we put $\partial_t u \equiv \partial u/\partial t$, $\partial_\nu u \equiv \partial u/\partial \nu$ etc.

Consider now the problem

$$\begin{aligned}
\partial_t u_\varepsilon &= \Delta u_\varepsilon + f_\varepsilon(x,t) && \text{in } \Omega_T,\\
\partial_\nu u_\varepsilon &= \vartheta_\varepsilon(x,t) - \sigma_\varepsilon(x,t)u_\varepsilon && \text{on } \mathcal{N}_T^\varepsilon,\\
u_\varepsilon &= 0 && \text{on } \mathcal{D}_T^\varepsilon,\\
\partial_\nu u_\varepsilon &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T,\\
u_\varepsilon &= u_0^\varepsilon && \text{on } \Omega \times \{t=0\}
\end{aligned} \tag{7}$$

under the following assumptions:

(A) $f_\varepsilon, f, f^1 \in L_2(\Omega_T)$ and such that

$$\frac{f_\varepsilon - f}{\varepsilon} \rightharpoonup f^1 \qquad \text{in} \quad L_2(\Omega_T);$$

(B) σ_ε, $\partial_t\sigma_\varepsilon \in L_\infty(\Gamma_T)$ for any ε and there exists a positive constant C independent of ε such that $\|\sigma_\varepsilon\|_{L_\infty(\Gamma_T)} \le C$;

(C) $\vartheta_\varepsilon, \vartheta$, $\partial_t\vartheta_\varepsilon \in L_2(\Gamma_T)$ and such that

$$\vartheta_\varepsilon \rightharpoonup \vartheta \qquad \text{in} \quad L_2(\Gamma_T);$$

(D) u_0^ε, $u_0 \in W_2^1(\Omega)$, $u_0 = 0$ on Γ, $u_0^\varepsilon = 0$ on $\mathcal{D}^\varepsilon$, $u^1 \in L_2(\Omega)$ and such that

$$\frac{u_0^\varepsilon - u_0}{\varepsilon} \rightharpoonup u_0^1 \qquad \text{in} \quad L_2(\Omega).$$

We prove that asymptotic expansion (2) holds in the sense that

$$\mathcal{O}(\varepsilon) \longrightarrow 0$$

weakly in $L_2(\Omega_T)$ and strongly in $L_2(\Omega_T^*)$ for any subdomain $\Omega^* \subset \Omega$ with a positive distance from Γ, and, comparing to (3),

$$\frac{u_\varepsilon - u}{\varepsilon}(x,t) \rightharpoonup \omega_0(x)\,(\vartheta(x,t) - \partial_\nu u(x,t)) \tag{8}$$

(weakly in) in $L_2(\Gamma_T)$. Here, similarly as above (see (4) and (5) above) u is the unique solution of the problem

$$\begin{aligned}
\partial_t u &= \Delta u + f(x,t) && \text{in } \Omega_T,\\
u &= 0 && \text{on } \Gamma_T,\\
\partial_\nu u &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T,\\
u &= u_0 && \text{on } \Omega \times \{t=0\},
\end{aligned} \tag{9}$$

u^1 is the unique very weak solution of the problem

$$\begin{aligned} \partial_t u^1 &= \Delta u^1 + f^1(x,t) && \text{in } \Omega_T, \\ u^1 &= \omega_0(x)\,(\vartheta(x,t) - \partial_\nu u(x,t)) && \text{on } \Gamma_T, \\ \partial_\nu u^1 &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T, \\ u^1 &= 0 && \text{on } \Omega \times \{t=0\}, \end{aligned} \tag{10}$$

and the function $\omega_0(x)$ is defined for $x \in \Gamma$ as follows:

$$\omega_0(x) \equiv \frac{1}{\ell^2}\int_0^\ell \int_0^\ell \varpi(x;\bar y,0)\,d\bar y\ ,$$

$\varpi = \varpi(x;y)$ is the unique bounded nonnegative solution of the problem

$$\begin{aligned} \sum_{k=1}^{3} \frac{\partial}{\partial y_k}\left(\sum_{j=1}^{3} \gamma_{jk}(x)\frac{\partial\varpi}{\partial y_j}(x;y)\right) &= 0 && y \in \mathbb{R}^3_+\ , \\ \varpi(x;\bar y,0) &= 0 && \bar y \in D\ , \\ -\frac{\partial\varpi}{\partial y_3}(x;\bar y,0) &= 1 && \bar y \in N\ , \end{aligned} \tag{11}$$

where

$$\mathbf{C}(x) = (\gamma_{jk})_{j,k=1,2,3}\ ,$$

$$\mathbf{C}(x) \equiv \frac{1}{\sqrt{1+a_1^2+a_2^2}} \begin{pmatrix} 1+a_2^2 & -a_1a_2 & 0 \\ -a_2a_1 & 1+a_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,$$

and

$$a_j \equiv \frac{\partial\theta}{\partial x_j}(\bar x)\ .$$

The function ϖ is 2ℓ-periodic in each of its variables y_1, y_2 and it is demonstrated that

$$\varpi(x;y) \ = \ \omega(x;\mathbf{E}^{-1}(x)y)\,,$$

where $\omega(x;z)$ is for each $x \in \Gamma$ the harmonic function in $z \in \mathbb{R}^2_+$ such that

$$a(\widehat{\mathbf{E}}(x)\bar z)\left(\frac{\partial\omega}{\partial z_3}(x;\bar z,0) + \lambda\right) + \left(1 - a(\widehat{\mathbf{E}}(x)\bar z)\right)\omega(x;\bar z,0) \ = \ 0 \tag{12}$$

and

$$\mathbf{E}^{-1}(x) \equiv \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{a_2}{\sqrt{a_1^2+a_2^2}} & -\frac{a_1}{\sqrt{a_1^2+a_2^2}} & 0 \\ \frac{a_1}{\sqrt{a_1^2+a_2^2}} & \frac{a_2}{\sqrt{a_1^2+a_2^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathbf{E}}(x) \equiv \begin{pmatrix} \frac{a_2}{\sqrt{a_1^2+a_2^2}} & \frac{a_1}{\sqrt{a_1^2+a_2^2}} \\ -\frac{a_1}{\sqrt{a_1^2+a_2^2}} & \frac{a_2}{\sqrt{a_1^2+a_2^2}} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

$$\lambda(x) = \left(1 + a_1^2 + a_2^2\right)^{1/4}.$$

4 A priori estimates

The first and basic step to prove the validity of the expansion (2) consists of a priori estimates, that can be summarized in the following

Theorem 1. *Assume that (A)–(D) are satisfied. Then there exists a positive constant C, independent of ε, such that*

$$\max_{0\le t\le T} \int_\Omega |u_\varepsilon - u|^2(x,t)\,dx + \int_0^T \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\,dx\,dt \le C\varepsilon\ ,$$

$$\int_0^T \int_\Gamma |u_\varepsilon - u|^2(x,t)\,dH^2(x)\,dt + \int_0^T \int_\Omega |u_\varepsilon - u|^2(x,t)\,dx\,dt \le C\varepsilon^2\ ,$$

$$\max_{0\le t\le T} \int_\Omega |u_\varepsilon - u|^2(x,t)\phi(x)\,dx + \int_0^T \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\phi(x)\,dx\,dt \le C\varepsilon^2$$

and

$$\operatorname*{ess\,sup}_{0\le t\le T} \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\phi^3(x)\,dx + \int_0^T \int_\Omega |\partial_t(u_\varepsilon - u)|^2(x,t)\phi^3(x)\,dx\,dt \le C\varepsilon^2\ ,$$

where ϕ is the principal eigenfunction of the problem

$$\begin{aligned} \Delta\phi + \mu\phi &= 0 && \text{in} \quad \Omega, \\ \phi &= 0 && \text{on} \quad \Gamma, \\ \partial_\nu \phi &= 0 && \text{on} \quad \partial\Omega \setminus \Gamma, \end{aligned}$$

with the corresponding principal eigenvalue $\mu = \mu_1 > 0$.

In the proof of Theorem 1 the following proposition plays an important role.

Proposition 2. *Let $v \in W_2^{1,0}(\Omega_T)$ be such that $v = 0$ on $\mathcal{D}_T^\varepsilon$. Then*

$$\int_0^T \int_\Gamma |v(x,t)|^2 dH^2(x) dt \le C\varepsilon \int_0^T \|v\|^2_{W_2^{1/2}(\Gamma)}(t)\ dt$$

and

$$\|v\|_{L_2(\Gamma_T)} \le c\|v\|_{W_2^{1,0}(\Omega_T)} \sqrt{\varepsilon}\ ,$$

where the positive constants C, c do not depend on ε and v.

Proof (of Proposition 2). We set

$$V(y,t) \equiv v(x(y),t),\ \ x(y) = (y_1, y_2, \theta(\bar{y}) + (d - \theta(\bar{y}))y_3/(d - \theta_0))$$

for $\bar{y} = (y_1, y_2) \in \Lambda$, $y_3 \in (0, d - \theta_0)$ and $\theta_0 = \max_{\bar{x} \in \overline{\Lambda}} \theta(\bar{x})$. Note that

$$v(x,t) = V(y(x),t), \qquad y(x) = (x_1, x_2, (d - \theta_0)(x_3 - \theta(\bar{x}))/(d - \theta(\bar{x})))$$

and $V(\bar{y}, 0, t) = 0$ for any $\bar{y} \in \Lambda$ such that $a(\varepsilon^{-1}\bar{y}) = 0$. Then it is not difficult to see that

$$\int_0^T \int_\Lambda |V(\bar{y},0,t)|^2 d\bar{y}\, dt \le \frac{\varepsilon \ell^3}{\delta^2 \pi} \int_0^T \int_\Lambda \int_\Lambda \frac{|V(\bar{y},0,t) - V(\bar{z},0,t)|^2}{|\bar{y} - \bar{z}|^3}\, d\bar{y}\, d\bar{z}\, dt\ .$$

As

$$\int_0^T \int_\Gamma |v(x,t)|^2 dH^2(x)\, dt = \int_0^T \int_\Lambda |V(\bar{y},0,t)|^2 \sqrt{1 + |\overline{\nabla}\theta(\bar{y})|^2}\, d\bar{y}\, dt$$

and $\|V\|^2_{W_2^{1/2}(\Lambda)} \le c\|v\|^2_{W_2^{1/2}(\Gamma)} \le C\|v\|^2_{W_2^1(\Omega)}$, the assertion of Proposition 2 follows.

Proof (of Theorem 1). Note first that $u_\varepsilon - u$ is a solution of the problem

$$\begin{aligned} \partial_t(u_\varepsilon - u) &= \Delta(u_\varepsilon - u) + (f_\varepsilon - f)(x,t) && \text{in } \Omega_T, \\ \partial_\nu(u_\varepsilon - u) &= g_\varepsilon(x,t) && \text{on } \mathcal{N}_T^\varepsilon, \\ u_\varepsilon - u &= 0 && \text{on } \mathcal{D}_T^\varepsilon, \\ \partial_\nu(u_\varepsilon - u) &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T, \\ u_\varepsilon - u &= u_0^\varepsilon - u_0 && \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{13}$$

where $g_\varepsilon(x,t) = \vartheta_\varepsilon(x,t) - \sigma_\varepsilon(x,t)u_\varepsilon - \partial_\nu u$. Testing the problem (13) by $u_\varepsilon - u$ and applying Proposition 2 we arrive at

$$|u_\varepsilon - u| \equiv \max_{0\le t\le T} \|(u_\varepsilon - u)(t)\|_{L_2(\Omega)} + \|\nabla(u_\varepsilon - u)\|_{L_2(\Omega_T)} \le \\ \|u_0^\varepsilon - u_0\|_{L_2(\Omega)} + 2\|f_\varepsilon - f\|_{L_2(\Omega_T)} + C\|g_\varepsilon\|_{L_2(\Gamma_T)}\sqrt{\varepsilon}\,.$$

As, however, $\|u_\varepsilon - u\|_{L_2(\Gamma_T)} \le C|u_\varepsilon - u|\sqrt{\varepsilon}$, due to our assumptions (A) and (D) we get $\|u_\varepsilon - u\|_{L_2(\Gamma_T)} \le C\varepsilon$.

Multiplying now the equation in the problem (13) by $(u_\varepsilon - u)\phi$ and integrating over Ω one easily gets the third estimate of Theorem 1. Denote next

$$U(y,t) \equiv (u_\varepsilon - u)(x(y),t) \quad \text{for} \quad y \in \Omega^* \equiv \Lambda \times (0, d-\theta_0)\,.$$

Then we obtain

$$\int_{\Omega^*} |U(y,t)|^2 dy \le C_\eta \int_\Lambda \int_\eta^{d-\theta_0} |U(\bar y, y_3, t)|^2 y_3\, dy_3\, d\bar y + C\int_{\Omega^*} |\partial_{y_3} U(y,t)|^2 y_3\, dy$$

for any $t \in (0,T)$ and fixed $\eta \in (0, d-\theta_0)$. It is very well known that there exist positive constants c, C such that $c \le -\partial_\nu \phi \le C$ on Γ. This together with the above estimate yield the estimate $\|u_\varepsilon - u\|_{L_2(\Omega_T)} \le C\varepsilon$. The last estimate we obtain by multiplying the equation in the problem (13) by $\phi^3\partial_t(u_\varepsilon - u)$ and by integrating.

The essentiall part of the proof of the convergence (8) is the uniqueness of the problem

$$\Delta_z \omega(x;z) = 0 \qquad \text{in } \mathbb{R}^3_+ \tag{14}$$

with the boundary condition (12) in the following class of solutions.

Definition 3. By a solution of Problem (14), (12) we mean a function $\omega \in W^{1,2}_{loc}(\mathbb{R}^3_+)$ satisfying

$$\begin{gathered} \int_0^R \int_{B_2(\bar y, L)} |\nabla\omega|^2(\bar x, x_3)\, d\bar x dx_3 \le CL^2, \\ \int_0^R \int_{B_2(\bar y, L)} |\omega|^2(\bar x, x_3)\, d\bar x dx_3 \le CL^2(R^2 + R), \\ \int_{B_2(\bar y, L)} |\omega|^2(\bar x, 0)\, dx' \le CL^2 \end{gathered} \tag{15}$$

for any $\bar y \in \mathbb{R}^2$ (the positive constant C does not depend on $\bar y, L, R$), and the integral identity

$$\int_{\mathbb{R}^3_+} \nabla\omega(x)\nabla\psi(x)\, dx \; = \mu \int_{\mathbb{R}^2} \psi(\bar x, 0)\, d\bar x$$

for any $\psi \in W^1_{2,loc}(\mathbb{R}^3_+)$, $\psi = 0$ on $\Gamma_D \equiv \{x = (\bar{x}, 0) \mid a(\widehat{\mathbf{E}}(\bar{x})) = 0\}$ with compact support in $\overline{\mathbb{R}}^3_+$. Note that $B_2(\bar{y}, L) = \{\bar{x} \in \mathbb{R}^2 \mid |\bar{x} - \bar{y}| < L\}$.

This problem was obtained as a limit as $\varepsilon \to 0$ after applying rescaling arguments for $(u_\varepsilon - u)/\varepsilon$ in any point $x \in \Gamma$.

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Periodic Problems for ODE's via Multivalued Poincaré Operators

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Abstract. We shall consider periodic problems for ordinary differential equations of the form

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(a), \end{cases} \tag{I}$$

where $f : [0, a] \times R^n \to R^n$ satisfies suitable assumptions.
To study the above problem we shall follow an approach based on the topological degree theory. Roughly speaking, if on some ball of R^n, the topological degree of, associated to (I), multivalued Poincaré operator P turns out to be different from zero, then problem (I) has solutions.
Next by using the multivalued version of the classical Liapunov-Krasnoselskǐ guiding potential method we calculate the topological degree of the Poincaré operator P. To do it we associate with f a guiding potential V which is assumed to be locally Lipschitzean (instead of C^1) and hence, by using Clarke generalized gradient calculus we are able to prove existence results for (I), of the classical type, obtained earlier under the assumption that V is C^1.
Note that using a technique of the same type (adopting to the random case) we are able to obtain all of above mentioned results for the following random periodic problem:

$$\begin{cases} x'(\xi, t) = f(\xi, t, x(\xi, t)), \\ x(\xi, 0) = x(\xi, a), \end{cases} \tag{II}$$

where $f : \Omega \times [0, a] \times R^n \to R^n$ is a random operator satisfying suitable assumptions.
This paper stands a simplification of earlier works of F. S. De Blasi, G. Pianigiani and L. Górniewicz (see: [7], [8]), where the case of differential inclusions is considered.

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1 Background in geometric topology

Throughout this note $\mathbb{R}^n$, $n \geq 1$, is an n-dimensional real Euclidean space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. A closed (resp. open) ball in $\mathbb{R}^n$ with center x and radius $r \geq 0$ is denoted by $B^n(x, r)$ (resp. $B_0^n(x, r)$). Furthermore we put:

$$B^n(r) = B^n(0, r), \qquad B_0^n(r) = B_0^n(0, r),$$
$$S^{n-1}(r) = B^n(r) \setminus B_0^n(r), \qquad \mathbb{P}^n = \mathbb{R}^n \setminus \{0\}.$$

$\mathbb{Z}$ stands for the set of all integers.

For $A \subset \mathbb{R}^n$ we denote by $\overline{A}$, the closure of A. If $A \subset \mathbb{R}^n$ is nonempty, we put

$$|A|^- = \inf\{\|a\| \mid a \in A\}.$$

As usual, $\varphi : X \to Y$ (resp. $\varphi : X \multimap Y$) denotes a single valued (resp. multivalued) map φ from X to Y.

In the sequel, any topological space X we consider is always supposed to be metric. When the clarity is not affected, by "space X" we mean "topological metric space X".

A space X is called *contractible* if there is a continuous homotopy $h : X \times [0, 1] \to X$ and a point $x_0 \in X$ such that:

$$h(x, 0) = x \qquad \text{for every } x \in X,$$
$$h(x, 1) = x_0 \qquad \text{for every } x \in X.$$

A nonempty compact space X is called an *R_δ-set* if there is a decreasing sequence $\{X_k\}$ of compact contractible spaces X_k such that

$$X = \bigcap_{k=1}^{+\infty} X_k.$$

A space X is called an *absolute neighbourhood retract* ($X \in ANR$) if, for every space Y and any closed set $C \subset Y$ and any continuous map $f : C \to X$, there is an open neighbourhood U of C in Y and a continuous map $g : U \to X$ such that:

$$g(x) = f(x) \qquad \text{for every } x \in C. \tag{3}$$

A space X is called an *absolute retract* ($X \in AR$) if, for any space Y and any closed $C \subset Y$ and any continuous map $f : C \to X$, there is a continuous map $g : Y \to X$ satisfying (3).

Clearly $X \in AR$ implies $X \in ANR$ (the converse is not true). Moreover, if $X, Y \in ANR$ then $X \times Y \in ANR$. It is also easy to verify that any $X \in AR$ is contractible.

A multivalued map $\varphi : X \multimap Y$ with nonempty values is called *upper semicontinuous* (u.s.c.), if $\{x \in X \mid \varphi(x) \subset U\}$ is open in X for each open $U \subset Y$.

As usual $C([a,b], \mathbb{R}^n)$ stands for the Banach space of all continuous maps $x : [a,b] \to \mathbb{R}^n$, endowed with the norm of uniform convergence. Clearly it holds $C([a,b], \mathbb{R}^n) \in AR$.

Now, following [11], (see also [6,7,10]), we recall some definitions of the topological degree for multivalued maps. Applications to periodic problems (comp. (9)) will be given later, in Sections 3 and 4.

For any $X \in ANR$ we set

$$J(B^n(r), X) = \{F : B^n(r) \multimap X \mid F \text{ is u.s.c. with } R_\delta\text{-values}\}.$$

For any continuous $f : X \to \mathbb{R}^n$, when $X \in ANR$, we put

$$J_f(B^n(r), \mathbb{R}^n) = \{\varphi : B^n(r) \multimap \mathbb{R}^n \mid \varphi = f \circ F \text{ for some } F \in J(B^n(r), X), \\ \text{and } \varphi(S^{n-1}(r)) \subset \mathbb{P}^n\}.$$

Finally, we define

$$CJ(B^n(r), \mathbb{R}^n) = \bigcup \{J_f(B^n(r), \mathbb{R}^n) \mid f : X \to \mathbb{R}^n \text{ is continuous,} \\ \text{with } X \in ANR\}.$$

It is well known (see: [11,6,7,10]) that the notion of topological degree for multivalued maps in the class $CJ(B^n(r), \mathbb{R}^n)$ is available. To define it we need an appropriate notion of homotopy in $CJ(B^n(r), \mathbb{R}^n)$.

Definition 1. Let $\varphi_1, \varphi_2 \in CJ(B^n(r), \mathbb{R}^n)$ be two maps of the form

$$\varphi_1 = f_1 \circ F_1 \qquad B^n(r) \overset{F_1}{\multimap} X \xrightarrow{f_1} \mathbb{R}^n$$
$$\varphi_2 = f_2 \circ F_2 \qquad B^n(r) \overset{F_2}{\multimap} X \xrightarrow{f_2} \mathbb{R}^n.$$

We say that φ_1 and φ_2 are *homotopic* in $CJ(B^n(r), \mathbb{R}^n)$ if there exists an u.s.c. R_δ-valued homotopy $\chi : B^n(r) \times [0,1] \multimap X$ and a continuous homotopy $h : X \times [0,1] \to \mathbb{R}^n$ satisfying:

(i) $\chi(u,0) = F_1(u), \quad \chi(u,1) = F_2(u) \qquad$ for every $u \in B^n(r)$,

(ii) $h(x,0) = f_1(x), \quad h(x,1) = f_2(x) \qquad$ for every $x \in X$,

(iii) for every $(u,\lambda) \in S^{n-1}(r) \times [0,1]$ and $x \in \chi(u,\lambda)$ we have $h(x,\lambda) \neq 0$.

The map $H : B^n(r) \times [0,1] \multimap \mathbb{R}^n$ given by

$$H(u,\lambda) = h(\chi(u,\lambda),\lambda)$$

is called a *homotopy* in $CJ(B^n(r), \mathbb{R}^n)$ between φ_1 and φ_2.

By using approximation results for multivalued maps (see: [11,6,7,10]) one can prove the following theorem concerning the construction of a degree for maps $\varphi \in CJ(B^n(r), \mathbb{R}^n)$.

Theorem 2. *There exists a map*

$$\mathrm{Deg}\,: CJ(B^n(r), \mathbb{R}) \to \mathbb{Z},$$

called topological degree function, satisfying the following properties:

(i) *If* $\varphi \in CJ(B^n(r), \mathbb{R}^n)$ *is of the from* $\varphi = f \circ F$ *with* F *single valued and continuous, then* $\mathrm{Deg}\,(\varphi) = \deg\,(\varphi)$, *when* $\deg\,(\varphi)$ *stands for the ordinary Brouwer degree of the single valued continuous map* $\varphi : B^n(r) \to \mathbb{R}^n$.

(ii) *If* $\mathrm{Deg}\,(\varphi) \neq 0$, *where* $\varphi \in CJ(B^n(r), \mathbb{R}^n)$, *then there exists* $u \in B_0^n(r)$ *such that* $0 \in \varphi(u)$.

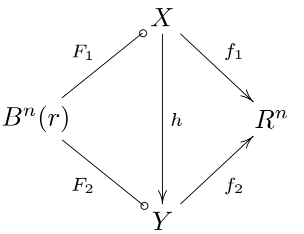
(iii) *If* $\varphi \in CJ(B^n(r), \mathbb{R}^n)$ *and* $\{u \in B^n(r) \mid 0 \in \varphi(u)\} \subset B_0^n(\widetilde{r})$ *for some* $0 < \widetilde{r} < r$, *then the restriction* $\widetilde{\varphi}$ *of* φ *to* $B^n(\widetilde{r})$ *is in* $CJ(B^n(\widetilde{r}), \mathbb{R}^n)$ *and* $\mathrm{Deg}\,(\widetilde{\varphi}) = \mathrm{Deg}\,(\varphi)$.

(iv) *Let* $\varphi_1, \varphi_2 \in CJ(B^n(r), \mathbb{R}^n)$ *be two maps of the form*

$$\varphi_1 = f_1 \circ F_1, \qquad B^n(r) \overset{F_1}{\multimap} X \xrightarrow{f_1} \mathbb{R}^n$$

$$\varphi_2 = f_2 \circ F_2, \qquad B^n(r) \overset{F_2}{\multimap} Y \xrightarrow{f_2} \mathbb{R}^n,$$

where $X, Y \in ANR$. *If there exists a continuous map* $h : X \to Y$ *such that the diagram*



is commutative, that is $F_2 = h \circ F_1$ *and* $f_1 = f_2 \circ h$, *then* $\mathrm{Deg}\,(\varphi_1) = \mathrm{Deg}\,(\varphi_2)$.

(v) *If* φ_1, φ_2 *are homotopic in* $CJ(B^n(r), \mathbb{R}^n)$, *then* $\mathrm{Deg}\,(\varphi_1) = \mathrm{Deg}\,(\varphi_2)$.

2 Construction of a random topological degree

For details concerning this section we recommend [8] where the present state of a random topological degree and a random periodic problem for differential inclusions is presented.

Let (Ω, Σ) be a measurable space and $\varphi : \Omega \multimap X$ be a multivalued mapping with nonempty values; φ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset A\} \in \Sigma$ for every closed A in a metric space X.

If X is a metric space we shall use the symbol $\mathcal{B}(X)$ to denote the *Borel σ-algebra* on X. Then $\Sigma \otimes \mathcal{B}(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathcal{B}(X)$. We say that a single valued map $v : X \to Y$ is a selection of $\varphi : X \multimap Y$ (written $v \subset \varphi$) provided $v(x) \in \varphi(x)$ for every $x \in X$.

The following lemma is crucial in what follows:

Lemma 3 ([8]). *Let X be a separable metric space, A be a closed subset of X and $\varphi : \Omega \times A \multimap X$ be a measurable mapping (with respect to the σ-algebra $\Sigma \otimes \mathcal{B}(A)$) with compact nonempty values. Assume further that for every $\omega \in \Omega$ the set $\operatorname{Fix} \varphi_\omega = \{x \in X \mid x \in \varphi(\omega, x)\}$ is compact and nonempty. Then the map $F : \Omega \multimap X$ defined by:*

$$F(\omega) = \operatorname{Fix} \varphi_\omega \qquad \textit{for every } \omega \in \Omega,$$

has a measurable selection, where $\varphi_\omega(x) = \varphi(\omega, x)$ for every $x \in A$.

Sketch of proof. First, let us define the function $f : \Omega \times A \to [0, +\infty)$ as follows:

$$f(\omega, x) = \operatorname{dist}(x, \varphi(\omega, x)) = \inf\{d(x, y) \mid y \in \varphi(\omega, x)\}$$

for every $\omega \in \Omega$ and $x \in A$. Since φ is measurable one can get that f is measurable.

Now, it is obvious that the graph

$$\Gamma_F = \{(\omega, x) \in \Omega \times X \mid x \in F(\omega)\}$$

of F is equal to

$$f^{-1}(0) = \{(\omega, x) \in \Omega \times A \mid f(\omega, x) = 0\}.$$

It implies that Γ_F is a measurable subset of $\Omega \times X$ and consequently by virtue of Aumann's selection theorem there exists a measurable selection of F. □

Definition 4. A multivalued map $\varphi : \Omega \times Y \multimap X$ with compact nonempty values is called a *random operator* provided φ is measurable and satisfies the following condition:

$$\varphi(\omega, \cdot\,) : Y \multimap X \text{ is u.s.c.for every } \omega \in \Omega. \tag{4}$$

Now, assume that $Y \subset X$ and $\varphi : \Omega \times Y \multimap X$ is a random operator. We say that φ has a *random fixed point* provided there exists a single valued measurable map $\xi : \Omega \to Y$, called the random fixed point of φ, such that:

$$\xi(\omega) \in \varphi(\omega, \xi(\omega)) \qquad \text{for every } \omega \in \Omega.$$

We let

$$\operatorname{Fix}^{ra}(\varphi) = \{\xi : \Omega \to Y \mid \xi \text{ is a random fixed point of } \varphi\}.$$

In view of Lemma 3 it is easy to see that in many cases existence of deterministic fixed points implies existence of random fixed points. Namely, we have:

Proposition 5. *Let X be a separable AR-space and $\varphi : \Omega \times X \multimap X$ be a random operator. Assume further that φ has R_δ-values and $\overline{\varphi_\omega(X)}$ is compact for every $\omega \in \Omega$. Then φ has a random fixed point.*

Sketch of proof. In fact, in view of Schauder Fixed Point Theorem (see [10]) we get that $\operatorname{Fix}\varphi_\omega$ is compact and nonempty. Then the map $F : \Omega \multimap X$, $F(\omega) = \operatorname{Fix}\varphi_\omega$ has a measurable selection $\xi \subset \mathcal{F}$ (comp. Lemma 3). Of course, $\xi \in \operatorname{Fix}^{ra}(\varphi)$. □

Note (comp. [4,10,11]) that Proposition 5 can be formulated in many other cases.

Below, we would like to show that the topological degree theory considered in Section 1 can be taken up onto the random case (see: [7,8]).

According to Section 1 we shall use the following notations.

For any $X \in ANR$ we let:

$$J^{ra}(\Omega \times B^n(r), X) = \{F : \Omega \times \mathcal{B}^n(r) \multimap X \mid F \text{ is random operator with } R^\delta\text{-values}\};$$

for any continuous $f : X \to R^n$ we let

$$J_f^{ra}(\Omega \times B^n(r), \mathbb{R}^n) = \{\varphi : \Omega \times B^n(r) \multimap \mathbb{R}^n \mid \varphi = f \circ F \text{ for some } F \in J^{ra}(\Omega \times B^n(r), X) \text{ and } \varphi(\Omega \times S^{n-1}(r) \subset \mathbb{P}^n\};$$

finally, we define

$$CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n) = \bigcup \{J_f^{ra}(\Omega \times B^n(r), \mathbb{R}^n) \mid f : X \to \mathbb{R}^n \text{ is continuous and } X \in ANR\}.$$

In the set $CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n)$ we can introduce the appropriate notion of homotopy (comp. Section 1 for deterministic case or [8]).

Now, observe that if $\varphi \in CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n)$, then $\varphi_\omega \in CJ(\Omega \times B^n(r), \mathbb{R}^n)$ for every $\omega \in \Omega$ and, consequently, topological degree $\operatorname{Deg}(\varphi_\omega)$ of φ_ω is well defined (see Section 1 or [7]). Therefore we are allowed to define:

Definition 6. We define a multivalued map $\mathcal{D} : CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n) \multimap \mathbb{Z}$ by putting

$$\mathcal{D}(\varphi) = \{\operatorname{Deg}(\varphi_\omega) \mid \omega \in \Omega\}.$$

Then the map $\mathcal{D}$ is called the *random topological degree* on $CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n)$; we say that the random topological degree $\mathcal{D}(\varphi)$ of φ is different from zero (written $\mathcal{D}(\varphi) \neq 0$) if and only if $\operatorname{Deg}(\varphi_\omega) \neq 0$ for every $\omega \in \Omega$.

Finally, let us remark that Theorem 2 holds true for random operators. We recommend also [6] and [10] for further possible consequences of the random topological degree.

3 The Poincaré operator

In this section we define the Poincaré translation map along trajectories of ordinary differential equations ([1,2,3,4,5,7,8,9,10,11,12]): A map $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is called *Carathéodory* if it satisfies:

$$t \to f(t,x) \text{ is measurable for every } x \in \mathbb{R}^n, \tag{5}$$
$$x \to f(t,x) \text{ is continuous for almost all } t \in [0,a], \tag{6}$$
$$\|f(t,x)\| \le \mu(t)(1 + \|x\|) \text{ for every } (t,x) \in [0,a] \times \mathbb{R}^n,$$
$$\text{where } \mu : [0,a] \to [0,+\infty) \text{ is an integrable function.} \tag{7}$$

For a Carathéodory map $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ we shall consider the following two problems:
Cauchy problem

$$\begin{cases} x'(t) = f(t,x(t)), \\ x(0) = x_0, \end{cases} \tag{8}$$

and
Periodic problem

$$\begin{cases} x'(t) = f(t,x(t)), \\ x(0) = x(a), \end{cases} \tag{9}$$

where a solution $x : [0,1] \to \mathbb{R}^n$ is an absolutely continuous map such that:

$$x'(t) = f(t,x(t)) \text{ for almost all } t \in [0,a].$$

For each $x_0 \in \mathbb{R}^n$ we denote by

$$S^f(x_0) = \{x : [0,1] \to R^n \mid x \text{ is a solution of (8)}\}$$

and by

$$P^f = \{x : [0,a] \to R^n \mid x \text{ is a solution of (9)}\}.$$

We recall the well known result so called Aronszajn Theorem (comp. [7,10,12]).

Theorem 7 (Aronszajn). *If $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map, then the map*

$$S_f : \mathbb{R}^n \multimap C([0,a], \mathbb{R}^n)$$

defined by $S_f(x) = S^f(x)$, for every $x \in R^n$, is an u.s.c. map with R_δ-values, where $C([0,a], \mathbb{R}^n)$ is a Banach space of continuous mappings with the usual max-norm.

Now, we wish to study problem (9). To do it we shall consider the diagram:

$$\mathbb{R}^n \overset{S_f}{\multimap} C([0,a], \mathbb{R}^n) \xrightarrow{e_t} \mathbb{R}^n,$$

where for every $t \in [0,a]$ the map e_t is defined by $e_t(x) = x(0) - x(t)$.

For any $t \in [0,a]$ the map $P_f^t = e_t \circ S_f$ is called the *Poincaré translation map* associated to the problem (8).

The following proposition is self-evident:

Proposition 8. *Problem* (9) *has a solution if and only if there is* $x \in \mathbb{R}^n$ *such that* $0 \in P_f^a(x)$.

In what follows we can assume, without loss of generality, that $0 \notin P_f^a(x)$ for every $x \in \mathbb{R}^n$ such that $\|x\| = r$ for some $r > 0$. Then we have:

Theorem 9. *If* $\mathrm{Deg}\,(P_f^a) \neq 0$*, then* $P^f \neq \emptyset$*, where we consider* P_f^a *as a mapping in* $CJ(B^n(r), \mathbb{R}^n)$ *for* r *given above.*

Proof. Since S_f is u.s.c. with R_δ-values and e_a is continuous, we have that

$$P_f^a \in CJ(B^n(r), \mathbb{R}^n), \text{ for } X = C([0,a], \mathbb{R}^n)$$

being an AR-space. □

Therefore by our assumption $\mathrm{Deg}\,(P_f^a)$ on $B^n(r)$ is well defined. Consequently, our result follows from Theorem 2, (ii) and Proposition 8.

In order to show that $\mathrm{Deg}\,(P_f^a) \neq 0$ we shall use the *guiding potential* introduced by Liapunov (comp. [16,17,18]) and subsequently developed by Krasnoselskĭ (comp. [13]), Mawhin [14] and others (see: [7,8,10,15]).

4 Guiding potentials

The guiding potential method has been recently employed in [10,15] to study periodic problems (9). Unlike these papers, where the guiding potential V is supposed to be C^1, here we assume that V is only locally Lipschitzean (see: [7]). We recall some notions of Clarke generalized gradient calculus [9] we shall need later.

Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitzean function. For $x_0 \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, let $V^0(x_0, v)$ be the generalized directional derivative of V at x_0 in the direction v, that is

$$V^0(x_0, v) = \limsup_{\substack{x \to x_0 \\ t \to 0+}} \frac{V(x + tv) - V(x_0)}{t}.$$

Then the *generalized gradient* $\partial V(x_0)$ of V at x_0 is defined by

$$\partial V(x_0) = \{y \in \mathbb{R}^n \mid \langle y, v\rangle \leq V^0(x_0, v) \text{ for every } v \in \mathbb{R}^n\}.$$

The map $\partial V : \mathbb{R}^n \multimap \mathbb{R}^n$ defined by the above equality is u.s.c. with nonempty compact convex values ([9, pp. 27, 29]). If, in addition, V is also convex, then $\partial V(x_0)$ coincides with the subdifferential of V at x_0 in the sense of convex analysis ([10, p. 36]), that is

$$\partial V(x_0) = \{y \in \mathbb{R}^n \mid \langle y, x - x_0\rangle \leq V(x) - V(x_0) \text{ for every } x \in \mathbb{R}^n\}.$$

If V is C^1, then $\partial V(x_0)$ reduces to the singleton set $\{\mathrm{grad}\, V(x_0)\}$ ([10, p. 33]).

Definition 10. Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitzean. If, for some $r_0 > 0$, V satisfies

$$\langle \partial V(x), \partial V(x)\rangle^- > 0 \qquad \text{for every } \|x\| \geq r_0, \tag{10}$$

then V is called a *direct potential*, where $\langle \partial V(x), \partial V(x)\rangle^- = \inf\{\langle u, v\rangle \mid u, v \in \partial V(x)\}$. If, in addition, V is convex and instead of (4.1) satisfies

$$0 \notin \partial V(x) \qquad \text{for every } \|x\| \geq r_0, \tag{11}$$

then V is called a *direct convex potential*.

Observe that Definition 10 implies (11), the converse is not true in general. Moreover, if V is C^1, then either (10) or (11) is equivalent to saying that $\operatorname{grad} V(x) \neq 0$ for every $\|x\| \geq r_0$. In view of that, the above definition can be interpreted as a generalization of the definition of a direct potential V in the sense of Krasnoselskiĭ [13] (see also [14,15,18]), where V is assumed to be a C^1 function.

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a direct potential. Observe that $\partial V \in CJ(B^n(r), \mathbb{R}^n)$ if $r \geq r_0$, and thus, by Theorem 2, the topological degree $\operatorname{Deg}(\partial V)$ is well defined and independent of r. Hence, it makes sense to define the index $\operatorname{Ind}(V)$ of the direct potential V, by putting

$$\operatorname{Ind}(V) = \operatorname{Deg}(\partial V),$$

when ∂V stands for the restriction of ∂V to $B^n(r), r \geq r_0$. Of course the definition of $\operatorname{Ind}(V)$ is analogous if V is direct convex potential.

Proposition 11 ([6]). *If $V : \mathbb{R}^n \to \mathbb{R}^n$ is a direct potential (or a direct convex potential) satisfying $\lim_{\|x\| \to +\infty} V(x) = +\infty$, then $\operatorname{Ind}(V) = 1$.*

A connection between the notion of direct potential and ordinary differential equations is given by the following:

Definition 12. Let $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map. A direct potential $V : \mathbb{R}^n \to \mathbb{R}$ is called a guiding potential for f if there exists $r_0 > 0$ such that:

$$\langle f(t, x), \partial V(x)\rangle^- \geq 0 \;\; \text{for every } t \;\text{ and }\; \|x\| \geq r_0. \tag{12}$$

Our main result of this section is the following:

Theorem 13. *Assume that $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map and $V : \mathbb{R}^n \to \mathbb{R}$ is a guiding potential for f such that $\operatorname{Ind}(V) \neq 0$. Then $P^f \neq \emptyset$.*

In the proof of Theorem 13 we proceed analogously as in the proof of (4.4) in [12] (comp. also [6]). By the homotopy property of the topological degree finally we obtain $\operatorname{Deg}(P^a_f) = |\operatorname{Ind}(V)|$ and therefore our result follows from Theorem 9.

All technical details are omitted here and we left them to the reader.

5 The random case

In this section we would like point out that all results of section 3 and 4 can be formulated in the random case. We shall restrict our considerations to showing formulations and some ideas of the proofs (we recommend for more details [8]).

Definition 14. Let $f : \Omega \times [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a single valued map such that

$$f(\,\cdot\,,\,\cdot\,,x) : \Omega \times [0,a] \to \mathbb{R}^n \text{ is measurable,} \tag{13}$$

$$f(\omega,t,\,\cdot\,) : \mathbb{R}^n \to \mathbb{R}^n \text{ is continuous,} \tag{14}$$

$$\begin{aligned}&\|f(\omega,t,x)\| \le \mu(\omega,t)(1+\|x\|) \text{ for every } t,\ \omega \text{ and } x, \text{ where}\\ &\mu\colon \Omega \times [0,a] \to [0,+\infty) \text{ is a map such that } \mu(\,\cdot\,,t) \text{ is mea-}\\ &\text{surable and } \mu(\omega,\,\cdot\,) \text{ is Lebesgue integrable.}\end{aligned} \tag{15}$$

Then f is called a random Carathéodory operator.

Now, for given random Carathéodory operator and a measurable map $\xi_0 : \Omega \to \mathbb{R}^n$ we consider the following Cauchy problem:

$$\begin{cases} x'(\omega,t) = f(\omega,t,x(\omega,t)) \\ x(\omega,0) = \xi_0(\omega), \end{cases} \tag{16}$$

where the solution $x : \Omega \times [0,a] \to \mathbb{R}^n$ is a map such that $x(\,\cdot\,,t)$ is measurable, $x(\omega,\,\cdot\,)$ is absolutely continuous and the derivative $x'(\omega,t)$ is considered with respect to t.

Moreover we consider the following *random periodic problem*:

$$\begin{cases} x'(\omega,t) = f(\omega,t,x(\omega,t)), \\ x(\omega,t) = x(\omega,a). \end{cases} \tag{17}$$

To solve it we need the random Poincaré translation operator. Observe that for every $\omega \in \Omega$ and $y \in \mathbb{R}^n$ we can consider the following Cauchy problem:

$$\begin{cases} x'(t) = f_\omega(t,x(t)) = f(\omega,t,x(t)), \\ x(0) = y. \end{cases} \tag{18}$$

Now, we define the following map:

$$\begin{aligned} &P : \Omega \times \mathbb{R}^n \multimap C([0,a],\mathbb{R}^n), \\ &P(\omega,y) = S^{f_\omega}(y). \end{aligned}$$

We have:

Proposition 15 ([8]). *Under the above assumptions the map P is a random operator.*

Note that for the proof of Proposition 15 we need the Fubini and Kuratowski Ryll-Nardzewski Selection Theorem (see [8]).

Consequently, the map

$$P_a = e_a \circ P \tag{19}$$

is called the random Poincaré operator associated to (17).

Now we are able to formulate Proposition 8, Theorem 9 and results of Section 4 for the random periodic problem. We refer to do it to the reader (comp. [8]).

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A New Finite Element Approach for Problems Containing Small Geometric Details

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Abstract. In this paper a new finite element approach is presented which allows the discretization of PDEs on domains containing small micro-structures with extremely few degrees of freedom. The applications of these so-called *Composite Finite Elements* are two-fold. They allow the efficient use of multi-grid methods to problems on complicated domains where, otherwise, it is not possible to obtain very coarse discretizations with standard finite elements. Furthermore, they provide a tool for discrete homogenization of PDEs without requiring periodicity of the data.

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Keywords. Finite Elements, Shortley-Weller discretization, complicated boundary

1 Introduction

Before the mid-sixties the finite difference method was the standard discretization method for differential equations. The following two severe drawbacks of finite differences lead to the development of the finite element method. First, the use of Cartesian difference quotients made the treatment of complicated and curved boundaries difficult and many technical tricks have to be employed to overcome this problem. Furthermore, it turned out that only the variational setting of the continuous problem leads to satisfactory existence and uniqueness results in appropriate function spaces, usually the convergence results of FDM require too much smoothness. Nowadays, we know that the question whether a discretization method is a FDM or a FEM is often only a matter of interpretation. In numerical linear algebra where one is interested in the algebraic properties of the linear system as, e.g., the M-matrix property, it is, in many case, very useful to interpret the discretization as a discrete, FD-like method while for the estimates of the discretization error the powerful apparatus of finite elements is employed.

An advantage, however, of FDM is the easy regular structure of the grid. Hence, the matrix pattern has a very regular structure, too. We know that this is very essential in the performance of iterative solvers as, e.g. ILU-like methods, while in an a priori unstructured FE mesh, sometimes, big effort has to be spent to find an advantageous numbering of the grid points. Furthermore, the simple structure of the matrix pattern makes the implementation of FDM much easier compared to FEM. Additionally, the efficient use of high performance computers as, e.g., vector computers, favorites such simple data structures.

On the other hand, the FEM has big advantages compared to FDM, namely, it provides a powerful apparatus for convergence analysis and is very flexible with respect to an appropriate geometric discretization of the domain allowing adaptive refinement strategies and proper resolution of the boundary.

However, the latter mentioned feature is true, only, if the grid size is small enough resolving essentially all micro-structures of the domain and differential equation. Very coarse discretizations (step size much larger than the geometric details) are not possible. In the context of homogenization and in order to apply multi-grid methods where the efficiency depends on how coarse the coarsest grid can be chosen this is a severe drawback. The Shortley-Weller FDM [14] which is in the literature since 1938 allows that the Cartesian grid overlap the boundary and appropriate weights are introduced in the difference quotients. The first multi-grid computations [3] use this discretization method in order to get very coarse coarse-grid approximations.

Since recently, various approaches have been presented in the literature concerning coarsening strategies for finite element spaces or, more general, discretizations with only few degrees of freedom which have already the asymptotic accuracy. In [1], [2], and [9], approaches are presented which can be used in the context of BPX-multigrid methods and hierarchical basis multigrid methods.

An approach which is based on pure algebraic considerations is the so-called *Algebraic Multigrid Method (AMG)* where only the information of the system matrix is used to obtain matrices of lower dimension. For details see [11]. A further related paper in this context is [10].

Composite Finite Elements were first presented by the authors in [8] and [6] where the aim was to define finite element spaces which have the asymptotic approximation property and the possibly low number of unknowns is independent of the shape of the domain. They can be used for both pure Galerkin discretization and in combination with standard multigrid methods and are not necessarily linked to a special solver.

In the present paper, we will, in the light of the Shortley-Weller discretization, define a new class of finite elements which is appropriate to resolve complicated geometries with very few degrees of freedom.

The paper is organized as follows. First, we recapitulate the principle of the Shortley-Weller method within an elementary setting. Then, we will introduce the new class of finite elements called *Composite Finite Elements* which resolves complicated boundaries with a very small number of degrees of freedom satisfying the usual asymptotic approximation property. We will show that the implementation of this method is very easy and the application to 3-d problems does not involve further difficulties compared to the 2-d version.

2 Shortley-Weller Finite Difference Discretization

In this section we recall the principles of the Shortley-Weller method for finite difference discretization of partial differential equations (PDEs) on domains having complicated boundary. The basic principles of this method will be used for the design of the new class of *Composite Finite Elements*.

In order to approximate the second derivative of a function u at a point $x \in \mathbf{R}$ using a non-uniformly spaced grid, Newton's divided second differences are employed

$$\begin{aligned} -u''(x) &\approx \frac{2}{h_1+h_2}\left(\frac{u(x)-u(x-h_1)}{h_1} - \frac{u(x+h_2)-u(x)}{h_2}\right) \\ &= -\frac{2}{(h_1+h_2)\,h_2}u(x+h_2) + \frac{2}{h_1 h_2}u(x) - \frac{2}{(h_1+h_2)\,h_1}u(x-h_1) \end{aligned}$$

Symbolically, the *matrix stencil* is given by

$$L_h = \left[-\frac{2}{(h_1+h_2)\,h_2}, \frac{2}{h_1 h_2}, -\frac{2}{(h_1+h_2)\,h_1}\right]. \tag{1}$$

The use of non-uniform spaced Cartesian grids for finite difference approximation is necessary if non-rectangular geometries as depicted in Figure 1 occur. A coarse

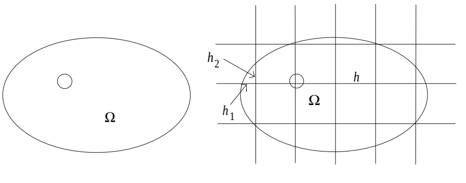

Fig. 1. Domain Ω with curved boundary and a small hole. The Cartesian grid does not fit in the domain and defines local stepsizes h_j near the boundary.

Cartesian grid will overlap the domain substantially. Instead of deforming the

Cartesian grid we use 2-d analogues of (1). The arising system matrix $\mathbf{L}_h$ has favorable properties. $\mathbf{L}_h$ is an M-matrix and has special stability properties (see [5, Theorem 4.8.4]) which can be expressed by

$$\left\|\mathbf{L}_h^{-1}\right\|_\infty \leq \frac{d^2}{8}.$$

However, difficulties arise if the micro-structures of the grid are not visible on the coarse grid. This would arise if, e.g., a hole lies in the interior of a grid cell and no Cartesian line of the grid intersects the hole. To overcome this problem we consider a hierarchy of Cartesian grids τ_ℓ of step size h_ℓ satisfying

$$h_0 = O(1) \approx \operatorname{diam}(\Omega),$$
$$h_\ell = 2^{-\ell} h_0.$$

We assume that $\ell_{\max}$ is such that $\tau_{\ell_{\max}}$ resolves all necessary details of the domain. Hence, the matrix $\mathbf{L}_{\ell_{\max}}$ can be generated by using the Shortley-Weller scheme. Matrices corresponding to coarser grids are then extracted from the fine grid matrix by introducing prolongations $p_{\ell\leftarrow\ell-1}$ and restrictions $r_{\ell-1\leftarrow\ell}$ linking grid functions on different grids $\tau_{\ell-1}$ and τ_ℓ with each other. Having defined these operators the coarse grid matrices are given recursively by the Galerkin product

$$\mathbf{L}_{\ell-1} = r_{\ell-1\leftarrow\ell} \mathbf{L}_\ell p_{\ell\leftarrow\ell-1}.$$

In standard cases, the prolongation and restriction can be defined, e.g., via interpolation in the following way. First, we consider the one-dimensional case which is illustrated in Figure 2.

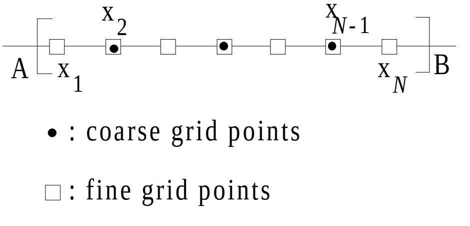

Fig. 2. Domain $\Omega = [A, B]$ with non-fitting fine and coarse grids.

The prolongation in the case of homogeneous Dirichlet boundary conditions is given for all fine grid points x_i by interpolating the neighbouring coarse grid values.

$$[p_{\ell\leftarrow\ell-1} u](x_i) = \begin{cases} u(x_i) & \text{if } x_i \text{ is also a coarse grid point,} \\ \frac{1}{2}\left(u(x_{i-1}) + u(x_{i+1})\right) & \text{otherwise and } i \neq 1, N, \\ \frac{\|x_1 - A\|}{\|x_2 - A\|} u(x_2) & i = 1, \\ \frac{\|x_{N-1} - B\|}{\|x_N - B\|} u(x_{N-1}) & i = N. \end{cases}$$

In more than one-dimension one has to interpolate sequentially in all directions. We state that in regular situation, i.e., in the case of domain-fitting grids, the prolongation is the bilinear interpolation. In any case the restriction $r_{\ell-1\leftarrow\ell}$ is defined as the adjoint of $p_{\ell\leftarrow\ell-1}$ with respect to the weighted Euclidean scalar product:

$$\langle u, v\rangle = \frac{1}{N}\sum_{i=1}^{N} u\left(x_i\right)\bar{v}\left(x_i\right).$$

An important feature of the prolongation and restriction above is that the sparsity of the system matrix is preserved and the regular distribution of the non-zero entries as well. If $\mathbf{L}_\ell$ is given by a 9-point stencil, i.e., 9 non-vanishing entries per matrix line, then, the same is true for $\mathbf{L}_{\ell-1}$.

Using these system matrices $\{\mathbf{L}_\ell\}_{0\leq\ell\leq\ell_{\max}}$ in a multi-grid method one observes the typical convergence rates even if the coarse grid contains only one degree of freedom and the domain contains many very small geometric details (cf. [3]).

The purpose of this section was to elucidate some key principles how very coarse discretizations of domains having complicated micro-structures can be obtained. The consideration was quite elementary but it will turn out that the principles can be used to define a new class of finite elements which includes the advantages of the Shortley-Weller FDM but can be applied to a much bigger class of problems.

3 Composite Finite Elements

In this section we will introduce so-called *Composite Finite Elements*. First, we will explain how grids can be generated such that geometric coarsening is straightforward. Then, the finite element spaces are defined on these coarsened grids as subspaces of the fine grid space by specifying appropriate inter-grid prolongations. The following considerations do not depend on the space dimension and hence are formulated in an abstract way.

We start recalling some basic definitions of finite element spaces. Let τ denote a partitioning of a domain Ω into small elements $\{K_j\}_{1\leq j\leq n}$. The finite element space V corresponding to this grid is defined as

$$V = \left\{u \in \mathcal{C}^k\left(\Omega\right) : u\mid_K \text{ is a polynomial of maximal degree } p \text{ for all } K \in \tau\right\}.$$

Let $\Theta = \{x_j\}_{1\leq j\leq N}$ denote the set of nodal points and $\{\Phi_i\}_{1\leq i\leq N}$ the corresponding Lagrangian nodal basis of V given by

$$\begin{aligned} &\Phi_i \in V, \\ &\Phi_i\left(x_j\right) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2}$$

Then, each function $u \in V$ has a unique basis representation by

$$u\left(x\right) = \sum_{i=1}^{N} \mathbf{u}_i \Phi_i\left(x\right) \tag{3}$$

with $\mathbf{u}_i = u(x_i)$. Equation (3) provides a canonical interpretation of a (discrete) vector of nodal values $\mathbf{u} \in \mathbf{R}^N$ as a finite element function.

In the following we will describe a method how a sequence of grids can be constructed such that geometric coarsening is straightforward.

3.1 Construction of the Grids and Definition of Composite Finite Elements

The following formal setting is illustrated in Figures 3-6.

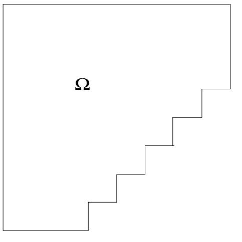

Fig. 3. Domain Ω containing a rough boundary piece.

First, we have to construct a sequence of auxiliary grids $\{\tilde{\tau}_\ell\}_{0\leq\ell\leq\ell_{\max}}$. Let Q_0 be a rectangle resp. a cuboid containing the domain Ω. Choose an arbitrary partitioning of Q_0 as the initial grid $\tilde{\tau}_0$. Refine $\tilde{\tau}_0$ for several times by any common refinement strategy as, e.g., combining the midpoint of triangles, the faces of hexahedrons, etc. to obtain a physically and logically nested sequence of grids $\{\tilde{\tau}_\ell\}_{0\leq\ell\leq\ell_{\max}}$.

This means that any element K of $\tilde{\tau}_\ell$ has a certain numbers of children given by

$$K' \in \tilde{\tau}_\ell \text{ is a } \textit{child} \text{ of } K \Leftrightarrow K' \subset K$$

and, vice versa, each element of $\tilde{\tau}_{\ell+1}$ has a uniquely determined parent in $\tilde{\tau}_\ell$. Note that the definition of $\tilde{\tau}_\ell$ does not include any adjustment process of the grids to the physical domain. However, in practical computations, one would generate grids $\tilde{\tau}_\ell$ which contain small elements in or near parts of Ω where a higher resolution is required. This can be done, e.g., by using error estimators or an a priori known grading function which controls the refinement strategy. We assume that $\tau_{\ell_{\max}}$ is fine enough such that nodal points lying close to the boundary of Ω can be moved onto the boundary without distorting the elements too much. Furthermore, we assume that there exists a subset of elements of the resulting grid which is a proper FE grid of the domain Ω. This mesh is denoted by $\tau_{\ell_{\max}}$. Note that the movement of grid points of $\tau_{\ell_{\max}}$ also is changing the shape of the elements on

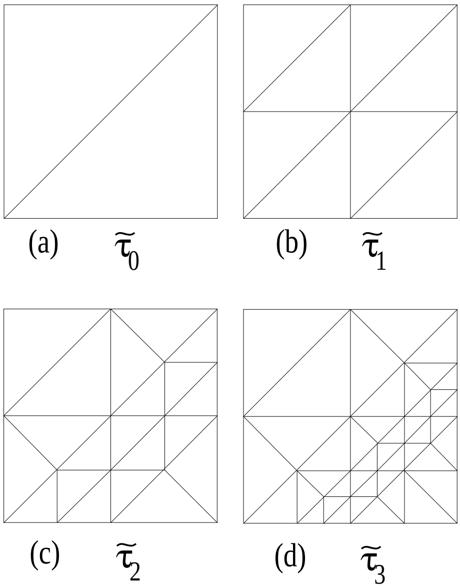

Fig. 4. Auxiliary grids $\{\tilde{\tau}_\ell\}_{0\leq\ell\leq3}$ which arise by refining a coarse grid with an appropriate refinement strategy. Note that no adjustment of the grid to the boundary of the domain takes place.

coarser levels. These distorted coarser grids are further reduced by cancelling all elements having zero cut with Ω. The resulting meshes are denoted by τ_ℓ.

The construction above implies that the elements are no longer physically nested. The situation, depicted in Figure 6, typically arise near the boundary where fine grid points have been moved.

Definition 1. An element $K \in \tau_\ell$ is said to be regular if the union of the (iterated) sons of K on the finest level is K.

Since $\tau_{\ell_{\max}}$ is a proper FE grid of Ω, the system matrix $\mathbf{L}_{\ell_{\max}}$ on this level is generated in the standard way. The coarser systems are defined recursively via the Galerkin product

$$\mathbf{L}_{\ell-1} = r_{\ell-1\leftarrow\ell}\mathbf{L}_\ell p_{\ell\leftarrow\ell-1}. \tag{4}$$

Since the restriction is again defined as the adjoint of $p_{\ell\leftarrow\ell-1}$ we have to specify only an appropriate choice of the inter-grid prolongation $p_{\ell\leftarrow\ell-1}$. This is done by using the interpretation (3) of a nodal vector as a grid function. A nodal vector $\mathbf{u}_\ell \in \mathbf{R}^{N_\ell}$ on level ℓ defines a continuous function u_ℓ by using the grid τ_ℓ and corresponding standard FE basis functions $\{\Phi_i^\ell\}_{1\leq i\leq N_\ell}$ (see (2)). The evaluation

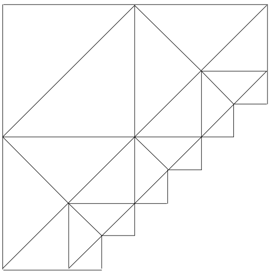

Fig. 5. Fine grid $\tau_{\ell_{\max}}$ with $\ell_{\max} = 3$. All triangles which lie outside of the domain are rejected. Note that in this example no movement of grid points was necessary.

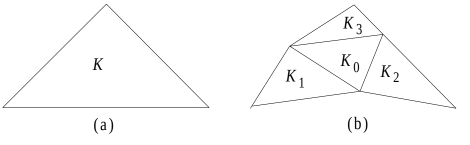

Fig. 6. Triangle K of τ_ℓ and logical children $\{K_i\}_{0\leq i\leq 3}$ of the finer level $\ell+1$.

of u_ℓ at the nodal points of the finer grid associates to any $\mathbf{u}_\ell \in \mathbf{R}^{N_\ell}$ a nodal vector $\mathbf{u}_{\ell+1} \in \mathbf{R}^{N_{\ell+1}}$. This defines the mapping $p_{\ell+1\leftarrow\ell} : \mathbf{R}^{N_\ell} \to \mathbf{R}^{N_\ell+1}$.

This prolongation can be interpreted as a convex interpolation in the following way. Let x be a nodal point of the grid $\tau_{\ell+1}$ which lies in a coarser element $K \in \tau_\ell$. Then, the prolonged nodal value at x is given by standard FE interpolation on K using the coarse-grid nodal values on K

$$\mathbf{u}_{\ell+1}(x) = \sum_{y\in\Theta_\ell\cap\bar{K}} \alpha_y(x)\,\mathbf{u}_\ell(y)$$

where $\alpha_y(x)$ are the coefficients of the FE interpolation.

In the case of homogeneous Dirichlet boundary conditions, we have to modify $p_{\ell\leftarrow\ell-1}$ such that $x \in \partial\Omega$ implies that $\mathbf{u}_{\ell+1}(x) = 0$ (see [12]).

The FE system matrices were generated recursively by using (4). Alternatively, it is possible to define a finite element space along with an appropriate basis such that the corresponding stiffness matrix equals $\mathbf{L}_\ell$. For this, let us consider the grid τ_ℓ and let x_j denote a nodal point of τ_ℓ. Define the unit nodal vector corresponding to this point by

$$\mathbf{e}_i = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Using the prolongation operators iteratively we can associate a fine grid nodal vector $\tilde{\mathbf{e}}_i$ with $\mathbf{e}_i$ by

$$\tilde{\mathbf{e}}_i = p_{\ell_{\max}\leftarrow\ell_{\max}-1} p_{\ell_{\max}-1\leftarrow\ell_{\max}-2} \cdots p_{\ell+1\leftarrow\ell} \mathbf{e}_i.$$

The finite element interpolation of the fine grid vector $\tilde{\mathbf{e}}_i$ links any $\mathbf{e}_i$ with a continuous function on Ω by

$$e_i(x) := \sum_{i=1}^{N_{\ell_{\max}}} \tilde{\mathbf{e}}_i \Phi_i^{\ell_{\max}}(x). \tag{5}$$

Note that e_i is a polynomial on each **fine**-grid element (provided $\Phi_i^{\ell_{\max}}$ are piecewise polynomials) while this is not true in general for the coarse grid elements. The *Composite Finite Element Space* is defined by

$$V_\ell = \operatorname{span}\{e_i(x) : 1 \leq i \leq N_\ell\}.$$

Remark 2. From the definition it follows that the Composite Finite Element Spaces are nested: $V_\ell \subset V_{\ell+1}$.

3.2 Approximation Property of V_ℓ

In many cases, the error analysis of Galerkin discretizations of PDEs leads to an estimate of the form

$$\|u - u_\ell\|_{H^1(\Omega)} \leq \left(1 + \frac{C_S}{\gamma}\right) \operatorname{dist}(u, V_\ell),$$

where u_ℓ denotes the solution of the Galerkin discretization and

$$\operatorname{dist}(u, V_\ell) := \inf_{w_\ell \in V_\ell} \|u - w_\ell\|_{H^1(\Omega)}.$$

The stability constant γ and continuity constant C_S mainly depend on the PDE on the continuous level. Obviously, the approximation property of the FE space, which is employed for the Galerkin discretization, plays a key role in the error estimate. In the following, we state that under relatively weak assumptions the asymptotic approximation property of finite elements carries over to composite finite element spaces independent of the (low) dimension of V_ℓ. The proof of the theorem was worked out in detail in [8] while more general situations as, e.g., the 3-d case and more general elements are treated in [13].

Theorem 3. *Let Ω be a 2-d domain with Lipschitz boundary, τ_ℓ denote a triangulation, and $h_\ell := \max_{\Delta\in\tau_\ell} \operatorname{diam}\Delta$ the step size of τ_ℓ. We assume that Φ_i of (2) are the piecewise linear "hat"-functions and*

(a) *τ_ℓ is quasi-uniform, i.e., $h_\ell \leq C \operatorname{diam}\Delta$, for all $\Delta \in \tau_\ell$,*

(b) τ_ℓ *is shape-regular, i.e.,* $\sup\{\operatorname{diam} S : S$ *is a ball contained in* $\Delta\} \geq Ch_\ell$ *for all* $\Delta \in \tau_\ell$*,*
(c) $h_{\ell+1} \leq \frac{2}{3} h_\ell$
(d) *the prolongation process is local, i.e.,* $\operatorname{diam}(\operatorname{supp} e_i) \leq Ch_\ell$ *with* e_i *given by* (5)*,*

with constants independent of ℓ *and* $\ell_{\max}$*.*
Then, for all $u \in H^2(\Omega)$ *there exists* $u_\ell \in V_\ell$ *such that*

$$\|u - u_\ell\|_{H^m(\Omega)} \leq Ch_\ell^{2-m} \|u\|_{H^2(\Omega)}, \qquad m \in \{0, 1\}. \tag{6}$$

Proof. The proof is essentially given in [8]. The only thing to check is that Assumption (d) above implies Assumption 2 in [8]. Since this is purely technical but straightforward we skip this detail here.

Hence, we have shown that V_ℓ has the asymptotic approximation property starting with extremely few degrees of freedom. In view of Figure 4(a), this means that the Galerkin discretization with composite finite elements on the grid τ_0 for the Poisson problem on Ω (cf. fig. 3) with Neumann boundary conditions satisfies

$$\|u - u_0\|_1 \leq Ch_0 \|u\|_2$$

with $h_0 = \operatorname{diam} \Omega$. The function u_0 is a function which lives only on the physical domain Ω, while the four degrees of freedom associated with u_0 are located at the corners of the square formed by the two coarse-grid triangles. Estimate (6) means that one is already in the asymptotic range, i.e., the error on the grid τ_1 is expected to be only half of the error of u_0.

Since the spaces V_ℓ are nested they are also well-suited to be used for defining coarse-grid approximations for multi-grid methods. The approximation property for multi-grid methods (cf. [4, Chapter 6]) directly follows from this fact.

3.3 Complexity of Composite Finite Elements

In this subsection we will investigate the complexity of generating the system matrix corresponding to the space V_ℓ. We recall that we assumed that the step sizes of the sequence of grids τ_ℓ satisfy

$$O(\operatorname{diam} \Omega) = h_0 > h_1 = \frac{h_0}{2} > h_2 = \frac{h_0}{4} > \ldots > h_\ell =: H > \ldots > h_{\ell_{\max}} =: h.$$

We assumed here for simplicity that the step size is reduced by a factor 2 in each step, while other contraction rates can be treated in the same way. If one is interested in the generation of the whole sequence of system matrices $\{\mathbf{L}_\ell\}_{0 \leq \ell \leq \ell_{\max}}$ one could use the Galerkin products. The complexity of generating the system matrix on the finest level is $O\left(h_{\ell_{\max}}^{-d}\right)$ where $d = 2, 3$ denotes the space dimension. Since the prolongation and restriction operators are local in the sense that the evaluation per nodal point requires $O(1)$ operations, we obtain that the generation of $\mathbf{L}_{\ell-1}$

from $\mathbf{L}_\ell$ needs $O\left(h_\ell^{-d}\right)$ operations. Together one obtains that the complexity of generating all system matrices is given by

$$\sum_{\ell=0}^{\ell_{\max}} h_\ell^{-d} = O\left(h_{\ell_{\max}}^{-d}\right),$$

i.e., does not increase the asymptotic complexity.

In some situations, however, one is interested only in the generation of a coarse-grid matrix $\mathbf{L}_\ell$ corresponding to a step size $H = h_\ell$ but would like to resolve the geometric details with a smaller step size $h = h_{\ell_{\max}}$. The following observation plays the key role. In the regular situation, where no grid points have been moved in the adjustment of the auxiliary fine grid $\tilde{\tau}_{\ell_{\max}}$ to the domain, the matrix $\mathbf{L}_\ell$ defined by the Galerkin product coincides with the matrix assembled directly on the grid τ_ℓ using the standard "coarse" finite element basis functions Φ_i^ℓ. Hence, the complexity of generating $\mathbf{L}_\ell$ is of order H^{-d}. This means that for elements $K \in \tau_m$ which are not distorted during the refinement process, i.e., are regular in the sense of Definition 1, the corresponding portions of $\mathbf{L}_\ell$, can be generated directly by using the standard FE basis function Φ_i^ℓ on K. Since the adjustment of elements to the boundary only takes place near the boundary nearly all elements are not distorted during the refinement process and there, the system matrix can be generated without prolonging up to the finest level $\ell_{\max}$.

In typical situation, only $O\left(h_m^{1-d}\right)$ elements of τ_m intersects the boundary of Ω and have to be refinement further. The computation of the so-called element matrix on an element $K \in \tau_m$ requires $O(1)$ operations. Symbolically, the algorithm reads as follows.

1. On $\tau_\ell : O\left(h_\ell^{-d}\right)$ elements are regular, i.e., not distorted on finer levels and the computation of the corresponding portions (element matrices) of $\mathbf{L}_\ell$ requires $O\left(h_\ell^{-d}\right)$ operations. $O\left(h_\ell^{1-d}\right)$ elements have to be refined further.
2. On $\tau_{\ell+1} : O\left(h_\ell^{1-d}\right)$ elements are involved. The computation of $O\left(h_\ell^{1-d}\right)$ corresponding portions of $\mathbf{L}_{\ell+1}$ needs $O\left(h_\ell^{1-d}\right)$ operations, while $O\left(h_{\ell+1}^{1-d}\right)$ elements have to be refinement further.

   $\vdots$
3. On $\tau_{\ell_{\max}} : O\left(h_{\ell_{\max}-1}^{1-d}\right)$ elements are involved. The computation of $O\left(h_{\ell_{\max}}^{1-d}\right)$ remaining element matrices needs $O\left(h_{\ell_{\max}-1}^{1-d}\right)$ operations.

The total operation count for generating $\mathbf{L}_\ell$ sums up to $O\left(H^{-d}\right) + O\left(h^{1-d}\right)$. A typical mesh which arise by this procedure is depicted in Figure 7. For a detailed study of the complexity of composite finite elements and implementation details, we refer to [7].

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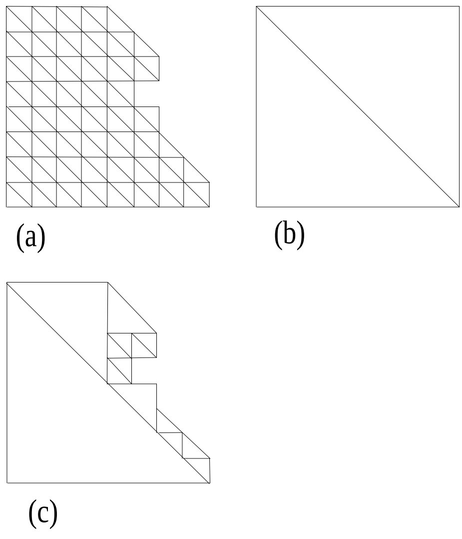

Fig. 7. Picture (a) shows the domain Ω together with the fine triangulation $\tau_{\ell_{\max}}$. In (b), the coarse triangulation τ_0 is depicted. Figure (c) shows the triangles corresponding to different grids which have to be generated in order to compute the entries of $\mathbf{L}_0$. Note that on any of the depicted triangles in (c) the usual FE basis functions are employed.

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On Small Solutions of Second Order Differential Equations with Random Coefficients

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Abstract. We consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \text{ if } t_{k-1} \le t < t_k, \text{ for } k = 1, 2, \dots ,$$

where $\{a_k\}$ is a given increasing sequence of positive numbers, and $\{t_k\}$ is chosen at random so that $\{t_k - t_{k-1}\}$ are totally independent random variables uniformly distributed on interval $[0, 1]$. We determine the probability of the event that all solutions of the equation tend to zero as $t \to \infty$.

AMS Subject Classification. 34F05, 34D20, 60K40

Keywords. Asymptotic stability, energy method, small solution

1 Introduction

The linear second order differential equation

$$x'' + a^2(t)x = 0 \tag{1}$$

describes the oscillation of a material point of unit mass under the action of the restoring force $-a^2(t)x$; function $a : [0, \infty) \to (0, \infty)$ is the square root of the varying elasticity coefficient a^2.

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Definition 1 (Ph. Hartman [8]). A function $t \mapsto x_0(t)$ existing and satisfying equation (1) on the interval $[0, \infty)$ is called a *small solution* of (1) if

$$\lim_{t\to\infty} x_0(t) = 0 \tag{2}$$

holds. The zero solution is called the *trivial small solution* of (1).

It is easy to see [10, p. 510] that if a is nondecreasing, then every solution of (1) is oscillatory and the successive amplitudes of the oscillation are decreasing. M. Biernacki [2] raised the question of the existence of a (nontrivial) solution whose amplitudes tend to zero, i.e., a small solution. H. Milloux answered this question by proving

Theorem A (H. Milloux [15]). *If $a : [0, \infty) \to (0, \infty)$ is differentiable, nondecreasing, and satisfies*

$$\lim_{t\to\infty} a(t) = \infty, \tag{3}$$

then equation (1) has a non-trivial small solution.

Milloux also provided an example of a step function a to show that one cannot conclude that *all* solutions are small.

Biernacki [2] raised also the following question: what additional conditions on a function a monotonously tending to infinity as t goes to infinity guarantee that all solutions are small? The first answer to this question was the famous Armellini-Tonelli-Sansone theorem (see, e.g., in [10]). It has been followed by many generalizations and improvements in the literature [3,9,10,13,14,16,17]. All of them require of the coefficient a to tend to infinity regularly. Roughly speaking this means that the growth of a cannot be located to a set with a small measure.

In this paper we are concerned with the case when the damping coefficient a in equation (1) is a step function. As is known such equations often serve as mathematical models in applications.

For example, let us consider the motion of the mathematical plain pendulum whose length changes by a given law $\ell = \ell(t)$. The position of the material point in the plain is described by the length $\ell(t)$ of the thread and the angle φ between the axis directed vertically downward and the thread. It is known [1,11] that the equation of the motion is

$$\varphi'' + \frac{g}{\ell(t)} \sin\varphi = 0, \tag{4}$$

where g denotes the constant of gravity. (No friction, the force of gravity acts only.) The "small oscillations" [1] are described by the linear second order differential equation

$$\varphi'' + \frac{g}{\ell(t)} \varphi = 0. \tag{5}$$

Consider the case when ℓ is a step function and $\ell(t) \to 0$ monotonously as $t \to \infty$. This is the situation when one has to lift a weight by a pulley and rope through a gape. The purpose is to guarantee $\lim_{t\to\infty} \varphi(t) = 0$.

In [12] the first author showed that the Milloux theorem can be generalized to step function coefficients, thus the existence of at least one solution with the desired property is guaranteed. However, this knowledge is useless from practical point of view. We would need a theorem guaranteeing *all* solutions to tend to zero as t goes to infinity. The Armellini-Tonelli-Sansone theorem cannot be applied because any step function can increase only irregularly: the growth of the function is located to a countable set, the function increases with jumps. Very recently the Armellini-Tonelli-Sansone theorem was generalized to impulsive systems [7] and step functions [5,6]. These theorems contain sophisticated conditions with requests of certain connections between different parameters of the step function coefficient. It is almost impossible to use these conditions for controlling the motions even if one can observe and measure the state variables during the motions, what, in general, cannot be assumed. (It is enough to mention the problem of pulling out used up graphit bars from a nuclear reactor, which can be modelled by equations similar to (5).) For this reason the first author [12] formulated the following practical problem: How many solutions are small if we do not require any additional condition on $\ell(t)$ beyond $\lim_{t\to\infty} \ell(t) = 0$? In other words, how often does it happen that $\lim_{t\to\infty} \varphi(t) = 0$?

To be more precise, let us suppose that the length $\ell(t)$ is of the form

$$\ell(t) := \ell_k, \qquad \text{if} \qquad t_{k-1} \le t < t_k,\ \ k = 1, 2, \dots ,$$

where $\{\ell_k\}_{k=1}^{\infty}$ is given, $\lim_{k\to\infty} \ell_k = 0$, and the sequence $\{t_k\}_{k=0}^{\infty}$ of the moments of pulling the rope is chosen "at random" such that $\lim_{k\to\infty} t_k = \infty$. *For an arbitrarily fixed pair of initial data φ_0, φ_0', what is the probability, that $\lim_{t\to\infty} \varphi(t) = 0$?*

In this paper we give an answer to this problem in the case when the differences $t_k - t_{k-1}$ $(k = 1, 2, \dots)$ are independent random variables uniformly distributed on interval $[0, 1]$. Namely, we prove that in this case $\lim_{t\to\infty} \varphi(t) = 0$ is almost sure (it is an event of probability 1).

2 Preliminaries and Results

Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers tending to infinity as k goes to infinity, and define $t_0 := 0$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that

$$0 < a_0 \le a_1 \le \dots \le a_k \le a_{k+1} \le \dots ,$$

and consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \ \text{ if } t_{k-1} \le t < t_k, \quad \text{for } k = 1, 2, \dots . \tag{6}$$

A function $x : [0, \infty) \to (-\infty, \infty)$ is a *solution of* (6) if it is continuously differentiable on $[0, \infty)$ and it solves the equation on every (t_{k-1}, t_k) for $k = 1, 2, \dots$.

Write (6) as a system of first order differential equations for a 2-dimensional vector (x, y), where $y := x'/a_k$. The resulting system is

$$x' = a_k y, \quad y' = -a_k x \qquad (t_{k-1} \le t < t_k; \quad k = 1, 2, \dots). \tag{7}$$

One has to be careful defining what it means that a function $t \mapsto (x(t), y(t))$ is a solution of (7) on the interval $[0, \infty)$. The function $t \mapsto x'(t) = a_k y(t)$ has to be continuous, so we require that the function $t \mapsto y(t)$ is continuous to the right for all $t \ge 0$ and satisfies $a_k y(t_k - 0) = a_{k+1} y(t_k)$ for $k = 1, 2, \dots$, where $y(t_k - 0)$ denotes the left-hand side limit of y at t_k. Accordingly, the system of first order differential equations for (x, y) equivalent with (6) is

$$\begin{aligned} &x' = a_k y, \quad y' = -a_k x \qquad (t_{k-1} \le t < t_k) \\ &y(t_k) = \frac{a_k}{a_{k+1}} y(t_k - 0), \qquad k = 1, 2, \dots . \end{aligned} \tag{8}$$

It is easy to see that introducing the polar coordinates (r, φ) by the equations $x = r \cos\varphi,\ y = r \sin\varphi$, we can rewrite system (7) into the form

$$r' = 0, \quad \varphi' = -a_k \qquad (t_{k-1} \le t < t_k, \quad k = 1, 2, \dots).$$

So, system (8) turns the plane uniformly around the origin for $t \in [t_{k-1}, t_k)$, and then contracts it along the y-axis by a_k/a_{k+1} at $t = t_k$. Introduce the notations

$$\tau_k := t_k - t_{k-1}, \quad \varphi_k := a_k \tau_k, \quad \alpha_k := \frac{a_k}{a_{k+1}},$$

$$T_k := \begin{pmatrix} 1 & 0 \\ 0 & \alpha_k \end{pmatrix} \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix}, \quad k = 1, 2, \dots; \qquad T_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then from (8) we obtain

$$\xi_k := \begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} = T_k T_{k-1} \dots T_2 T_1 \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \in \mathbb{R}^2, \quad k = 0, 1, 2, \dots . \tag{9}$$

Since $\alpha_k \le 1$, $k = 1, 2 \dots$, for every solution $t \mapsto (x(t), y(t))$ the limit

$$\omega := \lim_{t \to \infty} (x^2(t) + y^2(t)) = \lim_{k \to \infty} \|\xi_k\|^2 \tag{10}$$

exists and is finite, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

Suppose that $\tau_1, \tau_2, \dots, \tau_k, \dots$ *are totally independent random variables uniformly distributed on interval* $[0, 1]$. Limit ω is a function of the sequqnce $\{\tau_k\}_{k=1}^{\infty}$, so it is also random. Now we introduce the probability space where ω can be defined as a random variable.

For every natural number n, let $\mathcal{P}_n = (\Omega_n, \mathcal{A}_n, \mu_n)$ be the probability space with $\Omega_n := \prod_{k=1}^{n} [0, 1]$, the class $\mathcal{A}_n$ of Lebesgue measurable subsets of Ω_n, and the Lebesgue measure μ_n in Ω_n. By the Fundamental Theorem of Kolmogorov [4] there exists the infinite product probability space $\mathcal{P} = (\Omega := \prod_{k=1}^{\infty}[0,1], \mathcal{A}, \mu)$, having the following property:

$$\mu\Bigg(H \times \prod_{k=n+1}^{\infty}[0,1]\Bigg) = \mu_n(H) \quad \text{for every } H \in \mathcal{A}_n. \tag{11}$$

Limit ω defined by (10) is a random variable on probability space $\mathcal{P}$. Our purpose is to determine the probability

$$\mathbf{P}(\omega = 0 \text{ for all } \xi_0 \in \mathbb{R}^2).$$

Obviously, the event ($\omega = 0$ for all $\xi_0 \in \mathbb{R}^2$) is independent of the choices $\{\tau_k\}_{k=1}^{n}$ for every finite n. By Kolmogorov's Zero-Or-One Law, the probability of such an event equals either zero or one. The following theorems are in accordance with this law.

Theorem 2. *If* $\lim_{k\to\infty} a_k = \infty$, *then it is almost sure (i.e., it is an event of probability 1 in probability space* $\mathcal{P}$*) that*

$$\lim_{t\to\infty}\left(x^2(t) + \frac{(x'(t))^2}{a^2(t)}\right) = 0$$

for all solutions of equation (6)*.*

Corollary 3. *If* $\lim_{k\to\infty} a_k = \infty$, *then it is almost sure (i.e., it is an event of probability 1 in probability space* $\mathcal{P}$*) that*

$$\lim_{t\to\infty} x(t) = 0$$

for all solutions of equation (6)*.*

Theorem 4. *If* $\lim_{k\to\infty} a_k < \infty$, *then*

$$\lim_{t\to\infty}\left(x^2(t) + \frac{(x'(t))^2}{a^2(t)}\right) > 0$$

for every non-trivial solution x *of equation* (6)*.*

Corollary 5. *If* $\lim_{k\to\infty} a_k < \infty$, *then it is an impossible event in probability space* $\mathcal{P}$ *that there exists a non-trivial solution* x *of equation* (6) *with*

$$\lim_{t\to\infty} x(t) = 0.$$

3 Proofs

3.1 Proof of Theorem 2

Let $(x(0), y(0)) \in \mathbb{R}^2$ be fixed, and consider the solution of equation (8) starting from this point. If $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$, then for a fixed $k \geq 1$ we have

$$\|\xi_k\|^2 = \langle\xi_k, \xi_k\rangle = \langle T_k\xi_{k-1}, T_k\xi_{k-1}\rangle = \langle T_k^* T_k\xi_{k-1}, \xi_{k-1}\rangle \leq \Lambda_k\|\xi_{k-1}\|^2,$$

where T_k^* denotes the transposed of matrix T_k, and Λ_k denotes the greater eigenvalue of the symmetric matrix $T_k^*T_k$. The random variables $\xi_1, \xi_2, \dots, \xi_k$ are independent; consequently, for the expected values we obtain the inequality

$$\mathbf{E}(\|\xi_k\|^2) \leq \mathbf{E}(\Lambda_k)\mathbf{E}(\|\xi_{k-1}\|^2). \tag{12}$$

Now we compute $\mathbf{E}(\Lambda_k)$. First we determine the expected value of matrix $T_k^*T_k$:

$$\begin{aligned}
\mathbf{E}(T_k^*T_k) &= \int_0^1 \begin{pmatrix} \cos a_k\tau & -\sin a_k\tau \\ \sin a_k\tau & \cos a_k\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_k^2 \end{pmatrix} \begin{pmatrix} \cos a_k\tau & \sin a_k\tau \\ -\sin a_k\tau & \cos a_k\tau \end{pmatrix} d\tau \\
&= \int_0^1 \cos^2 a_k\tau d\tau \begin{pmatrix} 1 & 0 \\ 0 & \alpha_k^2 \end{pmatrix} + \int_0^1 \sin^2 a_k\tau d\tau \begin{pmatrix} \alpha_k^2 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + \int_0^1 \sin a_k\tau \cos a_k\tau d\tau \begin{pmatrix} 0 & \alpha_k^2 - 1 \\ \alpha_k^2 - 1 & 0 \end{pmatrix} \\
&= \frac{1+\alpha_k^2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin 2a_k}{4a_k}(1-\alpha_k^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\sin^2 a_k}{2a_k}(\alpha_k^2 - 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

It is easy to check that the greater eigenvalue of a symmetric matrix $(d_{ik})_{i,k=1}^2$ is determined by the formula

$$\frac{d_{11} + d_{22} + \sqrt{(d_{11} - d_{22})^2 + (2d_{12})^2}}{2}.$$

Λ_k is the greater eigenvalue of matrix $\mathbf{E}(T_k^*T_k)$; therefore,

$$\Lambda_k = \frac{1}{2}\left(1 + \alpha_k^2 + (1-\alpha_k^2)\left|\frac{\sin a_k}{a_k}\right|\right). \tag{13}$$

Applying inequality (12) for $k = 1, 2, \dots$ we obtain the estimate

$$\mathbf{E}(\|\xi_n\|^2) \leq \left(\prod_{k=1}^n \Lambda_k\right)\|\xi_0\|^2. \tag{14}$$

Now we prove

$$\prod_{k=1}^{\infty} \Lambda_k = 0. \tag{15}$$

This assertion is equivalent with

$$\sum_{k=1}^{\infty} \ln \left[1 - \frac{1-\alpha_k^2}{2} \left(1 - \left| \frac{\sin a_k}{a_k} \right| \right) \right] = -\infty.$$

This is obviously satisfied if $\liminf_{k\to\infty} \alpha_k < 1$. If $\lim_{k\to\infty} \alpha_k = 1$, then it is enough to show that

$$\sum_{k=1}^{\infty} (1-\alpha_k^2) \left(1 - \left| \frac{\sin a_k}{a_k} \right| \right) = \infty,$$

i.e., $\sum_{k=1}^{\infty}(1-\alpha_k) = \infty$. But this is equivalent with $\sum_{k=1}^{\infty} \ln \alpha_k = -\infty$, i.e.,

$$\lim_{n\to\infty} \prod_{k=1}^{n} \alpha_k = \lim_{n\to\infty} \frac{a_0}{a_{n+1}} = 0,$$

which was assumed.

From (14) and (15) it follows that $\lim_{n\to\infty} \mathbf{E}(\|\xi_n\|^2) = 0$. Then by Fatou's Lemma [4] and property (11) we have

$$\mathbf{E}(\omega) = \mathbf{E}(\lim_{n\to\infty}(\|\xi_n\|^2) = \int_{\Omega} \lim_{n\to\infty}(\|\xi_n\|^2\, d\mu \le \lim_{n\to\infty} \int_{\Omega} \|\xi_n\|^2\, d\mu$$
$$= \lim_{n\to\infty} \int_{\Omega_n} \|\xi_n\|^2\, d\mu_n = \lim_{n\to\infty} \mathbf{E}(\|\xi_n\|^2) = 0.$$

We have proved that for every fixed individual solution of (6) there holds $\mathbf{P}(\omega = 0)$. Since all solutions of the linear equation (6) can be represented as linear combinations of two fixed linearly independent solutions of the equation, this implies that

$$\mathbf{P}(\omega = 0 \text{ for all solutions of } (6)) = 0,$$

which completes the proof of Theorem 2.

3.2 Proof of Theorem 4

Suppose that $\lim_{n\to\infty} a_n =: a_\infty < \infty$. From the representation (9) and the definition of T_k we have

$$\|\xi_k\|^2 = \langle \xi_k, \xi_k \rangle = \langle T_k\xi_{k-1}, T_k\xi_{k-1} \rangle = \langle T_k^* T_k \xi_{k-1}, \xi_{k-1} \rangle \ge \alpha_k^2 \|\xi_k\|^2.$$

Iterating this estimate we obtain the inequality

$$\omega = \lim_{n\to\infty} \|\xi_n\|^2 \ge \left(\lim_{n\to\infty} \prod_{k=1}^{n} \alpha_k^2 \right) \|\xi_0\|^2 = \left(\lim_{n\to\infty} \frac{a_0^2}{a_{n+1}^2} \right) \|\xi_0\|^2 = \frac{a_0^2}{a_\infty^2} \|\xi_0\|^2 > 0,$$

whenever $\|\xi_0\|^2 > 0$. This completes the proof.

The proofs of Corollaries 3 and 5 are trivial, so they are omitted.

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Application of the Second Lyapunov Method to Stability Investigation of Differential Equations with Deviations

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Abstract. The paper presents overview of applications of A. M. Lyapunov's direct method to stability investigation of systems with argument delay. Methods of building Lyapunov-Krasovskiy funcionals for linear systems with constant coefficients are considered. Lyapunov quadratic forms are used to obtain applicable methods for stability investigation and estimation of solution convergence for linear stationary systems, as well as non-linear control systems and systems with quadratic and rational right hand sides.

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1 Introduction

The present paper is aimed at investigation of systems with deviating argument of delay type. The investigation is carried out using the second Lyapunov method. The following differential system with delay is considered

$$\dot{x}(t) = f(x(t), x(t-\tau)), \quad \tau > 0 \ . \tag{1}$$

Suppose that $x(t) \equiv 0$ is a solution of system (1), i.e. $f(0,0) \equiv 0$.

As opposed to ODE's, for which the Cauchy problem consists of finding a solution passing through the given point, equations with delay have an initial function. Thus, for (1) the Cauchy problem consists of finding a solution $x(t)$ that satisfies the initial condition $x(t) \equiv \varphi(t), \quad -\tau \leq t \leq 0$, where $\varphi(t)$ is a given initial function. Therefore, initial perturbations of the function $\varphi(t)$, $-\tau \leq t \leq 0$ are required to be small according to the definition of stability.

Definition 1. The solution $x(t) \equiv 0$ of system (1) is called stable according to Lyapunov if for an arbitrary $\varepsilon > 0$ there exists such $\delta(\varepsilon) > 0$ that $|x(t)| < \varepsilon$ when $t > 0$ if $\|x(0)\|_\tau < \delta(\varepsilon)$. Here $\|x(0)\|_\tau = \max_{-\tau \leq s \leq 0} \{|x(s)|\}$.

Definition 2. The solution $x(t) \equiv 0$ is called asymptotically stable if it is stable and the following condition holds

$$\lim_{t \to \infty} |x(t)| = 0 \ .$$

Definition 3. The solution $x(t) \equiv 0$ is exponentially stable if there exist such constants $N > 0$ and $\gamma > 0$ that for an arbitrary solution of the system the following estimate holds

$$|x(t)| \leq N \|x(0)\|_\tau \exp\{-\gamma t\}, \quad t \geq 0 \ .$$

System (1) cannot provide precise description of real objects. By using differential equations it is usually impossible to take into account all different factors that influence the system. Therefore, it is appropriate to consider a perturbed system in the form

$$\dot{x}(t) = f(x(t), x(t-\tau)) + q(x(t), x(t-\tau)) \ . \tag{2}$$

The following definitions of stability account for the influence of perturbation.

Definition 4. The solution $x(t) \equiv 0$ of system (1) is called stable under constantly acting perturbations when for an arbitrary $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ such that for an arbitrary solution $x_Q(t)$ of (2) the condition $|x_Q(t)| < \varepsilon$ when $t > 0$ holds if $\|x_Q(0)\|_\tau < \delta(\varepsilon)$ and $|q(x_Q(t), x_Q(t-\tau))| < \eta(\varepsilon)$.

Differential equations with delay (1) have many things in common with corresponding equations without delay. Therefore, many results from the movement stability theory for systems without delay were extended and adjusted to the equations in the form (1). One of the basic methods for investigation of system stability is the second Lyapunov method. Its application to systems with delay has been developed in two directions:

1. The first direction implies use of finite dimensional functions with an additional condition for the derivative. This is a so called B. S. Razumikhin condition [1,4].

2. The second method is a Lyapunov-Krasovskiy functional method, which has had more comprehensive theoretical ground [2,3,4].

Geometrical meaning of the Lyapunov function method involves finding the system of closed surfaces that contain the origin and are converging to it. The vector field of motion equations should be directed inside the areas limited by such surfaces. If a solution gets into such area limited by the surface, then it will never leave it again. These surfaces form level surfaces of a Lyapunov function.

For systems without argument deviation the speed vector on level surfaces is determined only by the present moment of time, i.e. by the point lying on the given surface. The speed in equations with deviating argument depends on the previous history as well; i.e. it depends on the point $x(t-\tau)$, which is usually hard to find. Therefore, it is logical to require negative definiteness of Lyapunov function derivative uniformly by the variable $x(t-\tau)$. However, this leads to an excessively sufficient character of the theorems, which in turn makes them inefficient for applications. Because of this, B. S. Razumikhin suggested to consider a previous history $x(t-\tau)$ to lie inside the level surface $v(x,t)=\alpha$ in order to be able to estimate the full derivative along system solutions. The standard technique of proving Lyapunov theorems on stability made such assumption both natural and logical. This led to an additional Razumikhin condition for the Lyapunov theorems, which included estimation of the character of Lyapunov function derivative on the curve that satisfies [1]

$$v(s,x(s)) < v(t,x(t)), \quad s<t.$$

The second approach was introduced by N. N. Krasovskiy. He suggested to consider sections $x(t+s)$, $-\tau \le s \le 0$ of the trajectory at each fixed time $t>0$ instead of functions with finite number of variables. Definitions of positive definiteness of corresponding functionals and of their derivatives on system solutions were introduced as well. Main Lyapunov theorems on stability (as well as asymptotic and exponential stability) were stated in terms of functionals and their derivatives [2].

Both methods are thought to have certain advantages and disadvantages. However, both methods have capacity for existence and further development according to opinions of many scientists.

2 Lyapunov-Krasovskiy Functional Method

Let us consider the basic idea of Lyapunov-Krasovskiy functional method. Denote vector-function defined on the interval $-\tau \le s \le 0$ for each fixed $t>0$ by $x(t+s)$. The functional $V[x(t),t]$ is determined on the vector-functions $x(t+s)$, $-\tau \le s \le 0$. Using introduced functionals N. N. Krasovskiy obtained theorems on stability and asymptotic stability of zero solution of system (1) with delay, which was analogous to the well known Lyapunov theorems.

In the theorems on stability (asymptotic stability, unstability) stated in terms of Lyapunov-Krasovskiy functional the following value (called right upper derivative number)

$$\bar{D}_+ V = \lim_{\Delta t \to +0} \sup \frac{1}{\Delta t} \left\{ V[x(t+\Delta t), t+\Delta t] - V[x(t),t] \right\}$$

played role of a function derivative dv/dt along solutions $x(t)$ of a system with delay.

We should draw our attention to the two steps in development of the Lyapunov-Krasovskiy functional method. The first step included development of a theoretical ground for the method. The second step used theoretical results to make theorems more applicable to construction of the functionals. Let us consider these two stages in more details.

The first step was to formulate theorems on stability and asymptotic stability, and invert them. All conditions of the theorems were formulated in terms of a uniform norm

$$\|x(t)\|_\tau = \sup_{-\tau \le s \le 0} \{|x(t+s)|\}$$

for a zero solution of system (1) with delay, which was similar to the well known Lyapunov's theorem.

The main results are as follows

Theorem 5 (Stability by Lyapunov). *Let differential equations of system* (1) *be such that there exists a functional* $V[x(t), t]$ *satisfying the following conditions:*

1. $a(\|x(t)\|_\tau) \le V[x(t), t]$,
2. $\bar{D}_+ V[x(t), t] \le 0$.

Here $a(r)$ *is a continuous non-decreasing function positive for all* $r > 0$ *and* $a(0) = 0$. *Then the zero solution* $x(t) \equiv 0$ *of system* (1) *is stable according to Lyapunov's definition.*

Theorem 6 (Asymptotic stability). *Let differential equations of system* (1) *be such that there exists a functional* $V[x(t), t]$ *satisfying the following conditions:*

1. $a(\|x(t)\|_\tau) \le V[x(t), t] \le b(\|x(t)\|_\tau)$,
2. $\bar{D}_+ V[x(t), t] \le -c(\|x(t)\|_\tau)$.

Here $a(r), b(r), c(r)$ *are continuous non-decreasing functions positive for all* $r > 0$ *and equal to zero at* $r = 0$. *Then the zero solution* $x(t) \equiv 0$ *of system* (1) *is asymptotically stable.*

It should be noted that the conditions of the above formulated theorems use the uniform metric, which essentially limits the number of differential systems for which functionals can be constructed in an explicit form. For example, for a linear stationary system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{3}$$

with constant matrices A and B and a functional in a quadratic form

$$V[x(t)] = x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t+s)Gx(t+s)ds,$$

where H, G are constant positive definite matrices it is impossible to find functions $a(r)$ and $c(r)$ that would satisfy theorem's conditions.

Therefore, the second step formulated stability theorems in terms of such norms, that are more convenient for constructing the functionals.

Theorem 7 (Asymptotic stability). *Let differential equations of system (1) be such that there exists a functional $V[x(t), t]$ satisfying the following conditions:*

1. $a(|x(t)|) \leq V[x(t), t] \leq b(\|x(t)\|_\tau)$,
2. $\bar{D}_+ V[x(t), t] \leq -c(|x(t)|)$.

Then the zero solution $x(t) \equiv 0$ of system (1) is asymptotically stable.

2.1 Quadratic Functionals in a General Form

Let us consider constructive methods for construction of Lyapunov-Krasovskiy functionals for linear stationary systems with delay (3). It is obvious that the natural form of a functional is a quadratic one, the same as for systems without delay. Yu. M. Repin constructed quadratic functionals in the following general form [5]

$$\begin{aligned} V[x(t)] = x^T(t)Hx(t) &+ \int_{-\tau}^{0} x^T(t+s)K(s)x(t)dt \\ &+ \int_{-\tau}^{0} x^T(t+s)G(s)x(t+s)ds \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s_1)M(s_1,s_2)x(t+s_2)ds_1 ds_2 \ . \end{aligned} \tag{4}$$

Here H is a constant quadratic $n \times n$ positive definite matrix; $K(s)$, $G(s)$, $M(s_1, s_2)$ are continuous matrices, and H and $M(s_1, s_2)$ are symmetric matrices. Functionals are chosen in such a way that

$$\frac{d}{dt}V[x(t)] = W[x(t)],$$

where

$$\begin{aligned} W[x(t)] = x^T(t)Qx(t) &+ x^T(t-\tau)Rx(t) + x^T(t-\tau)Sx(t-\tau) \\ &+ \int_{-\tau}^{0} x^T(t+s)D(s)x(t)ds + \int_{-\tau}^{0} x^T(t+s)E(s)x(t+s)ds \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s_1)F(s_1,s_2)x(t+s_2)ds_1 ds_2 \end{aligned} \tag{5}$$

for given matrices $Q, R, S, D(s), E(s), F(s_1, s_2)$. These matrices satisfy conditions ensuring negative definiteness of $W[x(s)]$ on system's solutions.

By taking a derivative of the functional (4) we obtain a system of algebraic equations that consists of ordinary matrix differential equations and partial differential equations

$$HA + A^T H + \frac{1}{2}[K(0) + K^T(0)] + G(0) = Q\,,$$

$$A^T K(s) - \frac{d}{ds} K(s) + M(s,0) = D(s), \quad -\frac{d}{ds} G(s) = E(s),$$

$$\frac{\partial M(s_1,s_2)}{\partial s_1} + \frac{\partial M(s_1,s_2)}{\partial s_2} = -F(s_1,s_2), \tag{6}$$

$$2HB - K(-\tau) = R, \quad B^T K(s) - M(-\tau, s) = 0 \ .$$

In some cases solutions of system (3) can be found, however in a general case the question of existence of a solution for such system cannot be addressed.

Simplified quadratic functional was proposed in the form [6]

$$V[x(t)] = x^T(t) H(0) x(t) + 2x^T(t) \int_{t-\tau}^{t} H(s-t+\tau) B x(s) ds$$
$$+ \int_{t-\tau}^{t} \int_{t-\tau}^{t} x^T(s_1) B^T H(s_2 - s_1) B x(s_2) ds_1 ds_2 \ .$$

Theorem 8. *Let there exist a matrix function $H(t)$, a solution of the matrix differential equation*

$$\ddot{H}(t) = A^T \dot{H}(t) - \dot{H}(t) A + A^T H(t) A - B^T H(t) B, \quad t \geq 0,$$

and let it satisfy

1. $\dot{H}(t) = A^T H(t) + B^T H(t-\tau), \quad t \geq 0,$
2. $H(t) = H^T(-t), \quad H(0) = H^T(0),$
3. $A^T H(0) + H(0) A + B^T H^T(\tau) + H(\tau) B = -C\,,$

where C is a positive definite matrix. If $H(t)$ is such that the functional $V[x(t)]$ satisfies bilateral estimates

$$a(|x(t)|) \leq V[x(t)] \leq b_1(|x(t)|) + b_2(\|x(t)\|_\tau),$$

then the system is asymptotically stable.

The important fact about this theorem is that the theorem can be reversed.

Theorem 9. *Let a linear system with a delay be asymptotically stable. Then there exists a quadratic functional $V[x(t)]$. Let a matrix function $H(t)$ be a solution of the ordinary differential equation*

$$\ddot{H}(t) = A^T \dot{H}(t) - \dot{H}(t) A + A^T H(t) A - B^T H(t) B, \quad t \geq 0,$$

and let it satisfy

1. $\dot{H}(t) = A^T H(t) + B^T H(t-\tau), \quad t \geq 0,$
2. $H(t) = H^T(-t), \quad H(0) = H^T(0),$

3. $A^T H(0) + H(0)A + B^T H^T(\tau) + H(\tau)B = -C\,,$

where C is a positive definite matrix. Then on solutions $x(t)$ of the system the functional $V[x(t)]$ satisfies bilateral estimates

$$a(|x(t)|) \le V[x(t)] \le b_1(|x(t)|) + b_2(\|x(t)\|_\tau),$$

and its full derivative satisfies

$$\dot{V}[x(t)] \le -\lambda_{\min}(C)|x(t)|^2 \ .$$

If we consider a functional in the form

$$V[x(t)] = x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t+s)Gx(t+s)ds,$$

then for an asymptotic stability of system (3) it is sufficient that such positive matrices H and G exist that the matrix

$$C[G,H] = \begin{bmatrix} -A^T H - HA - G & -HB \\ -B^T H & G \end{bmatrix}$$

is also positive definite.

Let us transform the problem of finding matrices H and G into an optimizational problem [7,8]

$$(G_0, H_0) = \arg \inf_{(G,H)\in \bar{L}_G^1 \times \bar{L}_H^1} \{\varphi_0(G,H)\},$$

where $\varphi_0(G,H) = -\lambda_{\min}[C(G,H)]$, $\lambda_{\min}(\bullet)$ is minimal eigenvalue of the matrix $C[G,H]$; $\bar{L}_G^1, \bar{L}_H^1$ are sets of positive definite matrices G and H that lie within a unit circle.

The Lagrange function is constructed in the form

$$\begin{aligned} \mathcal{L}(G,H,u) =& \varphi_0(G,H) + u_1\varphi_1(G) + u_2\varphi_2(G) + u_3\varphi_3(H) \\ & + u_4\varphi_4(H), \quad u_i \ge 0, i = \overline{1,4}; \\ \varphi_1(G) =& \lambda_{\max}(G) - 1, \quad \varphi_2(G) = -\lambda_{\min}(G), \\ \varphi_3(H) =& \lambda_{\max}(H) - 1, \quad \varphi_4(H) = -\lambda_{\min}(H) \ . \end{aligned}$$

Theorem 10. *For a function $\varphi_0(G,H)$ to reach its minimal value, it is necessary and sufficient for the point (G_0, H_0, u_0), $u_0^T = (u_1^0, u_2^0, u_3^0, u_4^0)$ to be a saddle point of the Lagrange function.*

The following theorem provides constructive conditions for finding matrices G_0 and H_0 such that the Lyapunov-Krasovskiy functional from a given class resolves a stability question.

Theorem 11. *The Lyapunov-Krasovskiy functional with matrices G_0, H_0 resolves a problem of stability within a given class of functionals (i.e. it is the optimal functional in a given class) if and only if the vector $u_0^T = (u_1^0, u_2^0, u_3^0, u_4^0)$ exists such that*

1. *A gradient set R_L^0 of the Lagrange function $\mathcal{L}(G, H, u)$ on variables (G, H) at the point G_0, H_0, u_0 contains a pair of zero matrices, i.e. $(\theta, \theta) \in R_L^0$.*
2. *Conditions of additional non-stiffness hold:*

$$u_1^0 \varphi_1(G_0) = 0, \quad u_2^0 \varphi_2(G_0) = 0, \quad u_3^0 \varphi_3(H_0) = 0, \quad u_4^0 \varphi_4(H_0) = 0 \ .$$

3 Lyapunov Function Method with Razumikhin Condition

Proofs of main Lyapunov's theorems are based on estimate of a speed vector direction at the moment $x(t)$ on level surfaces $v(x, t) = \alpha$ of the Lyapunov function $v(x, t)$. In other words, the sign of $\dot{v}(x, t)$ is studied, where

$$\frac{dv(x(t), t)}{dt} = \frac{\partial v(x(t), t)}{\partial t} + \mathrm{grad}_x^T v(x(t), t) f(x(t), x(t - \tau)) \ . \tag{7}$$

For systems with argument deviation this expression is a functional that depends on the previous history $x(t - \tau)$. On the basis of the stability definition we can assume that points lie inside the area limited by level surfaces before points of the previous history leave the level surfaces. In other words, the condition $v(x(t - \tau), t - \tau) < v(x(t), t)$ holds.

B. S. Razumikhin proposed to find the estimate of functional (7) not for all curves that correspond to solutions $x(t)$ of the system, but only for those that leave areas limited by level surfaces, i.e. $v(x(s), s) < v(x(t), t), \quad s < t$.

Theorem 12. *Let for system (1) a continuously differentiable function $v(x, t)$ exist and satisfy the conditions:*

1. $a(|x|) \leq v(x, t)$,
2. $\frac{dv(x(t))}{dt} \leq 0$ *for curves $x(t)$ that satisfy $v(x(s), s) < v(x(t), t), \quad s < t$.*

Here $a(r)$ is a continuous non-decreasing function positive for all $r > 0$ and $a(0) = 0$. Then the zero solution $x(t) \equiv 0$ of the system (1) is stable according to Lyapunov.

Theorem 13. *Let for the system (1) a continuously differentiable function $v(x, t)$ exist and satisfy the conditions:*

1. $a(|x|) \leq v(x, t) \leq b(|x|)$,
2. $\frac{dv(x(t))}{dt} \leq -c(|x(t)|)$ *for curves $x(t)$ that satisfy $v(x(s), s) < v(x(t), t), \quad s < t$.*

Here $a(r)$, $b(r)$, $c(r)$ are continuous non-decreasing functions positive for all $r > 0$ and equal to zero at $r = 0$. Then the zero solution $x(t) \equiv 0$ of the system (1) is asymptotically stable.

3.1 Asymptotic Stability of Systems with One Delay

Suppose that the system without deviation (3)

$$\dot{x}(t) = (A + B)x(t) \tag{8}$$

is asymptotically stable. Stability investigation is performed using Lyapunov function in the form $v(x) = x^T Hx$, where H is a solution of the equation

$$(A + B)^T H + H(A + B) = -C \ . \tag{9}$$

Here C is an arbitrary positive definite matrix.

Denote $\varphi(H) = \lambda_{\max}(H)/\lambda_{\min}(H)$, where $\lambda_{\max}(\bullet), \lambda_{\min}(\bullet)$ are maximal and minimal eigenvalues of the matrix H [9,10].

Theorem 14. *Let the system (8) be asymptotically stable. If there exists a positive definite matrix H, which is a solution of (9), and if the inequality*

$$\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,) > 0 \tag{10}$$

is satisfied, then the system (3) is asymptotically stable for an arbitrary $\tau > 0$. Moreover, for an arbitrary solution $x(t)$ of the system (3) the condition $|x(t)| < \varepsilon$, $t > 0$ holds only if $\|x(0)\|_\tau < \delta(\varepsilon)$, where $\delta(\varepsilon) = \varepsilon/\sqrt{\varphi(H)}$.

Conditions of the Theorem 14 provide exponential decay of solutions of the system (3).

Theorem 15. *Let the system (8) be asymptotically stable. If a positive definite matrix H, which is a solution of the equation (9), exists and if an inequality (10) holds, then for solutions $x(t)$ of the system (3) the following inequality holds*

$$|x(t)| < \sqrt{\varphi(H)}\, \|x(0)\|_\tau \exp\{-\gamma t/2\}, \quad t > 0,$$

where

$$\gamma = \left\{ \frac{2}{\tau} \ln^{-1} \left[\frac{\lambda_{\min}(C) - 2|HB|}{2|HB|\sqrt{\varphi(H)}} \right] + \frac{\lambda_{\max}(H)}{\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,)} \right\}^{-1} .$$

Let the system (8) be asymptotically stable, but there is no such H that satisfies the inequality (10).

Theorem 16. *Let the system (8) be asymptotically stable. If $\tau < \tau_0$, where*

$$\tau_0 = \frac{\lambda_{\min}(C)}{2(|A| + |B|)|HB|\sqrt{\varphi(H)}}, \tag{11}$$

then the system (3) is also asymptotically stable. Also $|x(t)| < \varepsilon, \quad t > 0$, only if $\|x(0)\|_\tau < \delta(\varepsilon, \tau)$, where

$$\delta(\varepsilon, \tau) = (1 + |B|\tau)^{-1} \exp\{-|A|\tau\}\varepsilon/\sqrt{\varphi(H)} \ .$$

Theorem 17. *Let the system* (8) *be asymptotically stable. If* $\tau < \tau_0$*, where* τ_0 *is defined in* (11)*, then the following inequality holds*

$$|x(t)| < \begin{cases} \sqrt{\varphi(H)}\,(1+|B|\tau)\|x(0)\|_\tau \exp\{|A|\tau\}, & 0 \le t \le \tau, \\ \sqrt{\varphi(H)}\,(1+|B|\tau)\|x(0)\|_\tau \exp\{|A|\tau - \gamma t/2\}, & t > \tau\,, \end{cases}$$

where

$$\gamma = \left(1 - \frac{\tau}{\tau_0}\right)\left[\frac{\lambda_{\max}(H)}{\lambda_{\min}(C)} - \frac{(1-\tau/\tau_0)\tau}{\ln(\tau/\tau_0)}\right]^{-1} .$$

3.2 Estimation of Delay Influence on System Solution

A system in the form

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + Q(x(t), x(t-\tau)) \tag{12}$$

is called "perturbed" to (3) [11].

Theorem 18. *Let the system* (8) *be asymptotically stable and let there exist a positive definite matrix* H *such that it is a solution of the equation* (9) *and the inequality* (10) *holds. Then for an arbitrary solution* $x_Q(t)$ *of the system* (12) *the following holds:* $|x_Q(t)| < \varepsilon, \quad t > 0$*, if* $\|x_Q(0)\|_\tau < \delta(\varepsilon)$ *and* $|Q(x_Q(t), x_Q(t-\tau))| < \eta(\varepsilon)$*, where*

$$\delta(\varepsilon) = \varepsilon/\sqrt{\varphi(H)}\,, \quad \eta(\varepsilon) = \frac{\lambda_{\min}(C) - 2|HB|(1+\sqrt{\varphi(H)}\,)}{2|H|\sqrt{\varphi(H)}}\varepsilon \ .$$

Let there be no such matrix H that satisfies the inequality (10).

Theorem 19. *Let the system* (8) *be asymptotically stable. Then if* $\tau < \tau_0$*, where* τ_0 *is defined in* (11)*, the following holds for a solution* $x_Q(t)$ *of the system* (12)*:* $|x_Q(t)| < \varepsilon, \quad t > 0$*, only if* $\|x_Q(0)\| < \delta(\varepsilon,\tau)$*, and* $|Q(x_Q(t), x_Q(t-\tau)| < \eta(\varepsilon,\tau)$*, where*

$$\delta(\varepsilon,\tau) = (1-\zeta)(1+|B|\tau)^{-1}\exp\{-|A|\tau\}\varepsilon/\sqrt{\varphi(H)}\,,$$
$$\eta(\varepsilon,\tau) = \min\left\{\frac{\zeta}{\tau}e^{-|A|\tau}, \frac{\lambda_{\min}(C)(1-\tau/\tau_0)}{2(|HB|\tau + |H|)}\right\}\frac{\varepsilon}{\sqrt{\varphi(H)}},$$

where $0 < \zeta < 1$ *is an arbitrary fixed constant.*

Let us estimate the maximum deviation $\tau = \tau_{\max}$, such that the divergence $|x(t) - x_0(t)| < \varepsilon, \quad t > 0$ holds. Denote $x_0(t)$ to be a solution (8), and

$$q = |B(A+B)||x_0(0)| \ .$$

Theorem 20. *Let the system* (8) *be asymptotically stable, and let there exist* H *— a solution of* (9) *— satisfying* (10)*. Then for an arbitrary* $\varepsilon > 0$, $\delta < \varepsilon/\sqrt{\varphi(H)}$ *the following is true:* $|x(t) - x_0(t)| < \varepsilon$, $t > 0$ *only when* $\|x(0) - x_0(0)\|_\tau < \delta$, *and* $\tau \leq \tau_{\max}$, *where*

$$\tau_{\max} = \frac{\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,)}{2|H|q\varphi(H)}\varepsilon \ .$$

Let us introduce the following notations

$$M_1 = 1 - \delta\sqrt{\varphi(H)}\,/\varepsilon, \quad M_2 = |A| + |B|\sqrt{\varphi(H)}\,\delta/\varepsilon,$$
$$N_1 = \varepsilon\lambda_{\min}(C)/\varphi(H)q, \quad N_2 = |H| + \varepsilon\lambda_{\min}(C)/2\varphi(H)q\tau_0.$$

Theorem 21. *Let the system* (8) *be asymptotically stable. Then for any* $\varepsilon > 0$ *and* $\delta < \varepsilon/\sqrt{\varphi(H)}$ *we have* $|x(t) - x_0(t)| < \varepsilon$, $\quad t > 0$ *only if* $\|x(0) - x_0(0)\|_\tau < \delta$ *and* $\tau \leq \tau_{\max}$, *where*

$$\tau_{\max} = \min\left\{2M_1\left[\sqrt{M_2^2 + 4M_1\varphi(H)q/\varepsilon} + M_2\right]^{-1}, \right.$$
$$\left. N_1\left[\sqrt{N_2^2 + 2N_1|HB|} + N_2\right]^{-1}\right\} \ .$$

3.3 Absolute Stability of "Direct" Control Systems with Delay

Consider the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + b_0 f(\sigma[t]) + b_1 f(\sigma[t-\tau]), \\ \sigma[t] = c_0^T x(t) + c_1^T x(t-\tau) \ . \end{cases} \tag{13}$$

Function $f(\sigma)$ satisfies the Lipshitz condition with a constant L and a sector $(0, k)$; i.e.

$$f(\sigma)(K\sigma - f(\sigma)) > 0 \ . \tag{14}$$

Lyapunov function is used in the form

$$v(x) = x^T Hx + \beta\int_0^{\sigma(x)} f(\xi)d\xi, \quad \sigma(x) = c^T x, \quad c = c_0 + c_1 \ .$$

Matrix H is found from the equation (9). For the function $v(x)$ the following condition holds:

$$\lambda_{\min}(\tilde{H})|x|^2 \leq v(x) \leq \lambda_{\min}(\tilde{H})|x|^2,$$

where

$$\lambda_{\min}(\tilde{H}) = \begin{cases} \lambda_{\min}(H), \quad \beta \geq 0, \\ \lambda_{\min}(H + \beta k c c^T/2), \quad \beta < 0\,; \end{cases}$$

$$\lambda_{\max}(\tilde{H}) = \begin{cases} \lambda_{\max}(H + \beta k c c^T/2), \quad \beta \geq 0, \\ \lambda_{\max}(H), \quad \beta < 0\ . \end{cases}$$

Definition 22. The system (13) is absolutely stable if the solution $x(t) \equiv 0$ is stable for an arbitrary function $f(\sigma)$ that satisfies (14).

Denote

$$\varphi(\tilde{H}) = \lambda_{\max}(\tilde{H})/\lambda_{\min}(\tilde{H}), \quad p_1 = 2(|HB| + L|Hb_0||c_1| + L|Hb_1||c_0|),$$
$$p_2 = |\beta c^T B| + |\beta c^T b_0|L|c_1| + |\beta c^T b_1|L|c_0|, \quad b = b_0 + b_1,$$
$$q_1 = 2L|Hb_1||c_1|, \quad q_2 = |\beta c^T b_1|L_1|c_1|, \quad c = c_0 + c_1,$$

$$\tilde{C}_1 = \begin{bmatrix} -[(A+B)^T H + H(A+B)] & \vdots & -[Hb + (\beta(A+B)^T + E)c/2] \\ -(p_1 + q_1 + (p_2+q_2)/2\xi^2) & \vdots & \\ \times(1 + \sqrt{\varphi(\tilde{H})})E & \vdots & \\ \cdots & \cdots & \cdots \\ -[Hb + (\beta(A+B)^T + E)c/2]^T & \vdots & 1/k - \beta b^T c - (p_2+q_2)\xi^2 \\ & \vdots & \times(1 + \sqrt{\varphi(\tilde{H})})/2 \end{bmatrix}.$$

Theorem 23. *Let matrix H and a parameter β be such that $\lambda_{\min}(\tilde{H}) > 0$, and let there exist such ξ that $\tilde{C}_1$ is positive definite. Then the system (13) is absolutely stable for any $\tau > 0$. In such case $|x(t)| < \varepsilon, \quad t > 0$ only when $\|x(0)\|_\tau < \delta(\varepsilon)$, where $\delta(\varepsilon) = \varepsilon/\sqrt{\varphi(\tilde{H})}$.*

When conditions of the theorem hold, solutions of the system decay.

Theorem 24. *Let matrix H and a parameter β be such that $\lambda_{\min}(\tilde{H}) > 0$ and $\tilde{C}_1$ exists and is positive definite. Then for solutions $x(t)$ of the system (13) the following holds*

$$|x(t)| < \sqrt{\varphi(\tilde{H})}\|x(0)\|_{2\tau} \exp\{-\gamma t/2\}, \quad t > 0,$$

where

$$\gamma = \min\left\{\frac{\gamma_1\lambda_{\min}(\tilde{C}_1)}{\gamma_1\lambda_{\max}(\tilde{H}) + \lambda_{\min}(\tilde{C}_1)}, \gamma_2\right\},$$

$$\gamma_1 = \frac{2}{\tau}\ln\left\{\left[\sqrt{[(p_1+p_2)+2(q_1+q_2)]^2 + 4\lambda_{\min}(\tilde{C}_1)(q_1+q_2)/\sqrt{\varphi(\tilde{H})}} - (p_1+p_2)\right]/2(q_1+q_2)\right\},$$

Introduce the following designations

$$M(0) = |A| + |B|\,|K|(|b_0| + |b_1|)(|c_0| + |c_1|),$$
$$N(0) = p_1 + 2q_1 + \sqrt{(p_1+2q_1)^2 + (p_2+2q_2)^2},$$

$$\tilde{C}_2 = \begin{bmatrix} -[(A+B)^T H + H(A+B)] & \vdots & -[Hb + (\beta(A+B)^T + E)c/2] \\ \cdots & \cdots & \cdots \\ -[Hb + (\beta(A+B)^T + E)c/2]^T & \vdots & 1/k - \beta b^T c \end{bmatrix}.$$

Theorem 25 ([13]). *Let matrix H and a parameter β be such that $\lambda_{\max}(\tilde{H}) > 0$, and let $\tilde{C}_2$ be positive definite. Then, when $\tau < \tau_0$, where*

$$\tau_0 = \frac{2\lambda_{\max}(\tilde{C}_2)}{M(0)N(0)\sqrt{\varphi(\tilde{H})}}, \tag{15}$$

the system (13) is absolutely stable. Moreover, $|x(t)| < \varepsilon, \quad t > 0$ if $\|x(0)\|_{2\tau} < \delta(\varepsilon,\tau)$, where

$$\delta(\varepsilon,\tau) = [(1+\bar{R}\tau)e^{\bar{L}\tau}]^{-2}\varepsilon/\sqrt{\varphi(\tilde{H})}$$

Theorem 26. *Let matrix H and a parameter β be such that $\lambda_{\min}(\tilde{H}) > 0$ and $\tilde{C}_2$ is positive definite. Then if $\tau < \tau_0$, where τ_0 is defined in (15), the following inequality holds for solutions $x(t)$ of the system (13)*

$$|x(t)| < \begin{cases} \sqrt{\varphi(\tilde{H})}\|x(0)\|_{2\tau}(1+R\tau)^2\exp\{2\bar{L}\tau\}, & 0 \le t \le 2\tau, \\ \sqrt{\varphi(\tilde{H})}\|x(0)\|_{2\tau}(1+R\tau)^2\exp\{2\bar{L}\tau - \gamma t/2\}, & \tau > 2\tau, \end{cases}$$

where

$$\gamma = \frac{\gamma_1\lambda_{\min}(\tilde{C}_2)(1-\tau/\tau_0)}{\gamma_1\lambda_{\max}(\tilde{H}) + \lambda_{\min}(\tilde{C}_2)(1-\tau/\tau_0)},$$

and γ_1 *is a root of the equation*

$$1 - [M(\gamma)N(\gamma)][M(0)N(0)]^{-1}e^{\gamma\tau/2} = 0,$$

$$\begin{aligned} M(\gamma) &= |A| + |B|e^{\gamma\tau/2} + K(|b_0| + |b_1|e^{\gamma\tau/2})(|c_0| + |c_1|e^{\gamma\tau/2}), \\ N(\gamma) &= (p_1 + 2q_1e^{\gamma\tau/2}) + \sqrt{(p_1 + 2q_1e^{\gamma\tau/2})^2 + (p_2 + 2q_2e^{\gamma\tau/2})^2}, \\ \bar{R} &= |B| + K(|b_0||c_1| + |b_1||c_0| + |b_1||c_1|), \\ \bar{L} &= |A| + K|b_0||c_0| \ . \end{aligned}$$

3.4 Differential Systems with a Quadratic Right-Hand Side

Difference-differential equation with a quadratic right-hand side

$$\begin{aligned} \dot{x}(t) = Ax(t) + Bx(t-\tau) + X^T(t)D_1x(t) + X^T(t)D_2x(t-\tau) \\ +X(t-\tau)D_3x(t-\tau) \end{aligned} \tag{16}$$

recently became very popular. Here $X(t), D_i$, $i = \overline{1,3}$ are rectangular $n^2 \times n$ matrices in the form

$$\begin{aligned} X(t) &= \{X_1(t), X_2(t), \ldots, X_n(t)\} \\ D_j^T &= \{D_{1j}, D_{2j}, \ldots, D_{nj}\} \ . \end{aligned}$$

Here $X_k(t)$, where $k = \overline{1,n}$, are quadratic matrices that have a vector $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ in place of a kth column, and other elements are zero. D_{ij} are symmetric matrices that define quadratic ith rows.

Theorem 27. *Let there exist such a matrix H that (10) holds. Then the solution $x(t) \equiv 0$ of the system (3) is asymptotically stable at any $\tau > 0$. The sphere U_R that lies in the area of asymptotic stability has the radius*

$$R = \frac{\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,)}{2\lambda_{\max}(H)\sum_{i=1}^{3}|D_i|(\sqrt{\varphi(H)}\,)^i} \ .$$

For solutions $x(t)$ from the sphere U_R the following convergence estimate holds:

$$\begin{aligned} |x(t)| &< \frac{R\sqrt{\varphi(H)}\,\|x(0)\|_\tau \exp\{-\gamma t/2\}}{R - \|x(0)\|_\tau[1 - \exp\{-\gamma t/2\}]}, \quad t > 0, \\ \gamma &= [\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,)]/\lambda_{\max}(H) \ . \end{aligned}$$

Theorem 28. *Let the system (8) be asymptotically stable. Then for $\tau < \tau_0$, where τ_0 is denoted in (11), solution $x(t) \equiv 0$ of the system (16) will also be asymptotically stable. For such solutions $x(t)$ that satisfy the condition $\|x(\tau)\|_\tau < \bar{R}\zeta$, $0 < \zeta < 1$ the following convergence estimate holds:*

$$|x(t)| \le \begin{cases} \|x(t)\|_\tau, & 0 \le t \le \tau, \\ \frac{\bar{R}\zeta\sqrt{\varphi(H)}\,\|x(\tau)\|_\tau \exp\{-\gamma t/2\}}{\bar{R}\zeta - \|x(\tau)\|_\tau[1-\exp\{-\gamma t/2\}]}, & t > \tau \ . \end{cases}$$

Here

$$\bar{R} = \frac{\lambda_{\min}(C)(1-\tau/\tau_0)}{2\sum_{i=1}^{3}\left[|HB||D_i|(\sqrt{\varphi(H)}\,)^3 + \lambda_{\max}(H)|D_i|(\sqrt{\varphi(H)}\,)^i\right]},$$

γ *is a solution of a special equation.*

3.5 Differential Systems with Rational Right-Hand Sides

Recently developed mathematical models of ordinary differential equations with rational right-hand sides were found adequate for description of various models in biology and medicine. The systems have the form [15,16]

$$\dot{x}(t) = [E + X(t)D_1 + X(t-\tau)D_2]^{-1}[Ax(t) + Bx(t-\tau)]. \tag{17}$$

Theorem 29. *Let there exist a symmetric positive definite matrix H that satisfies (10). Then the solution $x(t) \equiv 0$ of the system (17) is asymptotically stable for an arbitrary delay $\tau > 0$. The asymptotic stability region contains the ball $U_R = \{x : |x| \le R\}$, where*

$$R = \frac{[\lambda_{\min}(C) - 2|HB|(1+\sqrt{\varphi(H)}\,)]/\sqrt{\varphi(H)}}{(|D_1| + |D_2|\sqrt{\varphi(H)}\,)[[\lambda_{\min}(C) - 2|HB|(1+\sqrt{\varphi(H)}\,)] + 2|H|(|A| + |B|\sqrt{\varphi(H)}\,)]}.$$

Theorem 30. *Let the system (8) be asymptotically stable. Then for all $\tau < \tau_0$, where*

$$\tau_0 = \frac{\lambda_{\min}(C)(1-\zeta)^3}{2(|A|+|B|)|H|\sqrt{\varphi(H)}\,[|B| + (|D_2||A_1| - |D_1||B|)R\zeta]}.$$

Then the solution $x(t) \equiv 0$ of the system (17) is asymptotically stable. The asymptotic stability region contains a ball with the radius

$$R = \min\left\{\frac{1}{(|D_1|+|D_2|)\sqrt{\varphi(H)}}, \frac{\lambda_{\min}(C)/\sqrt{\varphi(H)}}{[2|HB| + \lambda_{\min}(C)]|D_1 + D_2|}\right\}.$$

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Quadratic Functionals: Positivity, Oscillation, Rayleigh's Principle

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Abstract. In this paper we give a survey on the theory of quadratic functionals. Particularly the relationships between positive definiteness and the asymptotic behaviour of Riccati matrix differential equations, and between the oscillation properties of linear Hamiltonian systems and Rayleigh's principle are demonstrated. Moreover, the main tools form control theory (as e.g. characterization of strong observability), from the calculus of variations (as e.g. field theory and Picone's identity), and from matrix analysis (as e.g. l'Hospital's rule for matrices) are discussed.

AMS Subject Classification. 49N10, 34A30, 34C10, 93B07, 93C05

Keywords. Quadratic functional, Hamiltonian system, Riccati equation, oscillation, observability, Rayleigh's principle, eigenvalue problem, linear control system

1 Introduction

This article presents a survey on the theory of quadratic functionals as described in a recent book by W. Kratz [13]. This theory is based mainly on the work by M. Morse and W. T. Reid (see [16] and [18]). We introduce the necessary notions and formulate the central results, but without any proofs. The setup of the paper is as follows.

In the next section we introduce the necessary notation and basic concepts, namely: We consider *quadratic functionals* for state and control functions, which satisfy a linear differential system (called the *equations of motion*), and for which the state function satisfies additionally some linear and homogeneous boundary condition. Classical methods of the calculus of variations lead to a *self-adjoint eigenvalue problem* consisting of a *linear Hamiltonian system* and boundary conditions (including the corresponding Euler equation and the natural boundary conditions). These eigenvalue problems contain e.g. the well-known Sturm-Liouville problems. The study of oscillation properties of the Hamiltonian system requires the concept of *conjoined bases* and their *focal points*, which includes in a certain sense the basic notion of disconjugacy. Moreover, the central notions of *controllability* and *strong observability* (or observability with unknown inputs) from control theory play a key role in the theory as well as *Riccati matrix differential equations* corresponding to linear Hamiltonian systems.

In Section 3 we formulate the main results concerning the positivity of quadratic functionals. Theorem 4 states a *Reid Roundabout Theorem*, which includes e.g. the well-known Jacobi condition from the calculus of variations as a special case. Theorem 5 concerns the positivity of a quadratic functional depending on a parameter. It states essentially that the positivity of the functional for small values of the parameter is equivalent to a certain asymptotic behaviour of the corresponding Riccati equation, and that it is also equivalent to strong observability of the underlying linear system.

In Section 4 we present in Theorem 7 the central oscillation theorem for Hamiltonian systems. The next Theorem 9 states the basic properties of the corresponding eigenvalue problem, i.e., existence of eigenvalues, Rayleigh's principle, and the expansion theorem.

Finally, we describe in Section 5 the main tools for the proofs. These tools include results from the calculus of variations (as e.g. Picone's identity), from matrix analysis (as e.g. properties of monotone matrix-valued functions), from linear control theory (as e.g. a canonical form for controllable systems), and from functional analysis (Ehrling's lemma). We formulate explicitly two basic results, namely a substitute of l'Hospital's rule for matrices in Theorem 10 and a characterization of strong observability for time-dependent systems in Theorem 11.

2 Notation and basic concepts

First we introduce *quadratic functionals*

$$\mathcal{F}(x) := \int_a^b \{x^T C x + u^T B u\}(t)dt + \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix}^T S_1 \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix} , \qquad (1)$$

and *bilinear forms*

$$\langle x, y\rangle_0 := \int_a^b \{x^T C_0 y\}(t)dt \ , \ \langle x, y\rangle := \langle x, y\rangle_0 + \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix}^T S_0 \begin{pmatrix} -y(a) \\ y(b) \end{pmatrix} , \qquad (2)$$

where x (or (x,u)) is (A,B)-*admissible* , i.e., the so-called *equations of motion* $\dot{x} = Ax + Bu$ hold on $\mathcal{I} := [a,b]$ for some *control* u with $Bu \in C_s(\mathcal{I})$ (i.e., Bu is piecewise continuous on $\mathcal{I}$), and where it satisfies boundary conditions of the form $\begin{pmatrix} -x(a) \\ x(b) \end{pmatrix} \in \mathcal{V}$, we write $x \in \tilde{\mathcal{R}}$, and where the same holds for y. Throughout we impose the following assumptions on the given data:

(A) $A(t), B(t), C(t), C_0(t)$ are real $n\times n$-matrix-valued functions, which are piecewise continuous on $\mathbb{R}$, $B(t), C(t), C_0(t)$ are symmetric, and $B(t), C_0(t)$ are non-negative definite (we write $B(t) \geq 0$, $C_0(t) \geq 0$) for $t \in \mathbb{R}$. $\mathcal{V} \subset \mathbb{R}^{2n}$ is a subspace of $\mathbb{R}^{2n}$, S_0 and S_1 are real and symmetric $2n \times 2n$-matrices, and $S_0 \geq 0$.

Let R_2, S_2 be $2n \times 2n$-matrices, such that $\mathcal{V} = \text{Im } R_2^T$ and $S_2 R_2^T = 0$, $\text{rank }(R_2, S_2) = 2n$ (see [13, Corollary 3.1.3]). We put

$$R_1(\lambda) \;:=\; R_2(S_1 - \lambda\, S_0) + S_2 \quad , \quad R_1 := R_1(0) \tag{3}$$

for $\lambda \in \mathbb{R}$. By Im, rank, ker we denote the *image*, *rank*, *kernel* of a matrix, and I denotes the *identity matrix* of corresponding size.

The pair (x,u) is stationary for the functional $\mathcal{F}$ if it satisfies the natural boundary conditions and the *Euler equations* $\dot{u} = Cx - A^T u$. These Euler equations lead together with the equations of motion to the linear *Hamiltonian system*

$$\dot{x} \;=\; Ax + Bu \quad , \quad \dot{u} \;=\; Cx - A^T u\,, \tag{H}$$

and the natural boundary conditions together with the given boundary conditions $\tilde{\mathcal{R}}$ lead to the self-adjoint boundary conditions (B_λ) below with $\lambda = 0$.

We need the following basic notions (see [13]).

Definition 1.

(i) (X,U) is called a *conjoined basis* of (H), if $X(t), U(t)$ are real $n \times n$-matrix-valued solutions of (H) with

$$\text{rank }(X^T(t), U^T(t)) \;\equiv\; n \quad , \quad X^T(t)U(t) - U^T(t)X(t) \;\equiv\; 0 \;\text{ on }\; \mathbb{R}\,.$$

(ii) Two conjoined bases (X_1,U_1) and (X_2,U_2) are called *normalized conjoined bases* of (H) if $X_1^T(t)U_2(t) - U_1^T(t)X_2(t) \equiv I$ on $\mathbb{R}$; and $(\tilde{X}_1,\tilde{U}_1), (\tilde{X}_2,\tilde{U}_2)$ denote the special normalized bases of (H), which satisfy the initial conditions

$$\tilde{X}_1(a) \;=\; \tilde{U}_2(a) \;=\; 0 \quad , \quad \tilde{U}_1(a) \;=\; -\tilde{X}_2(a) \;=\; I\,,$$

and then $(\tilde{X}_1,\tilde{U}_1)$ is called the *principal solution at a* .

(iii) A point $t_0 \in \mathbb{R}$ is called a *focal point* of X (or (X,U)) for a conjoined basis (X,U), if $X(t_0)$ is non-invertible, and the dimension of the kernel of $X(t_0)$ is called its multiplicity.

(iv) The pair (A, B) is called *controllable* on $\mathcal{J}$, if $\dot{v} = -A^T(t)v$, $B^T(t)v(t) \equiv 0$ on some non-degenerate interval $\tilde{\mathcal{J}} \subset \mathcal{J}$ always implies that $v(t) \equiv 0$ on $\tilde{\mathcal{J}}$ (see also [6, Definition 1.2.4] for "uniformly controllable").

(v) The triple (A, B, C_0) (or the *linear system*

$$\dot{x} = Ax + Bu \quad , \quad y = C_0 x \tag{LS}$$

with state x, input u, and output y) is called *strongly observable* on $\mathcal{J}$ if $\dot{x} = A(t)x + B(t)u$, $C_0(t)x(t) \equiv 0$ on some non-degenerate interval $\tilde{\mathcal{J}} \subset \mathcal{J}$ for some function u with $Bu \in C_s(\tilde{\mathcal{J}})$ always implies that $x(t) \equiv 0$ on $\tilde{J}$.

Given any conjoined basis (X, U) of (H) there exists another conjoined basis (X_2, U_2) such that $(X_1 = X, U_1 = U)$, (X_2, U_2) are normalized conjoined bases (see [13, Proposition 4.1.1]). By [13, Theorem 4.1.3] controllability of (A, B) is the same as saying that the focal points of every conjoined basis of (H) are isolated. Obviously, strong observability of (A, B, C_0) means that the bilinear form $\langle \cdot, \cdot \rangle_0$ is an inner product on the space of all (A, B)-admissible functions. Moreover, it is well-known (see [19] or [13]) that, for any conjoined basis (X, U) of (H), the quotient $Q(t) := U(t)X^{-1}(t)$ satisfies the *Riccati matrix differential equation*

$$\dot{Q} + A^T Q + QA + QBQ - C = 0 \,, \tag{R}$$

whenever $X(t)$ is invertible.

The investigation of extremal values if the so-called *Rayleigh quotient* $R(x) := \mathcal{F}(x)/\langle x, x \rangle$ leads to functions x and reals λ, where the functional

$$\mathcal{F}(x, \lambda) := \mathcal{F}(x) - \lambda \langle x, x \rangle \tag{4}$$

is stationary. Hence, these values λ are the eigenvalues of the *eigenvalue problem* $(\mathcal{E})$, which consists of the Hamiltonian system

$$\dot{x} = Ax + Bu \,, \; \dot{u} = (C - \lambda C_0)x - A^T u \,, \tag{H$_\lambda$}$$

and of the $2n$ linear and homogeneous boundary conditions

$$R_1(\lambda) \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix} + R_2 \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} = 0 \,, \text{ i.e. }, (x, u) \in \mathcal{R}(\lambda) \,. \tag{B$_\lambda$}$$

Note that (H)=(H$_0$), and that $(x, u) \in \mathcal{R}(\lambda)$ implies that $x \in \tilde{\mathcal{R}}$. Moreover, as above, there corresponds to (H$_\lambda$) a Riccati equation, namely:

$$\dot{Q} + A^T Q + QA + QBQ - C + \lambda C_0 = 0 \,. \tag{R$_\lambda$}$$

Remark 2. If the matrix $B(t)$ is *positive definite* for $t \in \mathcal{I}$, then the functional $\mathcal{F}$ and the equation of motion, i.e., $u = B^{-1}(\dot{x} - Ax)$, reduce to a quadratic functional occurring as *second variation* in the classical calculus of variations, which satisfies the strengthened Legendre condition. Moreover, our eigenvalue problems $(\mathcal{E})$ include the self-adjoint *Sturm-Liouville problems* of even order as a special case, where e.g. rank $B(t) =$ rank $C_0(t) \equiv 1$.

3 Positivity

In this section we derive criteria for the positive definiteness of the functionals $\mathcal{F}(\cdot)$ and $\mathcal{F}(\cdot, \lambda)$.

Definition 3. The functional $\mathcal{F}$ is called *positive definite*, we write $\mathcal{F} > 0$, if $\mathcal{F}(x) > 0$ for all (A, B)-admissible x with $x \in \tilde{\mathcal{R}}$ and $x(t) \not\equiv 0$ on $\mathcal{I}$.

Our first result includes the classical *Jacobi condition* from the calculus of variations, and it is often called "Reid Roundabout Theorem" (see [1], [2], [4], [5], [18]).

Theorem 4. *Assume* (A)*, and suppose that the pair* (A, B) *is controllable on* $\mathcal{I}$*. Then* $\mathcal{F} > 0$ *if and only if the following two assertions hold:*

(i) $\tilde{X}_1(t)$ *possesses no focal point in* $(a, b]$*.*

(ii) *The matrix* $M := R_2\{S_1 + \tilde{M}\}\, R_2^T$ *is positive definite on* $\operatorname{Im} R_2$*, where the matrix* $\tilde{M}$ *is defined by*

$$\tilde{M} := \begin{pmatrix} -\tilde{X}_1^{-1}\tilde{X}_2 & \tilde{X}_1^{-1} \\ \left(\tilde{X}_1^{-1}\right)^T & \tilde{U}_1\tilde{X}_1^{-1} \end{pmatrix}(b)\ . \tag{5}$$

This result is [13, Theorem 2.4.1]. Note that the matrices $\tilde{M}$ and M are symmetric and that assertion (ii) is empty (i.e., always satisfied), if $R_2 = 0$. The connection between the Hamiltonian system (H) and the Riccati equation (R) yields quite easily that the assertion (i) is equivalent with:

(i') The Riccati equation (R) possesses a symmetric solution $Q(t)$ on $\mathcal{I}$.

Moreover, in the case of so-called "separated boundary conditions" there is another result [13, Theorem 2.4.2], which uses only one conjoined basis (rather than $(\tilde{X}_1, \tilde{U}_1)$ and $(\tilde{X}_2, \tilde{U}_2)$ as above) depending on the boundary conditions (see also [5]).

Our next result concerns the positivity of $\mathcal{F}(\cdot, \lambda)$ for sufficiently small values of λ, and it is contained in the recent paper [15, Theorem 2]. It requires additional smoothness assumptions on the given data, i.e., for $n \geq 2$,

$$A \in C_s^{2n-3}(\mathbb{R})\ ,\ B \in C_s^{2n-3}(\mathbb{R})\ ,\ C_0 \in C_s^{2n-2}(\mathbb{R})\ . \tag{A1}$$

Theorem 5. *Assume* (A), (A1)*, and suppose that the pair* (A, B) *is controllable on* $\mathbb{R}$*. Then the following statements are equivalent:*

(i) *The linear system* (LS) *is strongly observable on* $\mathbb{R}$*.*

(ii) *For all non-degenerate intervals* $\mathcal{I} = [a,b]$ *and symmetric matrices* Q_0 *the solution* $Q(t;\lambda)$ *of* (R_λ) *with the initial condition* $Q(a;\lambda) \equiv Q_0$ *exists on* $\mathcal{I}$*, if* λ *is sufficiently small, and*

$$\lim_{\lambda \to -\infty} Q(b;\lambda) \; = \; \infty$$

(i.e., all eigenvalues of the symmetric matrix $Q(b;\lambda)$ *tend to infinity as* $\lambda \to -\infty$*).*

(iii) *For all non-degenerate intervals* $\mathcal{I} = [a,b]$*, subspaces* $\mathcal{V} \subset \mathbb{R}^{2n}$*, symmetric matrices* S_1 *and* S_0 *with* $S_0 \geq 0$*, there exists* $\lambda_0 \in \mathbb{R}$ *such that (see* (4)*)*

$$\mathcal{F}(\cdot,\lambda) > 0 \quad \textit{for all} \quad \lambda \leq \lambda_0 \ ,$$

and then, moreover,

$$\min\{R(x) \; = \; \mathcal{F}(x)/\langle x,x\rangle : x \ \textit{ is } \ (A,B)\textit{-admissible} \ \ , \, x \in \tilde{\mathcal{R}} \, , \, x \not\equiv 0\}$$

exists.

Remark 6. The assertion (iii) has the following interpretation in terms of the "optimal linear regulator problem" in control theory (see [10]), namely:
For given data the following LQ-problem with output energy constraints possesses a minimum. Minimize the quadratic functional $\mathcal{F}(x)$ for (A,B)-admissible $x \in \tilde{\mathcal{R}}$ under the additional "output energy" constraint $\langle x,x\rangle = 1$.

4 Oscillation and Rayleigh's principle

In this section we formulate the main results on the oscillation of solutions of the Hamiltonian system (H) and on the eigenvalue problem ($\mathcal{E}$). The oscillation theorem follows immediately from [13, Theorem 7.2.2] by using assertion (ii) or (iii) of Theorem 5.

Theorem 7 (Oscillation). *Assume* (A)*,* (A1)*, and suppose that the pair* (A,B) *is controllable on* $\mathcal{I}$ *and that the linear system* (LS) *is strongly observable on* $\mathcal{I}$*. Let* (X,U) *be any conjoined basis of* (H)*, such that* $X(a)$ *and* $X(b)$ *are invertible. Then*

$$n_1 + n_2 \; = \; n_3 + n \ ,$$

where n_1 *denotes the number of focal points of* X *(including multiplicities) in* (a,b) *;*

n_3 *denotes the number of eigenvalues* ($\mathcal{E}$) *(including multiplicities), which are less than zero; and where*

n_2 *denotes the number of negative eigenvalues of the symmetric* $3n \times 3n$*-matrix* $\mathcal{M}$*, which is defined by*

$$\mathcal{M} := \mathcal{R}_2 \begin{pmatrix} \Delta & -X^{-1}(a) & -X^{-1}(b) \\ -(X^{-1})^T(a) & -U(a)X^{-1}(a) & 0 \\ -(X^{-1})^T(b) & 0 & U(b)X^{-1}(b) \end{pmatrix} \mathcal{R}_2^T + \mathcal{R}_1 \mathcal{R}_2^T \ ,$$

where $\mathcal{R}_1 := \begin{pmatrix} 0 & 0 \\ 0 & R_1 \end{pmatrix}$, $\mathcal{R}_2 := \begin{pmatrix} I & 0 \\ 0 & R_2 \end{pmatrix}$, $\Delta := X^{-1}(a)X_2(a) - X^{-1}(b)X_2(b)$ *with conjoined basis* (X_2, U_2) *such that* (X, U) *and* (X_2, U_2) *constitute normalized conjoined bases of* (H).

Remark 8. If the matrix $X(a)$ or $X(b)$ is not invertible, then the same result holds but with a more complicated matrix $\mathcal{M}$, the definition of which needs more notation (see [13, Theorem 7.2.2]).

By using a generalized "Picone identity" [13, Theorem 1.2.1] this oscillation theorem is the main tool to derive Rayleigh's principle for our eigenvalue ($\mathcal{E}$) (see [13, Theorem 7.7.1 and Theorem 7.7.6]), namely:

Theorem 9 (Rayleigh's principle). *Assume* (A), (A1), *and suppose that the pair* (A, B) *is controllable on* $\mathcal{I}$ *and that the linear system* (LS) *is strongly observable on* $\mathcal{I}$. *Then the following statements hold:*

(i) *There* ***exist*** *infinitely many eigenvalues* λ_k *of the eigenvalue problem* ($\mathcal{E}$) *with* $\lambda_k \to \infty$ *(let* $-\infty < \lambda_1 \le \lambda_2 \le \cdots$ *denote these eigenvalues including multiplicities with corresponding orthonormal eigenfunctions* (x_1, u_1), $(x_2, u_2), \ldots$, *so that* $\langle x_k, x_\ell \rangle = \delta_{k\ell}$*).*

(ii) ***Rayleigh's principle*** *holds, i.e. for* $k = 0, 1, 2, \ldots$,

$$\lambda_{k+1} = \min\Big\{ R(x) = \frac{\mathcal{F}(x)}{\langle x, x \rangle} : x \text{ is } (A,B)\text{-admissible} , x \in \tilde{\mathcal{R}} , \ x \not\equiv 0 , \\ \text{and } \langle x, x_\nu \rangle = 0 \text{ for } \nu = 1, \ldots, k \Big\}.$$

(iii) *The* ***expansion theorem*** *holds, i.e.*

$$x = \sum_{\nu=1}^{\infty} \langle x_\nu, x \rangle x_\nu \ , \quad i.e., \quad \lim_{k\to\infty} \Big\| x - \sum_{\nu=1}^{k} \langle x_\nu, x \rangle x_\nu \Big\| = 0 \ ,$$

for all (A, B)*-admissible* x *with* $x \in \tilde{\mathcal{R}}$, *where* $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

5 Tools

In this section we discuss the main tools for the proof of our theorems cited above.

As already mentioned in the previous section a generalization of an identity due to *Picone* [17] (see also [21]) is the basis of the proof of Rayleigh's principle, i.e., Theorem 9, given in [13]. This extended version of Picone's formula can be derived from *field theory* (see e.g. [18]) as discussed in [13, Section 1.3].

The basic tools for the proof of the oscillation theorem, i.e., Theorem 7, come from *matrix analysis* (see [13, Chapter 3]). These results concern in particular limit and rank theorems for monotone matrix-valued functions (see [8], [11]). The basis for the limit theorem is a *substitute of l'Hospital's rule for matrices* [7, Theorem 1], namely:

Theorem 10. *Suppose that* X, U *are real* $n \times n$*-matrices such that it is fulfilled* $\operatorname{rank}(X^T, U^T) = n$ *and* $X^T U = U^T X$*. Then*

$$\lim_{S \to 0+} X(X + SU)^{-1} \; = \; 0 \, ,$$

where $S \to 0+$ *stands for* $S \to 0$ *and* $S > 0$.

This theorem together with monotonicity properties of the Riccati matrix differential equation (R) [13, Section 5.1] leads to the asymptotic behaviour of solutions of (R) (see [9] or [13, Chapter 6]).

Moreover, there are needed results from *linear control theory*. The asymptotics of Riccati equations requires, besides the results from matrix analysis above, in particular a certain *canonical form* of controllable pairs. While the Reid Roundabout Theorem, i.e., Theorem 4, may be proven by using mainly Picone's identity, the proof of Theorem 5 given in [15] depends essentially on two results. The first result is the following *characterization of strong observability* for time-dependent systems [14, Theorem 2].

Theorem 11. *Assume* (A1)*. Then the linear system* (LS) *is strongly observable on some interval* $\mathcal{I}$ *if and only if*

$$\operatorname{rank} S(t) \; = \; n + \operatorname{rank} T(t)$$

for $t \in \mathcal{I}$ *except on a nowhere dense subset of* $\mathcal{I}$*, where the matrix-valued functions* $S : \mathbb{R} \to \mathbb{R}^{n^2 \times n^2}$*,* $T : \mathbb{R} \to \mathbb{R}^{n^2 \times n(n-1)}$ *are defined as follows: First denote recursively* $C_k = C_k(t)$*,* $B_{\mu\nu} = B_{\mu\nu}(t)$ *by*

$$\begin{array}{llll} C_1 := C_0 \, , C_{\mu+1} := \dot{C}_\mu + C_\mu A & \textit{for} & \mu = 1, \dots, n-1, \\ B_{\mu+1,\mu} := C_0 B & \textit{for} & 0 \le \mu \le n-1 \, , \\ B_{\mu+1,0} := C_{\mu+1} B + \dot{B}_{\mu 0} & \textit{for} & 1 \le \mu \le n-1 \, , \\ B_{\mu+1,\nu} := B_{\mu,\nu-1} + \dot{B}_{\mu\nu} & \textit{for} & 1 \le \nu < \mu \le n-1 \, ; \end{array}$$

and (in block form) $S := [Q, T]$ *with*

$$Q \; := \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}, \qquad T \; := \begin{bmatrix} 0 & 0 & \dots & 0 \\ B_{10} & 0 & \dots & 0 \\ B_{20} & B_{21} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n-1,0} & B_{n-1,1} & \dots & B_{n-1,n-2} \end{bmatrix}.$$

This result reduces to [12, Theorem 2] or [13, Theorem 3.5.7] for time-invariant systems. In case $B = 0$, the theorem gives a characterization of controllability/observability (see [14, Theorem 1] and [6, Theorem 1.3.2 and Theorem 1.4.4]).

The second tool for the proof of Theorem 5 is a result from *functional analysis*, namely an application of the so-called Ehrling lemma (see [20, Lemma 8] or [3, 8.3]).

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Invariant Measures for Nonlinear SPDE's: Uniqueness and Stability*

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Abstract. The paper presents a review of some recent results on uniqueness of invariant measures for stochastic differential equations in infinite-dimensional state spaces, with particular attention paid to stochastic partial differential equations. Related results on asymptotic behaviour of solutions like ergodic theorems and convergence of probability laws of solutions in strong and weak topologies are also reviewed.

AMS Subject Classification. 60H15

Keywords. Stochastic evolution equations, invariant measures, ergodic theorems, stability

1 Introduction

The aim of the present paper is to review some recent results on uniqueness of invariant measures (that is, strictly stationary solutions) for nonlinear stochastic evolution equations (or, more generally, for stochastic differential equations in infinite-dimensional state spaces). Related asymptotic and ergodic properties of solutions like convergence of their probability laws to the invariant measure and ergodic theorems are also discussed.

The paper is divided into three parts: In Section 2, some existing results on strong and weak asymptotic stability of the invariant measure and its ergodic

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properties are recalled. By the strong asymptotic stability we mean convergence of probability laws of all solutions to the invariant measure in the norm defined by total variation of measures while the weak asymptotic stability means an analogous convergence in the weak (narrow) topology of the space of measures. It is obvious that both strong and weak asymptotic stability imply uniqueness of the invariant measure. Sections 3 and 4 contain more precise descriptions of some methods of proofs used in the papers listed in Section 2 for the respective cases of strong and weak asymptotic stabilities. In order to illustrate those methods, some typical statements and results are given. Note that the problem of existence of the invariant measure is not treated in the present paper; see for instance the monograph [20] by G. Da Prato and J. Zabczyk and the references therein.

It should be pointed out that the statements contained in Sections 3 and 4 are not always formulated in full generality. The authors' intention was to discuss some basic mathematical tools available and to avoid technical complications as much as possible. Some generalizations, improvements and applications of the presented results are referred to subsequently.

2 Review of existing results

A standard possibility to show uniqueness as well as the strong asymptotic stability (or the strong mixing property) of an invariant measure for a finite-dimensional nondegenerate stochastic differential equation is to utilize the usual correspondence between SDE's and PDE's; under suitable conditions (including, in particular, a sufficient nondegeneracy of the diffusion matrix of the SDE) the transitional densities coincide with the fundamental solution to a linear parabolic PDE (the Kolmogorov equation), which yields the strong Feller property (SFP) and the (topological) irreducibility (I) of the Markov process defined by the stochastic equation. Then the classical results of the ergodic theory of Markov processes, as developed by J. L. Doob, G. Maruyama and H. Tanaka, R. Z. Khas'minskiĭ and others (see e.g. [21], [48], [37], [22]) and later extended to more general state spaces (see the references in Section 3 and, in particular, Theorem 4), can be applied to obtain uniqueness of the invariant measure (provided it exists) as well as the the strong asymptotical stability.

For infinite-dimensional state spaces such mathematical tools are not easily available; the Lebesgue measure does not exist and equivalence of measures is in a sense a "rare" event (see, for instance, the discussion following Proposition 2 and the example at the beginning of Section 4). On the other hand, in the linear case when the transition probabilities are Gaussian measures it is possible to verify by direct computation (cf. Proposition 1) that in some important examples (typically, stochastic parabolic or parabolic-like equations) the strong asymptotic stability takes place.

There are several methods which have been used to prove similar results for nonlinear infinite-dimensional stochastic systems. At first, let us mention the approach based on verification of the strong Feller property and irreducibility of the induced Markov process which has been used in numerous papers that have appeared in recent years. We describe this method in detail in Section 3 while here we restrict ourselves to some bibliographical remarks. In early papers by B. Maslowski and R. Manthey ([50], [44]) the SFP has been proven via finite-dimensional approximations for semilinear systems under rather restrictive assumptions. Also, a controllability method to prove (I) was developed there. Those results were further extended by B. Maslowski in [52], in particular, certain smoothing properties of mild solutions to the infinite-dimensional backward Kolmogorov equation proven by G. Da Prato and J. Zabczyk ([15], see also [18]) were utilized to get the SFP for reaction-diffusion equations with additive noise. Alternatively, under different set of assumptions, the problem of equivalence of transition probabilities has been solved by means of a Girsanov type theorem in [51] and [52], cf. also [27] for an analogous but more difficult argument applied to the stochastic quantization equation. Let us mention that infinite-dimensional Kolmogorov equations have been treated very recently by many authors, their link to invariant measures of fairly general SPDE's was investigated in depth by A. Chojnowska-Michalik, B. Goldys and D. Gątarek, see [8], [28].

Another way of proving the SFP has emerged in the paper [11] by G. Da Prato, K. D. Elworthy and J. Zabczyk where a formula for directional derivatives of a Markov transition semigroup involving the L^2-derivative of the solution with respect to initial condition has been derived (cf. Proposition 9). This approach has been later extended by S. Peszat and J. Zabczyk [60] to be applicable to stochastic parabolic equations with multiplicative noise term, cf. also the already cited paper [28]. It also turned out to be useful in asymptotic analysis of various important particular systems studied in physics and chemistry, like stochastic Burgers and Navier-Stokes equations or stochastic Cahn-Hilliard equation (cf. [12], [10], [24] or [23]).

Tools from the Malliavin calculus were employed to establish the regularity of the transition semigroup (in particular the SFP) by M. Fuhrman in [26] (cf. also [14]).

Let us briefly mention some other methods of proving the strong asymptotic stability of invariant measures. S. Jacquot and G. Royer [36] used a general theory of Markov operators to prove geometric ergodicity (i.e. strong exponential stability of an invariant measure) for a particular but important stochastic parabolic equation. C. Mueller in [59] used an approach based on coupling techniques to prove strong asymptotic stability of the invariant measure for a nonlinear heat equation with multiplicative noise, defined on a circle (uniqueness of the invariant measure for this case had been proven earlier by R. Sowers in [62] by establishing suitable asymptotic stability of paths).

Very little seems to be known in the case of nonautonomous SPDE's, where the standard methods of ergodic theory are no longer available. A lower bound measure method developed in context of statistical analysis of deterministic dynamical systems has been used by B. Maslowski and I. Simão in [57] to investigate the limit behaviour (in variational norm) of Markov evolution operators corresponding to nonautonomous stochastic infinite-dimensional systems (cf. also a methodologically related paper [34]). Simulated annealing for stochastic evolution equations has been studied by S. Jacquot, see e.g. [35] where references to previous papers of the author can be found.

Results on the strong asymptotic stability, when available, usually provide us with a fairly complete description of the qualitative behaviour of solutions to the considered SPDE's. On the other hand, many stochastic equations with reasonable long-time behaviour can be never treated using the tools described above. So we shall discuss now methods for investigating the weak asymptotic stability that apply to different classes of SPDE's, including those with a degenerate noise.

As is known from the finite-dimensional case, uniqueness of an invariant measure may be obtained as a consequence of pathwise stability of the process, which, in turn, is often investigated by means of well developed Lyapunov techniques (see e.g. [38]). The Lyapunov functions methods were extended to semilinear SPDE's by A. Ichikawa in [32] (see also [31] for slight modifications), who found sufficient conditions for uniqueness, and further strengthened in [49] to yield stability as well. Later, these methods proved themselves applicable to nonhomogeneous boundary value problems for stochastic parabolic equations ([53], [54]). G. Leha and G. Ritter developed a rather general Lyapunov approach for establishing existence, uniqueness and attractiveness of invariant measures for Markov processes in topological spaces, that covers also some classes of stochastic infinite-dimensional differential equations (see [42], [43]). A recent paper [4] on uniqueness of an invariant measure for a stochastic parabolic variational inequality is virtually based on the same technique. We discuss the Lyapunov method in some detail in Section 4.

A special attention must be paid to the dissipativity method (sometimes also called "the remote start method") since most of recent results on invariant measures for SPDE's (both abstract theorems and results about important particular equations) seem to have been obtained using this procedure. The method was developed by G. Da Prato and J. Zabczyk in [16], [17], [19] for equations with additive noise and by them together with D. Gątarek in [13] for the multiplicative noise case; see the monographs [18], Chapter 11.5, [20] for a systematic account. (We list here only papers dealing with uniqueness and weak asymptotic stability, not the copious articles concerning applications of the dissipativity method to existence of invariant measures.) More factual description of the method is provided in Section 4.

Finally, we are going to list briefly other papers containing related results. R. Marcus in the early papers [45], [46], [47] considered stochastic parabolic equations with an additive noise under rather restrictive hypotheses and sketched a proof of the weak asymptotic stability of an invariant measure (using a procedure that can be viewed as a variant of the remote start trick). In particular, he investigated the case of the drift term having a potential, when the invariant measure may be given explicitly, see also [40] and [25] for uniqueness results in this direction. (These results are now partly covered by those based on the equivalence of transition kernels.) I. D. Chueshov and T. V. Girya proved existence and weak asymptotic stability of an invariant measure as a consequence of their results on inertial manifolds for parabolic SPDE's driven by additive noise ([9], [30]).

Uniqueness and stability theorems on invariant measures for semilinear stochastic parabolic equations, proved in the framework of the variational approach to SPDE's, can be found in [33] and [29], see also the book [65], §XII.7 and the references therein.

An analytic approach to invariant measures for infinite-dimensional stochastic systems, using logarithmic Sobolev inequalities (see the surveys [64] or [67] for references) or Dirichlet forms techniques (see [6], [5], [1], [7]), has found many applications to lattice systems (cf. e.g. [2], [3]). Applications to stochastic partial differential equations are up to now less frequent, see, however, the papers [58] and [39] in which ergodic properties of invariant measures for SPDE's are dealt with by means of Dirichlet forms.

3 Strong asymptotic stability

In the present section, some basic results on uniqueness, ergodicity and strong asymptotic stability of an invariant measure for stochastic evolution equations are listed and basic methods of their proofs are explained. By the strong asymptotic stability we understand convergence of probability laws of all solutions to a given stochastic evolution equation to the corresponding invariant measure in norm defined by the total variation of measures. In what follows, we denote by $|||\varrho|||$ the total variation of a signed measure ϱ and by $\mathcal{N}(m, U)$ the Gaussian measure with mean m and covariance operator U.

We start with the linear equation in which case the problem of strong asymptotic stability is in a sense much simpler than for the nonlinear equation. However, some "typical" difficulties (as well as differences between finite- and infinite-dimensional stochastic equations) can be seen already in that case.

Consider a linear stochastic equation of the form

$$\mathrm{d}Z_t = AZ_t \,\mathrm{d}t + \mathrm{d}W_t, \tag{1}$$

in a real separable Hilbert space $H = (H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ where $A : \mathrm{Dom}(A) \subseteq H \longrightarrow H$ is an infinitesimal generator of a strongly continuous semigroup $(e^{At}, t \geq 0)$ on H, W_t is a Wiener process on H defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ with an incremental covariance operator $Q \in \mathcal{L}(H)$. The operator Q is not necessarily nuclear (which means that W_t may be just cylindrical, not really H-valued, Wiener process). In the sequel we shall assume

$$\int_0^T \left\|e^{At} Q^{1/2}\right\|_{\mathrm{HS}}^2 \mathrm{d}t < \infty \tag{2}$$

for some $T > 0$, where $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm of an operator on H. It is well known that under the condition (2) the equation (1) has for any initial datum $Z_0 = x \in H$ a unique mild solution defined as a continuous H-valued process satisfying the variation of constants formula

$$Z_t = e^{At}x + \int_0^t e^{A(t-r)} \,\mathrm{d}W_r, \quad t \geq 0, \tag{3}$$

whose transition probabilities $P_t(x,\cdot)$ are Gaussian measures $\mathcal{N}(e^{At}x, Q_t)$ for $t \geq 0$, $x \in H$, where

$$Q_t = \int_0^t e^{Ar} Q e^{A^* r}\, \mathrm{d}r$$

is a nuclear operator (cf. [18] for basic results on the semigroup theory of stochastic evolution equations). An invariant measure μ^* for the Markov process induced by the equation (1) exists if and only if

$$\sup_{t\geq 0} \operatorname{Tr} Q_t < \infty \tag{4}$$

in which case $\mu^* = \mathcal{N}(0, Q_\infty)$, where $Q_\infty = \lim_{t\to\infty} Q_t$. In general, it can happen that μ^* is not the only invariant measure; the problem of uniqueness and characterization of all invariant measures has been treated in [66] (see also [18] and the references therein). As far as the strong asymptotic stability is concerned we expose the following result the proof of which can be found in [49] (for simplicity, we consider only the case $Q > 0$):

Proposition 1. *Assume (2) and let K be a linear subspace of H such that*

$$e^{At}x \in \operatorname{Im}\big(Q_t^{1/2}\big), \quad t > t_0(x) \geq 0 \tag{5}$$

and

$$\|Q_t^{-1/2} e^{At} x\| \longrightarrow 0, \quad t \to \infty, \tag{6}$$

for $x \in K$. Then

$$|||P_t(x,\cdot) - P_t(y,\cdot)||| \longrightarrow 0, \quad t \to \infty \tag{7}$$

for each $x, y \in K$. If, moreover, (4) holds true and $\mu^*(K) = 1$ then

$$|||P_t(x,\cdot) - \mu^*||| \longrightarrow \infty, \quad t \to \infty \tag{8}$$

for any $x \in K$. In particular, μ^* is the only invariant probability measure concentrated on K.

In particular examples, the condition (5) can be usually verified for $t_0 = 0$ (or for some t_0 independent of x) and for $K = H$. In this case the assumptions of Proposition 1 can be simplified as follows:

Proposition 2. *Assume (2) and (4) and let*

$$\operatorname{Im}\big(e^{At}\big) \subseteq \operatorname{Im}\big(Q_t^{1/2}\big) \tag{9}$$

be satisfied for $t > 0$. Then (8) holds true and, in particular, μ^* is the only invariant probability measure for the problem (1).

Note that Proposition 1 has been proven in [49] by direct computation using the Cameron-Martin formula for density of a Gaussian measure while Proposition 2 is a corollary of a more general statement given below (Theorem 4). The assumption (9) is an if and only if condition on the Gaussian transition probabilities $P_t(x,\cdot)$ to be equivalent (i.e., mutually absolutely continuous) for $t>0$, as can be seen easily by the Hájek-Feldman theorem. In some cases, (9) can be shown to be a necessary condition for the strong asymptotic stability (8) (see the example at the beginning of Section 4) and since it is rather restrictive (for example, the semigroup e^{At} satisfying (9) is necessarily Hilbert-Schmidt) it can be expected that the cases when the strong asymptotic stability takes place in the infinite-dimensional space H are rather "rare". However, it turns out that parabolic and parabolic-like stochastic equations with enough nondegenerate diffusion term are natural field for applications of Propositions 1 and 2 as may be seen from the simple example below.

Example 3. Assume that A is self-adjoint, negative, and has compact resolvent and denote by $\{e_j\}_{j\geq 1}$ the orthonormal basis of H such that

$$Ae_j = -\alpha_j e_j, \tag{10}$$

where $0<\alpha_j \longrightarrow \infty$, $j\in\mathbb{N}$. Assume that A, Q are such that for some $0<\lambda_j\leq \lambda_0<\infty$, we have

$$Qe_j = \lambda_j e_j, \quad j\in\mathbb{N}. \tag{11}$$

Then it is easy to check that the conditions (2) and (4) are satisfied if

$$\sum_{i=1}^{\infty}\frac{\lambda_i}{\alpha_i}<\infty, \tag{12}$$

which is sufficient for the mild solution Z_t of the equation (1) and the corresponding invariant measure μ^* to exist. The condition (9) verifying the strong asymptotic stability is now equivalent to the requirement that the sequence

$$\left\{\frac{\alpha_i}{\lambda_i}\exp(-2\alpha_i t)\right\}_{i\in\mathbb{N}} \tag{13}$$

is bounded for each $t>0$.

In particular, the process Z_t in the present example can represent a solution to a linear stochastic parabolic equation like, for instance, the equation

$$\frac{\partial u}{\partial t}(t,\xi)=\frac{\partial^2 u}{\partial \xi^2}(t,\xi)+\eta(t,\xi), \quad (t,\xi)\in\mathbb{R}_+\times(0,1), \tag{14}$$

with an initial condition $u(0,\xi)=x(\xi)$, $\xi\in(0,1)$, and the Dirichlet boundary conditions $u(t,0)=u(t,1)=0$, $t\in\mathbb{R}_+$, where η is a space-dependent noise, white in time. This can be achieved by the particular choice $H=L^2(0,1)$, and $A=\frac{\partial^2}{\partial\xi^2}$

with $\mathrm{Dom}(A) = H_0^1(0,1) \cap H^2(0,1)$. Now (13) can be viewed as a condition on the noise term (the covariance operator Q) for the strong asymptotic stability to hold. For example, if η represents a noise white in both space and time then we can take for Q the identity I and (13) is satisfied.

Our next aim is to describe a method based on the general ergodic theory of Markov processes that allows to prove the strong asymptotic stability (and, also, ergodic theorems) for nonlinear stochastic evolution equations. We shall utilize an abstract result stated in Theorem 4 below which was obtained independently by Stettner [63] and Seidler [61].

Theorem 4. *Let $\big((X_t)_{t\geq 0}, (\boldsymbol{P}_x)_{x\in H}\big)$ be a Markov process in a Polish space H with a transition probability function $P_t(x,\cdot)$, $t \geq 0$, $x \in H$, having an invariant probability measure μ^*. Assume that all the measures $P_t(x,\cdot)$, $t > 0$, $x \in H$, are equivalent. Then*

(i) for each bounded Borel function $\phi : H \longrightarrow \mathbb{R}$ and every $x \in H$ we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(X_t)\,\mathrm{d}t = \int_H \phi\,\mathrm{d}\mu^* \quad \boldsymbol{P}_x\text{-a.s.} \tag{15}$$

(ii) for every $x \in H$ we have

$$\|\!|P_t(x,\cdot) - \mu^*\|\!| \longrightarrow 0, \quad t \to \infty. \tag{16}$$

In particular, both (i) and (ii) imply that the invariant measure μ^* is unique.

The assertion (15) is known as the pointwise ergodic theorem (or the strong law of large numbers). As mentioned above, the condition (9) is an if and only if condition for the equivalence of transition probability functions for the linear equation (1). The simplest nonlinear case into which Theorem 4 can be applied is the one allowing reduction of the nonlinear problem to a linear one by means of a Girsanov type theorem. We shall present a simple result of this type now. Consider a stochastic semilinear equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + f(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \tag{17}$$

in the Hilbert space H, where A and W have the same meaning as in the equation (1) and $f : H \longrightarrow \mathrm{Im}\big(Q^{1/2}\big)$ satisfies

$$\|Q^{-1/2}\big(f(x) - f(y)\big)\| \leq K\|x - y\| \tag{18}$$

for a $K < \infty$ and all $x, y \in H$.

Theorem 5. *Assume (2), (9) and (18). Then $P_t(x,\cdot)$ and $\tilde{P}_t(x,\cdot)$ are equivalent measures for every $t > 0$, $x \in H$, where P and $\tilde{P}$ denote the transition probability functions for the solutions of the equations (17) and (1), respectively. If, moreover, there exists an invariant measure μ^* for the equation (17) then the strong asymptotic stability (16) holds true.*

Theorem 5 can be proven as a corollary of Theorem 4; note that (9) yields the equivalence of measures $\tilde{P}_t(x,\cdot)$ for $t > 0$, $x \in H$, and the equivalence $P_t(x,\cdot) \sim \tilde{P}_t(x,\cdot)$ follows from the Girsanov theorem. The condition (18) appears here because the Girsanov factor has the form

$$\exp\left\{\int_0^T \left\langle Q^{-\frac{1}{2}} f(Z_t), \mathrm{d}W_t\right\rangle - \frac{1}{2}\int_0^T \left\|Q^{-\frac{1}{2}} f(Z_t)\right\|^2 \mathrm{d}t\right\} \tag{19}$$

for $T > 0$ (cf. [18]). Theorem 5 can be generalized to cover also nonlinear terms which are not Lipschitz continuous or even only densely defined in H (cf. [52], [18]). The main disadvantage of this approach is that the inclusion $\mathrm{Im}(f) \subseteq \mathrm{Im}\big(Q^{1/2}\big)$ is required which makes the abstract results easily applicable only if Q is boundedly invertible, that is, just for a cylindrical Wiener process W (typically it can represent a space-time white noise; see however [56] for examples of stochastic parabolic equations in which a nonlinear term of the form $Q^{1/2} f$ occurs in a natural way in the drift part of the equation).

For equations of the form (17) where the covariance Q is "too degenerate" for the Girsanov theorem to be applied, it is sometimes possible to verify the equivalence of transition probability functions by means of Lemma 6 below which holds true even if the state space H is an arbitrary Polish space. Recall that a Markov process is called strongly Feller if its transition probability function $P_t(x,\Gamma)$ is continuous in the variable x for each fixed $t > 0$ and every Borel set Γ in H. Furthermore, the Markov process is called irreducible if $P_t(x,U) > 0$ holds for each $t > 0$, $x \in H$ and $U \neq \emptyset$, U open in H.

Lemma 6. *Assume that a Markov process is strongly Feller and irreducible. Then the measures $P_t(x,\cdot)$ are equivalent for $t > 0$, $x \in H$.*

Note that both the strong Feller property and irreducibility are of independent interest (for example, to investigate recurrence of the process, cf. [55] and [61]).

In the rest of the section we shall illustrate some methods which allow to verify irreducibility or the strong Feller property for stochastic evolution equations. At first we describe a method based on an argument of approximate controllability for a deterministic evolution equation, which yields the irreducibility property for the corresponding stochastic evolution equation. We again shall illustrate the method in the simple case (17) where A and W are as above and $f : H \longrightarrow H$ is assumed to be Lipschitz continuous. Note that the mild solution to the equation (17) with initial condition $X_0 = x \in H$ (which exists and is unique in this case) can be written $X_t = u(t,x;\tilde{Z})$, $t \geq 0$, where $\tilde{Z}$ solves the linear equation (1) with initial condition $\tilde{Z}_0 = 0$ and $u(t,x;\phi)$ is the solution of the integral equation

$$u(t,x;\phi) = e^{At}x + \int_0^t e^{A(t-r)} f(u(r,x;\phi))\,\mathrm{d}r + \phi(t), \quad t \in [0,T], \tag{20}$$

with $\phi \in \mathcal{C}_0([0,T];H) := \{g \in \mathcal{C}([0,T];H);\ g(0) = 0\}$. The method consists in finding a suitable space $\mathcal{X}$ of trajectories such that the paths of the Gaussian process $\tilde{Z}$ belong a.s. to $\mathcal{X}$, $\tilde{Z}$ induces a full Gaussian measure (that is, a measure whose closed support is the whole space) in $\mathcal{X}$, and $u(\cdot,x;\phi_n) \to u(\cdot,x;\phi)$ in $\mathcal{C}([0,T];H)$ as $\phi_n \to \phi$ in $\mathcal{X}$. The irreducibility of the transition probability function $P_t(x,\cdot)$ for the equation (17) follows easily. If the nonlinear term f is (globally) Lipschitz continuous on H it is natural to take $\mathcal{X} = \mathcal{C}_0([0,T];H)$ and it only remains to find conditions under which the measure induced by $\tilde{Z}$ in $\mathcal{X}$ is full. Since the closed support of a Gaussian measure is just the closure of its reproducing kernel we have a following result:

Theorem 7. *Define a mapping* $\mathcal{K} : L^2(0,T;H) \longrightarrow \mathcal{C}_0([0,T];H)$ *by*

$$\mathcal{K}\psi(t) := \int_0^t e^{A(t-r)} Q^{1/2} \psi(r)\,\mathrm{d}r, \quad t \in [0,T].$$

If f *is Lipschitz continuous and* $\operatorname{Im}(\mathcal{K})$ *is dense in* $\mathcal{C}_0([0,T];H)$ *then the transition probability function* $P_t(x,\cdot)$ *corresponding to the equation* (17) *is irreducible.*

Theorem 7 is a particular case of a result proven in [52] for the case of non-Lipschitz and densely defined nonlinear terms f, which is applicable to stochastic reaction-diffusion equations. More sophisticated versions of this method have been applied, for example, to stochastic Burgers equation [12], stochastic Cahn-Hilliard equation [10] and stochastic Navier-Stokes equation [24], [23].

Now we focus our attention on the strong Feller property of solutions to stochastic evolution equations. The usual procedure of verification of the strong Feller property in the finite-dimensional case utilizes the smoothing properties of the Kolmogorov equation. A similar theory for Kolmogorov backward equation in infinite dimensions is being developed in recent years ([18], [8], and others). The main tool to prove both existence and uniqueness of solutions and the required smoothing properties is the concept of mild solutions to the backward Kolmogorov equation, which we shall recall now. Basically, we follow the paper [8]. Assume (2), (4) and let $\mu = \mathcal{N}(0,Q_\infty)$ be the invariant measure for the linear equation (1) and T_t its Markov transition semigroup considered on the space $L^2(H,\mu)$, i.e., $T_t\phi(x) = \boldsymbol{E}_x\phi(Z_t)$, $t \geq 0$, $x \in H$, $\phi \in L^2(H,\mu)$. Further, denote by P_t the Markov transition semigroup defined by the nonlinear equation (17). Analogously to the finite-dimensional case it can be expected that, under suitable conditions, the semigroup P_t corresponds to solutions of the mild backward Kolmogorov equation

$$u(t,\cdot) = T_t\phi + \int_0^t T_{t-s}\langle f, Du(s,\cdot)\rangle\,\mathrm{d}s, \quad t > 0, \tag{21}$$

where D denotes the Fréchet derivative. A precise statement is formulated now ($\mathcal{C}_b(H)$ and $\mathcal{C}_b^1(H)$ denote the space of bounded continuous functions on H and its subspace of functions having bounded and continuous Fréchet derivative on H, respectively).

Theorem 8. *Let f be bounded and continuous, assume (2), (4), (9) and*

$$\int_0^T \|Q_t^{-1/2} e^{At}\|_{\mathcal{L}(H)} \,\mathrm{d}t < \infty \tag{22}$$

for some $T > 0$. Then for every bounded Borel function ϕ on H there exists a unique solution u to (21) and $u(t,\cdot) \in \mathcal{C}_b^1(H)$, $t > 0$. Moreover, $u(t,x) = P_t\phi(x) \equiv \boldsymbol{E}_x\phi(X_t)$, $t > 0$, $x \in H$, provided $\phi \in \mathcal{C}_b(H)$. In particular, we have

$$|||P_t(x,\cdot) - P_t(y,\cdot)||| \le \gamma(t)\|x - y\|, \quad t > 0,\ x, y \in H, \tag{23}$$

where $\gamma(t) := \sup\{\|DP_t\phi(z)\|;\ z \in H,\ \phi \in \mathcal{C}_b(H),\ |\phi| \le 1\} < \infty$, hence the strong Feller property holds true.

For the proof see [8] or (in certain earlier version) [18], [15]. The assumption of boundedness of f is not always essential for the strong Feller property and can be weakened by suitable truncation procedures (see [52], [28]) so that stochastic parabolic equations with polynomial-type nonlinearities could be included. The important assumption is (22) which is further strengthening of (9) and means certain "nondegeneracy of the noise" which, of course, is needed (even in finite-dimensional state space) for the strong Feller property to hold. It can be shown ([15]) that if the covariance Q is boundedly invertible then (22) is satisfied.

Theorem 8 is applicable only to equations with additive noise (if the diffusion term is a constant operator). Now we shall mention another method of establishing the strong Feller property, which is useful also in the case of multiplicative noise. The method was developed in [11] and is based on the so-called Elworthy formula which we present in the simple case of equation (17) where Q is assumed to be boundedly invertible and f is Lipschitz continuous and Gateaux differentiable on H with the Gateaux derivative continuous as a mapping from H into the space $\mathcal{L}(H)$ endowed with the strong operator topology.

Proposition 9. *Under the above hypotheses, we have that $P_t\phi \in \mathcal{C}^1(H)$ for each $t > 0$, ϕ bounded Borel, and*

$$\langle DP_t\phi(x), h\rangle = \frac{1}{t}\boldsymbol{E}_x\Big(\phi(X_t)\int_0^t \langle Q^{-1/2}X_s^h, \mathrm{d}W_s\rangle\Big) \tag{24}$$

holds for $x, h \in H$, where X_t^h denotes the directional derivative in the L^2-sense of the solution X_t to (17) in the direction $h \in H$.

For the proof see [11]. The usefulness of the formula (24) lies with the fact that it allows to estimate the value of $\|DP_t\phi(x)\|$ for a fixed $t > 0$, independently of $\phi \in \mathcal{C}_b(H)$, $|\phi| \le 1$ and the strong Feller property follows in the same way as in (23).

In fact, the method is applicable to more general cases as well as to some special equations which are rather difficult to handle (usually it is possible to use suitable approximations of the equation, which can be typically finite-dimensional approximations or approximations by smooth nonlinearities). Thus, in [60] the strong Feller property has been proven for stochastic semilinear equations with multiplicative noise (with boundedly invertible diffusion coefficients). In [12] and [10] the stochastic Burgers and Cahn-Hilliard, respectively, equations are treated. The 2-dimensional stochastic Navier-Stokes equation is dealt with in [24] and [23]. In all those cases the limit and ergodic properties of solutions listed in Theorem 4 are proved in respective state spaces.

4 Weak asymptotic stability

However efficient are the methods of investigating the long time behaviour of Markov processes based on the strong Feller property, they are relevant for a rather limited class of equations that are, roughly speaking, subject to a sufficiently nondegenerated noise. But such a nondegeneracy is necessary neither for the existence, nor for uniqueness and attractiveness of invariant measures. To indicate what may happen, let us consider a simple linear equation

$$\mathrm{d}Z = AZ\,\mathrm{d}t + \mathrm{d}W \tag{25}$$

in a separable Hilbert space H, where W is a Wiener process in H with a covariance operator Q and $A : \mathrm{Dom}(A) \longrightarrow H$ is a self-adjoint operator. Assume that the hypotheses (10)–(12) of Example 3 are satisfied. Denote by $P = P_t(x, \cdot)$ the transition function of the Markov process defined by (25). As above we set

$$Q_t = \int_0^t e^{Ar} Q e^{Ar}\,\mathrm{d}r, \quad 0 \le t \le \infty.$$

If (9) holds, that is

$$\mathrm{Im}\big(e^{At}\big) \subseteq \mathrm{Im}\big(Q_t^{1/2}\big) \quad \text{for each } t > t_0 \tag{26}$$

for a $t_0 \ge 0$, then the kernels $P_t(x, \cdot)$ are strong Feller and the theory discussed in Section 3 applies, so let us assume that (26) is violated. (Note that this is possible only in the "degenerate" case when Q is noninvertible, cf. [18], Remark B.9.) Then we can always find an $x_0 \in H$ satisfying

$$e^{At}x_0 \notin \mathrm{Im}\big(Q_t^{1/2}\big) \quad \text{for every } t > 0. \tag{27}$$

The semigroup (e^{At}) is exponentially stable, so there exists a unique invariant measure μ^* for (25), namely $\mu^* = \mathcal{N}(0, Q_\infty)$, see e.g. [18], Theorem 11.11(ii). At the same time, $P_t(x_0, \cdot) = \mathcal{N}(e^{At}x_0, Q_t)$, hence the measures $P_t(x_0, \cdot)$ and μ^* are mutually singular according to (27) and the Hájek-Feldman theorem (cf. e.g. [41], Theorems II.3.1 and II.3.4). This implies $\|\!|P_t(x_0, \cdot) - \mu^*\|\!| = 2$ and the measures $P_t(x_0, \cdot)$ cannot converge to the invariant measure in the total variation norm. Moreover, we see that nor the weaker assertion

$$\lim_{t\to\infty} P_t(x_0, B) = \mu^*(B) \quad \text{for any } B \subseteq H \text{ Borel} \tag{28}$$

holds true. Indeed, we know that there are Borel sets A_n, $n \geq 1$, such that $\mu^*(A_n) = 0$, $P_n(x_0, A_n) = 1$, so setting $B = \bigcup_{n\geq 1} A_n$ we obtain a counterexample to (28).

On the other hand we have

$$P_t(y, \cdot) \xrightarrow[t\to\infty]{\mathrm{w}^*} \mu^* \quad \text{for any } y \in H$$

by [49], Proposition 3.1, or [18], Theorem 11.11(i), therefore the invariant measure is globally asymptotically stable with respect to the narrow convergence. Hereafter, we denote by $\xrightarrow{\mathrm{w}^*}$ the narrow (or weak) convergence of finite (signed) Borel measures on H, that is,

$$\mu_\alpha \xrightarrow{\mathrm{w}^*} \mu \quad \text{if and only if} \quad \int_H f \,\mathrm{d}\mu_\alpha \longrightarrow \int_H f \,\mathrm{d}\mu \quad \forall f \in \mathcal{C}_b(H).$$

In finite-dimensional spaces, Lyapunov functions techniques are the basic tool for investigating stability properties of solutions to SDE's. A. Ichikawa [32] employed such an argument to establish uniqueness of an invariant measure for stochastic evolution equations, and later the procedure was extended to yield attractiveness as well, see the discussion in Section 2 above. The proofs based on Lyapunov functions have usually a lucid structure and lead, in a straightforward manner, to sufficient conditions for stability in terms of the coefficients of the equation. The known sufficient conditions, however, may be often too restrictive to cover interesting models. Furthermore, Itô's formula is not directly applicable to *mild* solutions of stochastic partial differential equations, nontrivial approximations are needed, and the class of admissible Lyapunov functions may be too narrow for useful applications, in particular if the Wiener process is cylindrical. Hence we content ourselves with stating a single typical result.

Let us consider a stochastic evolution equation

$$\mathrm{d}X_t = \bigl\{AX_t + f(X_t)\bigr\}\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t \tag{29}$$

in a separable Hilbert space H, where $A : \mathrm{Dom}(A) \longrightarrow H$ is an infinitesimal generator of a C_0-semigroup on H, W is a Wiener process in another (real, separable) Hilbert space U, with the covariance operator Q nuclear, and the mappings $f : H \longrightarrow H$, $\sigma : H \longrightarrow \mathcal{L}(U, H)$ are globally Lipschitz continuous. Denote by $\mathcal{C}^2(H)$ the set of all real valued functions on H having continuous the first and second Fréchet derivatives.

Theorem 10 ([49], Corollary 2.3). *Let there exist a function $V \in \mathcal{C}^2(H)$ satisfying:*

i) $V(0) = 0$ and

$$\inf_{\|y\|\geq r} V(y) > 0 \quad \textit{for any } r > 0;$$

ii) for some $k < \infty$, $p > 0$ and any $y \in H$ we have

$$V(y) + \|DV(y)\| + \|D^2V(y)\| \leq k\bigl(1 + \|y\|^p\bigr);$$

iii) there exists $a > 0$ such that

$$\big\langle DV(x-y), Ax - Ay + f(x) - f(y)\big\rangle \\ + \frac{1}{2}\operatorname{Tr}\big\{\big(\sigma(x)-\sigma(y)\big)^* D^2V(x-y)\big(\sigma(x)-\sigma(y)\big)Q\big\} \le -aV(x-y)$$

for all $x, y \in \operatorname{Dom}(A)$.

Then

$$\big(P_t(x,\cdot) - P_t(y,\cdot)\big) \xrightarrow[t\to\infty]{\mathrm{w}^*} 0$$

for any $x, y \in H$.

In particular, if there exists an invariant measure for (29) then it is globally asymptotically stable for the narrow convergence and, *a fortiori*, unique.

According to Corollary 2.8 in [49], the hypotheses of Theorem 10 are fulfilled with the natural choice $V = \|\cdot\|^p$ (for a suitable $p > 0$), provided

$$\langle Ax, x\rangle \le \beta\|x\|^2 \quad \text{for a } \beta\in\mathbb{R} \text{ and every } x\in\operatorname{Dom}(A), \\ \big\|Q^{1/2}\big(\sigma(x)-\sigma(y)\big)^*(x-y)\big\| \ge \alpha\|x-y\|^2 \quad \text{for an } \alpha \ge 0,$$

and

$$\beta + \operatorname{Lip}(f) + \frac{1}{2}\operatorname{Lip}(\sigma)^2 \operatorname{Tr} Q < \alpha^2,$$

$\operatorname{Lip}(\varUpsilon)$ denoting the Lipschitz constant of a mapping $\varUpsilon$.

As we have explained in Section 2, most of the recent results on the weak stability have been obtained by the "dissipativity method" of G. Da Prato and J. Zabczyk. To show the core of the method, we sketch here a proof of one of their results concerning a stochastic partial differential equation

$$\mathrm{d}X = \big(AX + f(X)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W \tag{30}$$

with an additive noise in a separable Hilbert space H. We assume that W is a standard cylindrical Wiener process in a Hilbert space U, $\sigma \in \mathcal{L}(U,H)$, and $A : \operatorname{Dom}(A) \longrightarrow H$ is a closed linear operator. To state the other hypotheses, we need a few additional definitions. If E is a Banach space, we denote by $\partial\|x\|_E$ the subdifferential of the norm $\|\cdot\|_E$ at the point $x \in E$. We say that a mapping $\gamma : \operatorname{Dom}(\gamma) \subseteq E \longrightarrow E$ is dissipative, provided for any $x, y \in \operatorname{Dom}(\gamma)$ there exists $z^* \in \partial\|x-y\|_E$ such that

$$z^*\big(\gamma(x) - \gamma(y)\big) \le 0.$$

A dissipative mapping γ is called m-dissipative, if $\operatorname{Im}(\lambda I - \gamma) = E$ for a $\lambda > 0$. Let $G \subseteq E$ be a subspace, a part γ_G of the mapping γ on G is defined by

$$\operatorname{Dom}(\gamma_G) = \{x \in \operatorname{Dom}(\gamma)\cap G;\ \gamma(x)\in G\}, \quad \gamma_G = \gamma \text{ on } \operatorname{Dom}(\gamma_G).$$

For completeness, we list here assumptions under which there exists a unique (generalized) mild solution of the equation (30) for any initial condition $X(0) = x \in H$, and (30) defines a Feller Markov process in H. We suppose:

1) There exists $\eta \in \mathbb{R}$ such that the mappings $A - \eta I$ and $f - \eta I$ are m-dissipative on H.

2) There exists a reflexive Banach space K densely and continuously imbedded in H, and $(A - \eta I)_K$, $(f - \eta I)_K$ are m-dissipative on K.

3) $\mathrm{Dom}(f) \supseteq K$ *and f maps bounded set in K into bounded sets in H.*

4) The process

$$W_A(t) = \int_0^t e^{A(t-r)} \sigma \, \mathrm{d}W(r), \quad t \geq 0,$$

is $\mathrm{Dom}(f_K)$*-valued, with paths continuous in H, and*

$$\sup_{t \in [0,T]} \big\{ \|W_A(t)\|_K + \|f(W_A(t))\|_K \big\} < \infty \quad \textit{almost surely}$$

for every $T > 0$.

Let us note that the introduction of an auxiliary space K is inevitable as interesting nonlinearities f are not defined (or do not behave well) on the basic state space H (compare Example 12 below).

Now we are prepared to state a theorem on existence and stability of an invariant measure (see Theorem 2.3 in [19], cf. also [20], Theorem 6.3.3).

Theorem 11. *Let there exist $\omega_1, \omega_2 \in \mathbb{R}$ such that $\omega \equiv \omega_1 + \omega_2 > 0$ and the mappings $A + \omega_1 I$, $f + \omega_2 I$ are dissipative on H. Suppose that*

$$\sup_{t \geq 0} \boldsymbol{E} \big\{ \|W_A(t)\|_H + \|f(W_A(t))\|_H \big\} < \infty.$$

Then there exists a unique invariant measure μ for (30) and for any $y \in H$ we have

$$P_t(y, \cdot) \xrightarrow[t \to \infty]{\mathrm{w}^*} \mu.$$

Moreover, there exists a constant $L < \infty$ such that

$$\left| \int_H g \, \mathrm{d}P_t(y, \cdot) - \int_H g \, \mathrm{d}\mu \right| \leq L \big(1 + \|y\|\big) e^{-\omega t/2} \mathrm{Lip}(g) \tag{31}$$

for any $y \in H$, $t > 0$ and any bounded Lipschitz function $g : H \longrightarrow \mathbb{R}$.

The procedure used in the proof, that is known as the "remote start method", yields in the present case existence and uniqueness of the invariant measure at the same time. We shall consider the equation (30) on the whole real line $\mathbb{R}$, that is, we shall work with solutions to

$$\mathrm{d}X_t = \big(AX_t + f(X_t)\big) \, \mathrm{d}t + \sigma \, \mathrm{d}\widetilde{W}_t, \tag{32}$$

where

$$\widetilde{W}(t) = \begin{cases} W(t), & t \geq 0, \\ Y(-t), & t < 0, \end{cases}$$

Y being a standard cylindrical Wiener process independent of W. Denote by $X(t; s, y)$, $t \geq s$, the unique solution of (32) with the initial datum $X(s; s, y) = y$. First, we derive an a priori estimate

$$\boldsymbol{E}\|X(t; s, y)\| \leq c + \|y\| \tag{33}$$

valid for all $s < 0$, $t \geq s$, $y \in H$. Setting

$$\Psi(t) = X(t; s, y) - \int_s^t e^{A(t-r)} \sigma \, \mathrm{d}\widetilde{W}(r)$$

we see that Ψ pathwise solves the equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = A\Psi + f\left(\Psi + \int_s^t e^{A(t-r)} \sigma \, \mathrm{d}\widetilde{W}(r)\right), \quad \Psi(s) = y.$$

Using the dissipativity hypothesis of Theorem 11 one easily finds that

$$\frac{\mathrm{d}^-}{\mathrm{d}t} \|\Psi(t)\| \leq -\omega \|\Psi(t)\| + \left\| f\left(\int_s^t e^{A(t-r)} \sigma \, \mathrm{d}\widetilde{W}(r)\right) \right\|,$$

which yields the desired estimate (33).

Analogously, for $v < s < 0$ one arrives at an estimate

$$\boldsymbol{E}\|X(t; s, y) - X(t; v, y)\| \leq e^{-\omega(t-s)} \big(2\|y\| + c\big), \quad t \geq s, \tag{34}$$

and it follows that the net $\{X(0; s, y), \ s \leq 0\}$ is Cauchy in $L^1(\Omega; H)$ as $s \to -\infty$. Let $p \in L^1(\Omega; H)$ be its limit, then the law μ of p is an invariant measure for (30): The L^1-convergence obviously implies the narrow convergence, therefore (P_t^* denoting the adjoint Markov semigroup)

$$P_t^* \delta_y = P_t(y, \cdot) = \mathrm{Law}(X(t; 0, y)) = \mathrm{Law}(X(0; -t, y)) \xrightarrow[t \to +\infty]{\mathrm{w}^*} \mathrm{Law}(p) = \mu,$$

and, since the Markov process solving (30) is Feller, we obtain

$$P_s^* \mu = P_s^* \Big(\lim_{t \to \infty} P_t^* \delta_y \Big) = \lim_{t \to \infty} P_{t+s}^* \delta_y = \mu$$

for any $s \geq 0$. The estimate (31) on the speed of convergence now follows from (34) in a straightforward way.

A similar theorem holds for equations with multiplicative noise, that is, for equations of the form (29), where W is now assumed to be a standard cylindrical Wiener process, see [13], Theorem 1, and [20], Theorem 6.3.2. We shall not cite the result precisely, let us only note that in this case the dissipativity assumption includes also the Yosida approximations $A_n = nA(nI - A)^{-1}$ of the operator A and reads as follows:

$$\left\langle A_n(x-y) + f(x) - f(y), x-y \right\rangle + \|\sigma(x) - \sigma(y)\|_{\mathrm{HS}}^2 \le -\varpi \|x-y\|^2$$

for a $\varpi > 0$ and any $x, y \in H$ and $n \in \mathbb{N}$.

We finish this section with an example which is very particular case of the example discussed in [19], Section 4, and in [20], §11.4, this example being based on Theorem 11.

Example 12. Let us consider a stochastic parabolic equation

$$\mathrm{d}X(t,\xi) = \left\{(\Delta - \alpha)X(t,\xi) + f(X(t,\xi))\right\}\mathrm{d}t + \mathrm{d}W(t,\xi), \quad \xi \in \mathbb{R},\ t \ge 0, \qquad (35)$$

where $\alpha > 0$ and W is a standard cylindrical Wiener process in $L^2(\mathbb{R})$. Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f = f_0 + f_1$, f_0 being (globally) Lipschitz continuous, $\xi \mapsto f_1(\xi) + b\xi$ is a continuous decreasing function for a $b \in \mathbb{R}$, and

$$|f_1(\xi)| \le c(1 + |\xi|^p)$$

for some $p \ge 1$, $c < \infty$ and every $\xi \in \mathbb{R}$. (For example, if f_1 is an odd degree polynomial with a negative leading coefficient,

$$f_1(\xi) = -\xi^{2k+1} + \sum_{j=0}^{2k} a_j \xi^j,$$

then the assumptions are satisfied.) Under the above hypotheses, there exists a unique (generalized) mild solution of (35) in the weighted space $L^2(\mathbb{R}; e^{-\varkappa|\xi|}\,\mathrm{d}\xi)$, for any $\varkappa > 0$. Moreover, suppose that f_1 is decreasing and

$$\alpha - \mathrm{Lip}(f_0) > 0.$$

Then there exists $\varkappa_0 > 0$ such that for any $\varkappa \in \,]0, \varkappa_0[$ the Markov process defined by (35) in the space $L^2(\mathbb{R}; e^{-\varkappa|\xi|}\,\mathrm{d}\xi)$ has a unique invariant measure, and an estimate of the type (31) holds for any $\omega \in \,]0, 2(\alpha - \mathrm{Lip}(f_0))[$.

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The Boundary-Value Problems for Laplace Equation and Domains with Nonsmooth Boundary

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Abstract. Dirichlet, Neumann and Robin problem for the Laplace equation is investigated on the open set with holes and nonsmooth boundary. The solutions are looked for in the form of a double layer potential and a single layer potential. The measure, the potential of which is a solution of the boundary-value problem, is constructed.

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Keywords. Laplace equation, Dirichlet problem, Neumann problem, Robin problem

Suppose that $G \subset R^m$ $(m \geq 2)$ is an open set with a non-void compact boundary ∂G such that $\partial G = \partial(\mathrm{cl}\ G)$, where cl G is the closure of G. Fix a nonnegative element λ of $\mathcal{C}'(\partial G)$ (the Banach space of all finite signed Borel measures supported in ∂G with the total variation as a norm) and suppose that the single layer potential $\mathcal{U}\lambda$ is bounded and continuous on ∂G. (In R^2 it means that $\lambda = 0$. If $G \subset R^m, (m > 2), \partial G$ is locally Lipschitz, $\lambda = f\mathcal{H}$, where $\mathcal{H}$ is the surface measure on ∂G and f is a nonnegative bounded measurable function, then $\mathcal{U}\lambda$ is bounded and continuous.) Here

$$\mathcal{U}\nu(x) = \int_{R^m} h_x(y)\ d\nu(y),$$

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where $\nu \in \mathcal{C}'(\partial G)$,

$$h_x(y) = (m-2)^{-1} A^{-1} |x-y|^{2-m} \text{ for } m > 2,$$

$$A^{-1} \log |x-y|^{-1} \text{ for } m = 2,$$

A is the area of the unit sphere in R^m.

If G has a smooth boundary,$u \in \mathcal{C}^1(\text{cl } G)$ is a harmonic function on G and

$$\frac{\partial u}{\partial n} + fu = g \text{ on } \partial G,$$

where $f, g \in \mathcal{C}(\partial G)$ (the space of all bounded continuous functions on ∂G equipped with the maximum norm) and n is the exterior unit normal of G then for $\phi \in \mathcal{D}$ (the space of all compactly supported infinitely differentiable functions in R^m)

$$\int_{\partial G} \phi g \, d\mathcal{H}_{m-1} = \int_G \nabla \phi \cdot \nabla u \, d\mathcal{H}_m + \int_{\partial G} \phi f u \, d\mathcal{H}_{m-1}. \tag{1}$$

Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized such that $\mathcal{H}_k$ is the Lebesgue measure in R^k. If we denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ on ∂G and by $N^G u$ the distribution

$$\langle \phi, N^G u \rangle = \int_G \nabla \phi \cdot \nabla u \, d\mathcal{H}_m \tag{2}$$

then (1) has the form

$$N^G u + f u \mathcal{H} = g \mathcal{H}. \tag{3}$$

Here $N^G u$ is a characterization in the sense of distributions of the normal derivative of u.

The formula (3) motivates the following definition of the solution of the Robin problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \text{ in } G, \\ N^G u + u\lambda &= \mu, \end{aligned} \tag{4}$$

where $\mu \in \mathcal{C}'(\partial G)$.

We introduce in R^m the fine topology, i.e. the weakest topology in which all superharmonic functions in R^m are continuous. This topology is stronger than the ordinary topology.

If u is a harmonic function on G such that

$$\int_H |\nabla u| \, d\mathcal{H}_m < \infty \tag{5}$$

for all bounded open subsets H of G we define the weak normal derivative $N^G u$ of u as a distribution

$$\langle \varphi, N^G u\rangle = \int_G \nabla\varphi \cdot \nabla u \; d\mathcal{H}_m$$

for $\varphi \in \mathcal{D}$.

Let $\mu \in \mathcal{C}'(\partial G)$.Now we formulate the Robin problem for the Laplace equation (4) as follows: Find a function $u \in L^1(\lambda)$ on cl G,the closure of G, harmonic on G and fine continuous in λ-a. a. points of ∂G for which ∇u is integrable over all bounded open subsets of G and $N^G u + u\lambda = \mu$.

The single layer potential $\mathcal{U}\nu$, where $\nu \in \mathcal{C}'(\partial G)$, has all these properties and if we look for a solution of the Robin problem in the form of the single layer potential we obtain the equation

$$N^G\mathcal{U}\nu + (\mathcal{U}\nu)\lambda = \mu.$$

It was shown by J. Král for $\lambda = 0$ (see [10]) and independently by Yu. D. Burago, V. G. Maz'ya (see [2]) and by I. Netuka ([20]) for a general λ that $N^G\mathcal{U}\nu + (\mathcal{U}\nu)\lambda \in \mathcal{C}'(\partial G)$ for each $\nu \in \mathcal{C}'(\partial G)$ if and only if $V^G < \infty$,where

$$V^G = \sup_{x\in\partial G} v^G(x),$$

$$v^G(x) = \sup\{\int_G \nabla\ \phi \cdot \nabla\ h_x\ d\mathcal{H}_m; \phi \in \mathcal{D}, |\phi| \le 1, spt\ \phi \subset R^m - \{x\}\}.$$

There are more geometrical characterizations of $v^G(x)$ which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \cup_{i=1}^k L_i$,where L_i are $(m-1)$-dimensional Ljapunov surfaces (i.e. of class $\mathcal{C}^{1+\alpha}$). Denote

$$\partial_e G = \{x \in R^m; \bar{d}_G(x) > 0, \bar{d}_{R^m-G}(x) > 0\}$$

the essential boundary of G, where

$$\bar{d}_M(x) = \limsup_{r\to 0_+} \frac{\mathcal{H}_m(M \cap U(x;r))}{\mathcal{H}_m(U(x;r))}$$

is the upper density of M at x, $U(x;r)$ is the open ball with the centre x and the radius r. Then

$$v^G(x) = \frac{1}{A}\int_{\partial U(0;1)} n(\theta, x)\ d\mathcal{H}_{m-1}(\theta),$$

where $n(\theta, x)$ is the number of all points of $\partial_e G \cap \{x + t\theta; t > 0\}$ (see [7]). It means that $v^G(x)$ is the total angle under which G is visible from the point x. This expression is a modification of the similar expression in [9]. Let us recall another characterization of $v^G(x)$ using a notion of an interior normal in Federer's sense.

If $z \in R^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in R^m; (x - z)\cdot\theta > 0\}$ has m-dimensional density zero at z

then $n^G(z) = \theta$ is termed the interior normal of G at z in Federer's sense. (The symmetric difference of B and C is equal to $(B - C) \cup (C - B)$.) If there is no interior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in R^m. The set $\{y \in R^m; |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial} G$. Clearly $\widehat{\partial} G \subset \partial_e G$.

If $\mathcal{H}_{m-1}(\partial_e G)$, the perimeter of G, is finite, then $\mathcal{H}_{m-1}(\partial_e G - \widehat{\partial} G) = 0$ and

$$v^G(x) = \int_{\widehat{\partial} G} |n^G(y) \cdot \nabla h_x(y)| \, d\mathcal{H}_{m-1}(y)$$

for each $x \in R^m$.

If G has a piecewise-$\mathcal{C}^{1+\alpha}$ boundary, then $V^G < \infty$. But there is a domain G with $\mathcal{C}^1$ boundary and $V^G = \infty$ (see [18]). On the other hand there is a domain G with $V^G < \infty$ and $\mathcal{H}_m(\partial G) > 0$. So open sets with a locally Lipschitz boundary and open sets with $V^G < \infty$ are incomparable.

Suppose now that $V^G < \infty$.Then the operator

$$\tau : \nu \mapsto N^G(\mathcal{U}\nu) + (\mathcal{U}\nu)\lambda$$

is a bounded linear operator on $\mathcal{C}'(\partial G)$ and

$$\tau\nu(M) = \int_{\partial G \cap M} \mathcal{U}\nu \, d\lambda + \int_{\partial G \cap M} d_G(x) \, d\nu(x) - \int_{\partial G} \int_{\partial G \cap M} n^G(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \, d\nu(x).$$

The Robin problem $N^G(\mathcal{U}\nu) + (\mathcal{U}\nu)\lambda = \mu$ leads to the equation

$$\tau\nu = \mu.$$

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ on $\widehat{\partial} G$. Then $\mathcal{H}(\partial G) < \infty$. If $\lambda = f\mathcal{H}$, $\nu = h\mathcal{H} \in \mathcal{C}'(\partial G)$, then

$$\tau(h\mathcal{H}) = (Th)\mathcal{H},$$

where

$$Th(x) = \frac{1}{2}h(x) - \int_{\partial G} n^G(x) \cdot \nabla h_y(x) h(y) \, d\mathcal{H}(y) + f(x)\mathcal{U}(h\mathcal{H})(x).$$

Theorem 1. *Let the Fredholm radius of $\big(\tau - (1/2)I\big)$ be greater than 2, $\mu \in \mathcal{C}'(\partial G)$. Then there is a harmonic function u on G, which is a solution of the Robin problem*

$$N^G u + u\lambda = \mu, \tag{6}$$

if and only if $\mu \in \mathcal{C}'_0(\partial G)$ *(the space of such* $\nu \in \mathcal{C}'(\partial G)$ *that* $\nu(\partial H) = 0$ *for each bounded component* H *of* cl G *for which* $\lambda(\partial H) = 0$*). If* $\mu \in \mathcal{C}'_0(\partial G)$ *then there is a unique* $\nu \in \mathcal{C}'_0(\partial G)$ *such that*

$$\tau\nu = \mu \tag{7}$$

and for such ν *the single layer potential* $\mathcal{U}\nu$ *is a solution of* (6)*. If*

$$\beta > \frac{1}{2}\,(V^G + 1 + \sup_{x \in \partial G} \mathcal{U}\lambda(x)), \tag{8}$$

then

$$\nu = \sum_{n=0}^{\infty} \left(\frac{\beta I - \tau}{\beta} \right)^n \frac{\mu}{\beta} \tag{9}$$

and there are $q \in (0,1)$*,* $C \in < 1, \infty)$ *such that*

$$\left\| \left(\frac{\tau - \beta I}{\beta} \right)^n \mu \right\| \le C q^n \|\mu\|$$

for $\mu \in \mathcal{C}'_0(\partial G)$ *and a natural number* n*. If* $\lambda = 0$ *then*

$$\nu = \mu + \sum_{n=0}^{\infty} (2\tau - I)^n (2\tau)\mu \tag{10}$$

and there are $q \in (0,1), C \in < 1, \infty)$ *such that*

$$\|(2\tau - I)^n (2\tau)\mu\| \le C q^n \|\mu\|$$

for $\mu \in \mathcal{C}'_0(\partial G)$ *and a natural number* n*.*

Remark 2. The condition that the Fredholm radius of $\big(\tau - (1/2)I\big)$ is greater than 2 does not depend on λ. In [15] it was shown that this condition has a local character. It is well-known that this condition is fulfilled for sets with a smooth boundary (of class $\mathcal{C}^{1+\alpha}$) (see [10]) and for convex sets (see [23]). J. Radon ([27]) proved this condition for open sets with "piecewise-smooth" boundary without cusps in the plane. R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in R^3 have this property (see [1], [12]). A. Rathsfeld showed in [28], [29] that polyhedral cones in R^3 have this property. (By a polyhedral cone in R^3 we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to R^2) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in R^3 we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons.) N. V. Grachev and V. G. Maz'ya obtained independently analogical result for polyhedral open sets with bounded boundary in R^3 (see [6]). (Remark that there is a polyhedral open set in R^3 which has not a locally Lipschitz boundary, for example $G = \{[x_1, x_2, x_3]; |x_1| < 3, |x_2| < 3, -3 < x_3 < 0\} \cup \{[3t, ty_2, ty_3]; 0 < t < 1, 1 < |y_2| < 2, 0 \leq y_3 < 1\}$. (The boundary of this set is not a graph of a function in a neighbourhood of the point $[0,0,0]$.)) The condition that the Fredholm radius of $\big(\tau - (1/2)I\big)$ is greater than 2 is fullfiled for $G \subset R^3$ with "piecewise-smooth" boundary, i.e. such that for each $x \in \partial G$ there are $r(x) > 0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x : U(x; r(x)) \to R^3$ of class $\mathcal{C}^{1+\alpha}$,$\alpha > 0$, such that $\psi_x(G \cap U(x; r(x))) = D_x \cap \psi_x(U(x; r(x)))$ (see [15]). N. V. Grachev and V. G. Maz'ya proved this condition for several types of sets with "piecewise-smooth" boundary in general Euclidean space (see [3,4,5]).

Remark 3. Let the Fredholm radius of $\big(\tau - (1/2)I\big)$ be greater than 2. Then it holds $\mathcal{H}_{m-1}(\partial G) < \infty$ and $\mathcal{H}$ is the restriction of $\mathcal{H}_{m-1}$ on ∂G. If $\lambda = f\mathcal{H}, \mu = g\mathcal{H} \in \mathcal{C}'(\partial G)$, then $\nu = h\mathcal{H}$, where $h \in L^1(\mathcal{H})$. If

$$\beta > \frac{1}{2}\,(V^G + 1 + \sup_{x \in \partial G} \mathcal{U}\lambda(x)),$$

then

$$h = \sum_{n=0}^{\infty} \left(\frac{\beta I - T}{\beta} \right)^n \frac{g}{\beta}$$

and there are $q \in (0,1)$, $C \in < 1, \infty)$ such that

$$\left\| \left(\frac{T - \beta I}{\beta} \right)^n g \right\| \leq C q^n \|g\|$$

for a natural number n and $g \in L^1(\mathcal{H})$ such that $g\mathcal{H} \in \mathcal{C}_0'(\partial G)$. If $f = 0$, then

$$h = g + \sum_{n=0}^{\infty} (2T - I)^n (2T) g$$

and there are $q \in (0,1), C \in < 1, \infty)$ such that

$$\|(2T - I)^n (2T) g\| \leq C q^n \|g\|$$

for a natural number n and $g \in L^1(\mathcal{H})$ such that $g\mathcal{H} \in \mathcal{C}_0'(\partial G)$.

Now, let us concentrate on the Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \text{ in } G, \\ u &= g \text{ on } \partial G, \end{aligned} \tag{11}$$

where $g \in \mathcal{C}(\partial G)$ is a continuous function on the boundary of G. Looking for a solution in the form of the double layer potential

$$W f(x) = \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \; d\mathcal{H}_{m-1}(y) \tag{12}$$

is a classical method. It was shown by J. Král and independently by Yu. D. Burago and V. G. Maz'ya that it is possible to define the double layer potential (12) on G as a continuously extendable function on cl G for each density $f \in \mathcal{C}(\partial G)$ if and only only if $V^G < \infty$. (This condition we obtained for the Robin problem, too.) Under this condition $n^G(y)$ in the expession (12) is the interior normal of G at y in Federer's sense. If we look for the solution of the Dirichlet problem (11) in the form of the double layer potential (12) with a continuous density on the boundary of G we obtain the integral operator

$$Df(x) = (1 - d_G(x))f(x) + \int_{\partial G} f(y)n^G(y) \cdot \nabla h_x(y)\, d\mathcal{H}_{m-1}(y).$$

on $\mathcal{C}(\partial G)$. The adjoint operator of D is the operator corresponding to the Neumann problem for the Laplace operator on the complementary domain to G. Noting that the Fredholm radius of $(D - \frac{1}{2}I)$ is equal to the Fredholm radius of $\big(\tau - (1/2)I\big)$ we obtain as a consequence of the theorem for the Neumann problem the following result:

Theorem 4. *Let $V^G < \infty$, the Fredholm radius of $(D - \frac{1}{2}I)$ be greater than 2. If the set $R^m - G$ is unbounded and connected and $g \in \mathcal{C}(\partial G)$, then the double layer potential*

$$Wf(x) = \int_{\partial G} f(y)n^G(y) \cdot \nabla h_x(y)\, d\mathcal{H}_{m-1}(y)$$

is a solution of the Dirichlet problem for the Laplace equation with the boundary condition g, where

$$f = g + \sum_{j=0}^{\infty} (2D - I)^j 2Dg.$$

The condition that the set $R^m - G$ is unbounded and connected is necessary for expressing the solution of the Dirichlet problem for the Laplace equation in the form of the double layer potential for each boundary condition. If we want to calculate the solution for an open set with holes we must modify a double layer potential. Suppose now that the dimension of the space R^m is greater than 2. If we look for a solution of the Dirichlet problem in the form of the sum of the single layer potential and the double layer potential with the same density we obtain the integral operator on the space of all continuous functions in ∂G the adjoint operator of which is the operator corresponding to some Robin problem for the Laplace equation on the complementary domain and we obtain the following result as a consequence of the theorem on the Robin problem.

Theorem 5. *Let $m > 2, V^G < \infty$, the Fredholm radius of $(D - \frac{1}{2}I)$ be greater than 2. If $g \in \mathcal{C}(\partial G)$ then $Wf + \mathcal{U}(f\mathcal{H})$ is a solution of the Dirichlet problem for the Laplace equation with the boundary condition g, where*

$$f = \sum_{n=0}^{\infty} \left(\frac{\beta I - V}{\beta} \right)^n \frac{g}{\beta},$$

$$Vg = Dg + \mathcal{U}(g\mathcal{H}),$$

$$\beta > \frac{1}{2}(V^G + 1 + \sup_{x\in\partial G} \mathcal{U}\mathcal{H}(x)).$$

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Singular Integral Inequalities and Stability of Semilinear Parabolic Equations

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Abstract. Using a method developed by the author for an analysis of singular integral inequalities a stability theorem for semilinear parabolic PDEs is proved.

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1 Introduction

Integral inequalities play an important role in the theory of differential, integral and integrodifferential equations. One can hardly imagine these theories without the well-known Gronwall inequality and its nonlinear version Bihari inequality [1]. However these inequalities are not directly applicable to integral equations with weakly singular kernels of the form

$$x(t) = \xi(t) + \int_0^t K(t,s) f(s, x(s)) ds, \ x \in X, \tag{1}$$

where X is a Banach space, $K(t,s) : X \to X$ is a linear operator satisfying the condition

$$||K(t,s)|| \leqq \frac{M}{(t-s)^\alpha} ||v||, \ v \in X, \tag{2}$$

for $t > s \geqq 0$, $\alpha > 0, M > 0$ are constants, ξ, f are continuous maps. Such equations appear e.g. in the geometric theory of parabolic differential equations. Basics of this theory are described in the well-known book by D. Henry [4] (see also the book by J. K. Hale [3]).

Many boundary value problems for parabolic PDEs can be written as a Cauchy initial value problem

$$\begin{gathered}\frac{du}{dt} + Au = f(t,u),\ u \in X,\\ u(0) = u_0 \in X,\end{gathered} \tag{3}$$

where X is an appropriate Banach space and $A : X \to X$ is a special linear operator, so called sectorial operator (for the definition see [4, Definition 1.3.1]). For any sectorial operator A there is a real number c such that if $A_1 = A + cI$, where I is the identity mapping, then Re $\sigma(A_1) > 0$ (i.e. Re $\lambda > 0$ for any $\lambda \in \sigma(A_1)$ — the spectrum of the operator A_1). One can define a fractional power A_1^α of A_1 as the inverse of $A_1^{-\alpha} := \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-A_1 t}dt$ for $\alpha > 0$. If $X^\alpha := D(A_1^\alpha)$ — the domain of A_1^α and $||x||_\alpha := ||A_1^\alpha x||, x \in X^\alpha$, then $(X^\alpha, ||.||_\alpha)$ is a Banach space (see [4]).

By [4, Theorem 1.3.4], if A is a sectorial operator then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geqq 0}$, $\frac{d}{dt}e^{-tA} = -Ae^{-tA}$ for $t > 0$ and if Re $\sigma(A) > b > 0$ then

$$||e^{-tA}u||_\alpha := ||A_1^\alpha e^{-tA}u|| \leqq \frac{d}{t^\alpha}e^{-bt}||u||,\ t > 0 \tag{4}$$

for any $u \in X^\alpha$, where $d > 0$ is a constant.

Definition 1 (see [3] and [7]). Let $A : X \to X$ be a sectorial operator and there is an $\alpha \in \langle 0, 1)$ such that the map $f : R \times X^\alpha \to X, (t,u) \mapsto f(t,u)$ is locally Hölder in t and locally Lipschitz in u. A solution of (3) on the interval $\langle 0, T)$ $(0 < T \leqq \infty)$ is a continuous function $u : \langle 0, T) \to X^\alpha$ with $u(0) = u_0 \in X^\alpha$ such that the map $f(., u(.)) : \langle 0, T) \to X, t \to f(t, u(t))$ is continuous, $u(t) \in D(A), t \in \langle 0, T)$ and u satisfies (3) on $(0, T)$.

By M. Miklavčič [7] a solution $u(t)$ of (3) in the sense of Definition 1 coincides with those solutions of the integral equations

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(s, u(s))ds,\ 0 < t \leqq T, \tag{5}$$

for which $u : \langle 0, T) \to X^\alpha$ is continuous and $f(., u(.)) : \langle 0, T) \to X, t \to f(t, u(t))$ is continuous.

If Re $\sigma(A) > b > 0$ then from (4), (5) it follows that

$$||u(t)||_\alpha \leqq \frac{ce^{-bt}}{t^\alpha}||u_0|| + de^{-bt}\int_0^t \frac{e^{bt}}{(t-s)^\alpha}||f(s, u(s))||ds. \tag{6}$$

If

$$||f(v)|| \leqq Q||v||_\alpha, v \in X^\alpha, \tag{7}$$

$a(t) = \frac{c}{t^\alpha}||u_0||, v(t) = ||u(t)||_\alpha e^{bt}$ and $c = dQ$, then (6) yields

$$v(t) \leqq a(t) + c\int_0^t (t-s)^{\beta-1} v(s)ds,\ t \in I = \langle 0, T),\tag{8}$$

where $\beta = 1 - \alpha,\ \alpha > 0$. By [4, Lemma 7.1.1]

$$v(t) \leqq \Theta \int_0^t E_\beta^{'}(\Theta(t-s))a(s)ds,\ t \in I,\tag{9}$$

where $\Theta = (c\Gamma(\beta))^{\frac{1}{\beta}}, R_\beta(z) = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}, \Gamma$ is the gamma-function and finally $E_\beta^{'}(z) = \frac{dE_\beta(z)}{dz}$.

The estimate (9) is obviously complicated and it is obtained in [4] by an iterative argument not applicable to the case of nonlinear integral inequalities. In the paper [6] the author developed a new method of a reduction of the inequality (8) as well as some nonlinear singular inequalities to the classical Gronwall and Bihari inequalities, respectively. Using this method we shall analyze an inequality of the form

$$\psi(t) \leqq a(t) + b(t)\int_0^t (t-s)^{\beta-1} s^{\gamma-1} \psi(s)^m ds,\ t \in I = \langle 0, T),\tag{10}$$

where $0 < T \leqq \infty$ and $m > 1$ with the aim to prove a stability theorem for the equation (3).

2 Stability theorem

First let us formulate a consequence of a result by G. Butler and T. Rogers published in [2] (see also [5, Theorem 1.3.8]) as the following lemma.

Lemma 2. *Let $a(t), b(t), K(t), \psi(t)$ be nonnegative, continuous function on $I = \langle 0, T)\,(0 < T \leqq \infty), \omega : \langle 0, \infty) \to R$ be a continuous, nonnegative and nondecreasing function, $\omega(0) = 0$, $\omega(u) > 0$ for $u > 0$ and let $A(t) = \max_{0 \leqq s \leqq t} a(s)$, $B(t) = \max_{0 \leqq s \leqq t} b(s)$. Assume that*

$$\psi(t) \leqq a(t) + b(t)\int_0^t K(s)\omega(\psi(s))ds, t \in I.\tag{11}$$

Then

$$\psi(t) \leqq \Omega^{-1}[\Omega(A(t)) + B(t)\int_0^t K(s)ds],\ t \in \langle 0, T_1\rangle,\tag{12}$$

where $\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}, v \geqq v_0 > 0, \Omega^{-1}$ is the inverse of Ω and $T_1 > 0$ is such that $\Omega(A(t)) + B(t)\int_0^T K(s)ds \in D(\Omega^{-1})$ for all $t \in \langle 0, T_1\rangle$.

Lemma 3. *Let $a(t), F(t), \psi(t), b(t)$ be continuous, nonnegative functions on $I = \langle 0, T)\,(0 < T \leqq \infty), \beta > 0, \gamma > 0, m > 1$ and $\psi(t)$ satisfies the inequality (10). Then the following assertions hold:*

(1) *If $\beta > \frac{1}{2}, \gamma > 1 - \frac{1}{2p}$ for some $p > 1$ and $\varepsilon > 0$ then*

$$\psi(t) \leqq e^{\varepsilon t} \Phi_\varepsilon(t), \tag{13}$$

where $\Phi_\varepsilon(t) = A_1(t)^{\frac{1}{2q}}[1 - (m-1)\Xi_1(t,\varepsilon)]^{\frac{1}{2q(1-m)}}$,
$\Xi_1(t,\varepsilon) = A_1(t)^{m-1} B_1(t,\varepsilon) \int_0^t F(s)^{2q} e^{2qm\varepsilon s} ds$,
$A_1(t) = 2^{2q-1} \max_{0 \leqq s \leqq t} a(s)^{2q}$,
$B_1(t,\varepsilon) = 2^{2q-1} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} \max_{0 \leqq s \leqq t} b(s)^{2q}, K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}$,
$L(\varepsilon) = \frac{\Gamma((2\gamma-2)p+1)}{(p\varepsilon)^{(2\gamma-2)p+1}}, \frac{1}{p} + \frac{1}{q} = 1$ and $t \in I$ is such that $\Phi_\varepsilon(t)$ is defined.

(2) *Let $\beta = \frac{1}{1+z}$ for some $z \geqq 1, \gamma > 1 - \frac{1}{kq}$, where $k > 0, q = z + 2$ and let $\varepsilon > 0$. Then*

$$\psi(t) \leqq e^{\varepsilon t} \Psi_\varepsilon(t), \tag{14}$$

where $\Psi_\varepsilon(t) = A_2(t)^{\frac{1}{rq}}[1 - (m-1)\Xi_2(t,\varepsilon)]^{\frac{1}{rq(1-m}}$,
$\Xi_2(t,\varepsilon) = A_2(t)^{m-1} B_2(t,\varepsilon) \int_0^t F(s)^{rq} e^{mqr\varepsilon s} ds$,
$A_2(t) = 2^{rq-1} \max_{0 \leqq s \leqq t} a(s)^{rq}$,
$B_2(t,\varepsilon) = 2^{rq-1} P(\varepsilon) \max_{0 \leqq s \leqq t} b(s)^{rq}$,
$P(\varepsilon) = (M(\varepsilon)N(\varepsilon))^{rq}, M(\varepsilon) = [\frac{\Gamma(1-\alpha p)}{(p\varepsilon)^{1-\alpha p)}}]^{\frac{1}{p}}$,
$N(\varepsilon) = [\frac{\Gamma(kq(\gamma-1)+1)}{(kq\varepsilon)^{kq(\gamma-1)+1}}]^{\frac{1}{kq}}, \alpha = 1 - \beta, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{k} + \frac{1}{r} = 1, p, q, r, k > 1$ and $t \in I$ is such that $\Psi_\varepsilon(t)$ is defined.

Proof. We shall repeat the same procedure as in the proof of [6, Theorem 4] however instead of inserting $e^t.e^{-t}$ into the integral on the right-hand side of (10) and then applying the Cauchy-Schwarz and Hölder inequality, respectively, we shall insert $e^\varepsilon.e^{-\varepsilon t}$ there. More precisely, under the assumption of the assertion (1) we obtain from (10) that

$$\psi(t) \leqq a(t) + b(t)[\int_0^t (t-s)^{2\beta-2} e^{2\varepsilon s} ds]^{\frac{1}{2}} [\int_0^t s^{2\gamma-2} F(s)^2 e^{-2\varepsilon s} \psi(s)^{2m} ds]^{\frac{1}{2}} \leqq$$
$$\leqq a(t) + b(t) e^{\varepsilon t} K(\varepsilon)^{\frac{1}{2}} [\int_0^t s^{2\gamma-2} F(s)^2 e^{-2\varepsilon s} \psi(s)^{2m} ds]^{\frac{1}{2}},$$

where $K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}$. Using the Hölder inequality with $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ we obtain

$$\psi(t) \leqq a(t) + b(t) e^{\varepsilon t} K(\varepsilon)^{\frac{1}{2}} [\int_0^t s^{(2\gamma-2)p} e^{-\varepsilon p s} ds]^{\frac{1}{2p}} [\int_0^t F(s)^{2q} \psi(s)^{2mq} ds]^{\frac{1}{2q}}$$

and since

$$\int_0^t s^{(2\gamma-2)p} e^{-p\varepsilon s} ds = \frac{1}{(p\varepsilon)^{(2\gamma-2)p+1}} \int_0^{p\varepsilon t} \sigma^{(2\gamma-2)p} e^{-\sigma} d\sigma < \\ < \frac{\Gamma((2\gamma-2)p+1)}{(p\varepsilon)^{(2\gamma-2)p+1}} := L(\varepsilon)$$

($(2\gamma-2)p+1 > [2(1-\frac{1}{2p})-2]p+1 > 0$, i.e. $\Gamma((2p-2)p+1)$ is a positive number) we have

$$\psi(t) \leqq a(t) + b(t) e^{\varepsilon t} K(\varepsilon)^{\frac{1}{2}} L(\varepsilon)^{\frac{1}{2p}} [\int_0^t F(s)^{2q} \psi(s)^{2mq} ds]^{\frac{1}{2q}}. \tag{15}$$

Since $(A_1 + A_2)^r \leqq 2^{r-1}(A_1^r + A_2^r)$ for any nonnegative real numbers A_1, A_2 and any real number $r > 1$ (see [6, (2), (3)]) we obtain from (15) that

$$\psi(t)^{2q} \leqq 2^{2q-1}[a(t)^{2q} + b(t)^{2q} e^{2q\varepsilon t} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} \int_0^t F(s)^{2q} e^{2qm\varepsilon s} e^{-2q\varepsilon s} \psi(s)^{2q})^m ds]. \tag{16}$$

If

$$v(t) = e^{-2q\varepsilon t} u(t)^{2q}, c(t) = 2^{2q-1} a(t)^{2q}, d(t) = 2^{2q-1} b(t)^{2q} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}}, \tag{17}$$

then (16) yields

$$v(t) \leqq c(t) + d(t) \int_0^t F(s)^{2q} e^{2qm\varepsilon s} v(s)^m ds.$$

Now we can apply Lemma 2, where $\omega(u) = u^m$, $\Omega(v) = \int_{v_0}^v \frac{dy}{\omega(y)} = \int_{v_0}^v y^{-m} dy = \frac{1}{m-1}[v^{1-m} - v_0^{1-m}]$, $\Omega^{-1}(z) = [(1-m)z + v_0^{1-m}]^{\frac{1}{1-m}}$ and we obtain the inequality

$$v(t) \leqq \Omega^{-1}[\Omega(A_1(t)) + B_1(t,\varepsilon) \int_0^t F(s)^{2q} e^{2qm\varepsilon s} ds] = \\ = A_1(t)[1 - (m-1)\Xi_1(t,\varepsilon)]^{\frac{1}{1-m}},$$

where $\Xi_1(t,\varepsilon), A_1(t), B_1(t,\varepsilon)$ are as in theorem. From this inequality and (17) the inequality (13) follows.

The proof of the inequality (14) is similar (see the proof of [6, Theorem 4]).

Theorem 4. *Let $A : X \to X$ be a sectorial operator, $Re\, \sigma(A) > b > 0, f$ be as in Definition 1 and let*

$$||f(t,u)|| \leqq t^\kappa \eta(t) ||u||_\alpha^m, \quad m > 1,\ \kappa \geqq 0 \tag{18}$$

for all $(t,u) \in R \times X^\alpha$, where $\eta : \langle 0, \infty) \to R$ is a continuous, nonnegative function. Then the following assertions hold:

(1) *Let* $0 < \alpha < \min\{\frac{1}{2}, \frac{\kappa}{m} + \frac{1}{2pm}\}$ *for some* $p > 1$ *and* $b > 0$ *be the number from the inequality* (4). *Let the function*

$$t \mapsto t^{2q\alpha} \int_0^t \eta(s)^{2q} e^{2q[(1-m)b+m\varepsilon]s} ds$$

is bounded on the interval $\langle 0, \infty)$ *for some* $0 < \varepsilon < b$, *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $u(t)$ *be a solution of the equation* (3) *satisfying* $u(0) = u_0 \in X^\alpha$, *where*

$$(m-1)2^{2q-1}(c||u_0||)^{2q(m-1)} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} (ct^\alpha)^{2q} \int_0^t \eta^{2q} e^{2q[(1-m)b+m\varepsilon]s} ds < 1,$$

where

$$K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}, L(\varepsilon) = \frac{\Gamma((2\gamma-2)p+1)}{(2\gamma-2)p+1}, \beta = 1-\alpha.$$

Then $u(t)$ *exists on the interval* $\langle 0, \infty)$ *and* $\lim_{t\to\infty} ||u(t)||_\alpha = 0$.

(2) *Let* $\frac{1}{2} \leqq \alpha < \min\{1, \frac{\kappa}{m} + \frac{1}{kqm}\}$ *for some* $k > 1$, *where* $\beta = 1-\alpha = \frac{1}{1+z}$, $z \geqq 1$, $q = z+2$ *and* $b > 0$ *is the number from the inequality* (4). *Assume that the function*

$$t \mapsto t^{rq\alpha} \int_0^t \eta(s)^{rq} e^{rq[(1-m)b+m\varepsilon]s} ds$$

is bounded on the interval $\langle 0, \infty)$ *for some* $0 < \varepsilon < b$, *where* $\frac{1}{k} + \frac{1}{r} = 1$. *Let* $u(t)$ *be a solution of the equation* (3) *satisfying* $u(0) = u_0$, *where*

$$(m-1)2^{rqm}(c||u_0||)^{rq(m-1)} P(\varepsilon) t^{rq\alpha} \int_0^t \eta(s)^{rq[(1-m)b+m\varepsilon]s} ds \begin{cases} < 1 & \text{for } rq(m-1) \text{ even}, \\ \neq 1 & \text{for } rq(m-1) \text{ odd}, \end{cases}$$

where $P(\varepsilon)$ *is the number defined in Lemma* 3. *Then* $u(t)$ *exists on the interval* $\langle 0, \infty)$ *and* $\lim_{t\to\infty} ||u(t)||_\alpha = 0$.

Proof. Under the assumptions of theorem there exists a solution of the equation (3) on an interval $I = \langle 0, T)(0 < T \leqq \infty)$ satisfying the condition $u(0) = u_0$. This solution satisfies the equation (5) and for $\alpha > 0$ the inequality (6) is satisfied. This inequality and the condition (18) yield

$$||u(t)||_\alpha \leqq \frac{ce^{-bt}}{t^\alpha} ||u_0|| + ce^{-bt} \int_0^t \frac{e^{bs} s^\kappa \eta(s)}{(t-s)^\alpha} ||u(s)||_\alpha^m ds, \quad t > 0$$

and if $\psi(t) = e^{bt} t^\alpha ||u(t)||_\alpha$ then

$$\psi(t) \leqq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{\gamma-1} F(s) \psi(s)^m ds, \tag{19}$$

where $a(t) = c||u_0||, b(t) = ct^\alpha, \beta = 1-\alpha, \kappa = 1+\kappa-\alpha m, F(t) = e^{(1-m)bt}\eta(t)$.

Let us prove the assertion (1). From the assumption it follows that $\alpha < \frac{1}{2}$, i.e. $\beta = 1-\alpha > \frac{1}{2}$ and $-\alpha m > -\kappa - \frac{1}{2p}$, i.e. $\gamma = 1+\kappa-\alpha m > 1-\frac{1}{2p}$. Thus the assumptions of Lemma 3 are satisfied. By the assertion (1) of this lemma we obtain that $\psi(t) \leqq e^{\varepsilon t}\Phi(t,\varepsilon)$, where

$$\Phi(t,\varepsilon) = A_1(t)^{\frac{1}{2q}}[1-(m-1)\Xi_1(t,\varepsilon)]^{\frac{1}{2q(1-m)}},$$
$$A_1(t,\varepsilon) = 2^{2q-1}(c||u_0||)^{2q},$$
$$\Xi_1(t,\varepsilon) = 2^{2q(m-1)}(c||u_0||)^{2q(m-1)}K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}}.t^{2q\alpha}\int_0^t \eta(s)^{2q}e^{2q[(1-m)b+m\varepsilon]s}ds,$$

$K(\varepsilon), L(\varepsilon)$ are defined in Lemma 3. Under the assumptions of theorem the function $\Phi(t,\varepsilon)$ is bounded on the interval $(0,\infty)$. Since $\psi(t) = e^{bt}t^\alpha||u(t)||_\alpha$, $0<\varepsilon<b$, we obtain that

$$||u(t)||_\alpha \leqq \frac{e^{-(b-\varepsilon)t}}{t^\alpha}\Phi(t,\varepsilon).$$

Thus the solution $u(t)$ of (3) exists on the interval $\langle 0,\infty)$ and $\lim_{t\to\infty}||u(t)||_\alpha = 0$. From the assumption of the assertion (2) it follows that $\beta = 1-\alpha \leqq \frac{1}{2}, -\alpha m > -\kappa - \frac{1}{kq}$, i.e. $\gamma = 1+\kappa-\alpha m > 1-\frac{1}{kq}$ and thus the assumptions of the assertion (2) of Lemma 3 are satisfied. Applying this lemma in the same way as in the proof of the assertion (1) one can prove the assertion (2).

Remark 5. M. Miklavčič in his paper [7] proved that if for some $0<\omega\leqq 1$, $0<\alpha<1$, $\alpha\omega p>1$, $\gamma>1$, $C>0$, $||t^\omega Ae^{-At}|| \leqq C$, $t \geqq 1$,

$$||f(t,x)|| \leqq C[||A^\alpha x||^p + (1+t)^{-\gamma}], \quad t \geqq 0,$$

whenever $||A^\alpha x|| + ||x||$ is small enough, then for small initial data there exist stable global solutions. Moreover, if the space X is reflexive (in this case $X = N(A) \oplus \overline{R(A)}$), then there exists $y \in N(A)$ such that $\lim_{t\to\infty}||x(t)-y||_\alpha = 0$. These results are obviously proved under different assumptions from those in our theorem.

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Fixed Point Theory for Closed Multifunctions

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Abstract. In this paper some new fixed point theorems of Ky Fan, Leray-Schauder and Furi-Pera type are presented for closed multifunctions.

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1 Introduction

This paper establishes some fixed point theorems for multivalued condensing maps with closed graph. In particular we obtain an analogue of (i). Ky Fan's Fixed Point Theorem, (ii). Leray-Schauder Alternative, and (iii). Furi-Pera Fixed Point Theorem, for such maps. The need for new fixed point theory for closed multifunctions arose out of the study of differential and integral inclusions (see [5,9] and their references). If our operator is compact then a well known result (see [1, page 465]) implies that we may use fixed point theory for upper semicontinuous (u.s.c.) maps. However a new theory is needed if our map is condensing and not compact. We initiated the study in [10,11]. This paper continues this study. In addition we simplify some of the proofs in [10].

For the remainder of this section we describe the type of maps which we will consider in section 2. Suppose X and Z are subsets of Hausdorff topological vector spaces E_1 and E_2 respectively and $F : X \to 2^Z$ a multifunction (here 2^Z denotes the family of nonempty subsets of Z). Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U, V)-approximate continuous selection [2,3] of F is a continuous function $s : X \to Z$ satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Z \qquad \text{for every } x \in X.$$

F is said to be **approximable** [3] if its restriction $F|_K$ to any compact subset K of X admits a (U,V)-approximate continuous selection for every open neighborhoods U and V of the origins in E_1 and E_2 respectively.

Definition 1. We say $F \in APCG(X,Y)$ if $F : X \to Cc(Y)$ is a closed (i.e. has closed graph), approximable map; here $Cc(Y)$ denotes the family of nonempty, closed subsets of Y.

Definition 2. We say $F \in ACG(X,Y)$ if $F : X \to CD(Y)$ is a closed map; here $CD(Y)$ denotes the family of nonempty, closed, acyclic (see [5]) subsets of Y.

Recall F is acyclic *if for every* $x \in X$, $H^m(F(x)) = \delta_{0m}\mathbb{Z}$, where $\{H^m\}$ denotes the Čech cohomology functor with integer coefficients.

We now recall two results from the literature.

Theorem 3 ([2,3]). *Let Q be a convex, compact subset of a locally convex Hausdorff linear topological space E and $F : Q \to C(Q)$ is a u.s.c., approximable map (here $C(Q)$ denotes the family of nonempty, compact subsets of Q). Then F has a fixed point.*

Let X be a Banach space and Ω_X the bounded subsets of X. The *Kuratowski measure of noncompactness* is the map $\alpha : \Omega_X \to [0,\infty]$ defined by

$$\alpha(Z) = \inf\,\{\epsilon > 0 :\ Z \subseteq \cup_{i=1}^{n} Z_i \ \text{ and } \ diam\,(Z_i) \le \epsilon\}\,; \quad \text{here } \ Z \in \Omega_X.$$

Let X_1 and X_2 be Banach spaces. A multivalued map $F : Y \subseteq X_1 \to X_2$ is said to be α-Lipschitzian if it maps bounded sets into bounded sets and if there exists a constant $k \ge 0$ with $\alpha(F(Z)) \le k\,\alpha(Z)$ for all bounded sets $Z \subseteq Y$. We call F a **condensing** map if F is α-Lipschitzian with $k = 1$ and $\alpha(F(Z)) < \alpha(Z)$ for all bounded sets $Z \subseteq Y$ with $\alpha(Z) \ne 0$.

Theorem 4 ([5]). *Let Q be a nonempty, closed, convex subset of a Banach space E. Suppose $F : Q \to CK(Q)$ is a u.s.c., condensing map with $F(Q)$ a subset of a bounded set in E (here $CK(Q)$ denotes the family of nonempty, compact, acyclic subsets of Q). Then F has a fixed point.*

Remark 5. All the results in this paper will be stated and proved when E is a Banach space (the extension to the case when E is a Fréchet space is immediate).

2 Fixed point theory

We begin this section by proving fixed point theorems of Ky Fan [12] type for $APCG$ and ACG maps.

Theorem 6. *Let Q be a nonempty, convex, closed subset of a Banach space E and suppose $F \in APCG(Q,Q)$ is a condensing map with $F(Q)$ a subset of a bounded set in Q. Then F has a fixed point in Q.*

Proof. Let $x_0 \in Q$. Then [5, Lemma A] guarantees a closed, convex set X with $x_0 \in X$ and

$$X = \overline{co}\,(F(Q \cap X) \cup \{x_0\}).$$

Since $F(Q) \subseteq Q$ implies $F(Q \cap X) \cup \{x_0\} \subseteq Q$ we have $X \subseteq Q$ and so $Q \cap X = X$. Thus

$$X = \overline{co}\,(F(X) \cup \{x_0\}).$$

Since F is condensing we have (using the properties of measure of noncompactness) that X is compact. Thus $F : X \to 2^X$ with X compact and convex. In addition the values of F are closed and $F|_X$ has closed graph. Now [1, page 465] implies $F|_X$ is u.s.c. Consequently $F|_X : X \to C(X)$ is a u.s.c., approximable map and X is convex and compact. Theorem 3 implies that F has a fixed point in X. □

Similarly we have the following result for ACG maps.

Theorem 7 ([11]). *Let Q be a nonempty, convex, closed subset of a Banach space E and suppose $F \in ACG(Q, Q)$ is a condensing map with $F(Q)$ a subset of a bounded set in Q. Then F has a fixed point in Q.*

Proof. Let $x_0 \in Q$ and construct a convex, compact set $X \subseteq Q$ (as in Theorem 6) with $F : X \to 2^X$. In addition the values of F are closed and acyclic and $F|_X$ has closed graph. Now [1] implies $F|_X$ is u.s.c. Consequently $F|_X : X \to CK(X)$ is a u.s.c. map and X is convex and compact. Theorem 4 (or indeed Ky Fan's Fixed Point Theorem [12]) implies that F has a fixed point in X. □

Remark 8. Note Theorem 6 and Theorem 7 can easily be extended to the Fréchet space setting.

We now prove a nonlinear alternative of Leray-Schauder type for ACG and $APCG$ maps. We proved such an alternative in [10]; however here we provide a simpler proof.

Theorem 9. *Let E be a Banach space with U an open, convex subset of E and $x_0 \in U$. Suppose $F \in ACG(\overline{U}, E)$ is a condensing map with $F(\overline{U})$ a subset of a bounded set in E. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*

(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda\, F(u) + (1-\lambda)\{x_0\}$.*

Proof. Without loss of generality assume $x_0 = 0$. Suppose $(A2)$ does not occur and F has no fixed points in ∂U. Let

$$H = \left\{x \in \overline{U} :\ x \in \lambda\, F(x) \quad \text{for some} \quad \lambda \in [0,1]\right\}.$$

Notice that $H \neq \emptyset$ is closed. To see this let (x_n) be a sequence in H (i.e. $x_n \in \lambda_n\, F(x_n)$ for some $\lambda_n \in [0,1]$) with $x_n \to x_0 \in \overline{U}$. Without loss of generality assume $\lambda_n \to \lambda_0 \in (0,1]$. Since $x_n \in H$ there exists $y_n \in F(x_n)$ with $x_n = \lambda_n\, y_n$. Now $x_n \to x_0$ and $y_n \to \frac{1}{\lambda_0}\, x_0$. The closedness of F implies $\frac{1}{\lambda_0}\, x_0 \in F(x_0)$ so $x_0 \in H$. Thus H is closed. In fact H is compact. To see this notice $H \subseteq \overline{co}\,(F(H) \cup \{0\})$ so if $\alpha(H) \neq 0$, we have

$$\alpha(H) \leq \alpha(F(H)) < \alpha(H),$$

a contradiction. Now since $H \cap \partial U = \emptyset$ there is a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial U) = 0$. Define the map J by

$$J(x) = \begin{cases} \mu(x)\, F(x), & x \in \overline{U} \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Now it is easy to check that $J : E \to CD(E)$ has closed graph. In addition $J : E \to CD(E)$ is condensing with $J(E)$ a subset of a bounded set in E. To see this note

$$J(A) \subseteq co\, \left(F(\overline{U} \cap A) \cup \{0\}\right)$$

for any subset A of E. Now Theorem 7 implies that there exists $x \in E$ with $x \in J(x)$. Also $x \in U$ since $0 \in U$. Thus $x \in \mu(x)\, F(x) = \lambda\, F(x)$ where $0 \leq \lambda = \mu(x) \leq 1$. Consequently $x \in H$, which implies $\mu(x) = 1$ and so $x \in F(x)$. □

Similarly we have the following nonlinear alternative of Leray-Schauder type for $APCG$ maps.

Theorem 10. *Let E be a Banach space with U an open, convex subset of E and $x_0 \in U$. Suppose $F \in APCG(\overline{U}, E)$ is a condensing map with $F(\overline{U})$ a subset of a bounded set in E. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*

(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda\, F(u) + (1-\lambda)\{x_0\}$.*

Proof. Without loss of generality assume $x_0 = 0$. Suppose $(A2)$ does not occur and F has no fixed points in ∂U. Let H, μ, J be as in Theorem 9. Now $J : E \to Cc(E)$ has closed graph and J is condensing with $J(E)$ a subset of a bounded set in E. Also an easy argument (see the ideas in [8]; note for any compact subset K of E we have that $F|_K$ is u.s.c. (see [1, page 465])) implies $J : E \to Cc(E)$ is approximable. Now Theorem 6 implies that there exists $x \in E$ with $x \in J(x)$. Also as in Theorem 9 we have $x \in F(x)$. □

Next we prove a new fixed point theorem of Furi-Pera type for ACG and $APCG$ maps. We discuss the case when E is a Hilbert space and then remark about the general situation.

Theorem 11. *Let Q be a closed, convex subset of a Hilbert space E with $0 \in Q$. In addition suppose $F \in APCG(Q,E)$ is a condensing map with $F(Q)$ a subset of a bounded set in E. Also assume*

$$\left.\begin{array}{l} \text{if } \{(x_j,\lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x \in \lambda\, F(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1,2,....\} \\ \text{with } \{\lambda_j\, F(x_j)\} \subseteq Q \text{ for each } j \ge j_0 \end{array}\right\} \qquad (1)$$

holds. Then F has a fixed point in Q.

Remark 12. If $F(\partial Q) \subseteq Q$ then (1) holds.

Proof. Define $r : E \to Q$ by $r(x) = P_Q(x)$ i.e. r is the nearest point projection on Q. Note r is nonexpansive. Consider

$$B = \{x \in E : \ x \in F\, r\,(x)\,\}.$$

Note $F\,r : E \to Cc(E)$ is a condensing map and $F\,r(E)$ is a subset of a bounded set in E. Also $F\,r : E \to Cc(E)$ has closed graph. To see this let (y_n) be a sequence in E with $y_n \to y_0$ and $v_n \in F\,r\,(y_n)$ is such that v_n converges to v_0. Let $z_n = r(y_n)$ and so $v_n \in F(z_n)$ and $z_n \to z_0 = r(x_0)$. Since F has closed graph $v_0 \in F(z_0)$ i.e. $v_0 \in F\,r\,(y_0)$. Finally notice $F\,r : E \to Cc(E)$ is an approximable map. To see this take any compact subset K of E. Note $r : K \to Q$ and $F : Q \to Cc(E)$. A result of [2, page 468] (follow the reasoning in Proposition 3.3; note $F|_{r(K)}$ is u.s.c. [1, page 465]) implies $F\,r : E \to Cc(E)$ is an approximable map. Theorem 6 implies $F\,r$ has a fixed point so $B \neq \emptyset$. We must show B is closed. To see this let (x_n) be a sequence in B (i.e. $x_n \in F\,r\,(x_n)$) with $x_n \to x_0 \in E$. Now since $F\,r$ has closed graph we have $x_0 \in F\,r\,(x_0)$ i.e. $x_0 \in B$. Thus B is closed. In fact B is compact. To see this notice $B \subseteq F\,r\,(B)$. If $\alpha(r(B)) \neq 0$ then

$$\alpha(B) \le \alpha(F\,r\,(B)) < \alpha(r(B)) \le \alpha(B),$$

a contradiction. Thus $\alpha(r(B)) = 0$ and so $\alpha(B) \le \alpha(F\,r\,(B)) \le \alpha(r(B)) = 0$ so B is compact.

It remains to show $B \cap Q \neq \emptyset$. Suppose this is not true i.e. suppose $B \cap Q = \emptyset$. Then there exists $\delta > 0$ with $dist\,(B,Q) > \delta$. Choose $N \in \{1,2,...\}$ such that $1 < \delta\, N$. Define

$$U_i = \left\{x \in E : \ d(x,Q) \ < \ \frac{1}{i}\right\} \quad \text{for} \quad i \in \{N, N+1, ...\};$$

here d is the metric induced by the norm. Fix $i \in \{N, N+1, ...\}$. Since there is $dist\,(B,Q) > \delta$ then $B \cap \overline{U_i} = \emptyset$. Now Theorem 10 implies (since $B \cap \overline{U_i} = \emptyset$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i \in \lambda_i\, F\,r\,(y_i)$. Consequently for each $j \in \{N, N+1, ...\}$ there exists $(y_j, \lambda_j) \in \partial U_j \times (0,1)$ with $y_j \in \lambda_j\, F\,r\,(y_j)$. In particular since $y_j \in \partial U_j$ we have

$$\{\lambda_j\, F\,r\,(y_i)\} \not\subseteq Q \quad \text{for each} \quad j \in \{N, N+1, ...\}. \qquad (2.2)$$

Next let us look at

$$D = \{ x \in E : \, x \in \lambda \, F \, r \, (x) \quad \text{for some} \quad \lambda \in [0,1] \, \}.$$

First notice D is closed. To see this let (x_n) be a sequence in D (i.e. $x_n \in \lambda_n \, F \, r \, (x_n)$ for some $\lambda_n \in [0,1]$) with $x_n \to x_0 \in E$ and without loss of generality assume $\lambda_n \to \lambda_0 \in (0,1]$. The closedness of $F \, r$ (see the argument in Theorem 9) implies $\frac{1}{\lambda_0} \, x_0 \in F \, r \, (x_0)$ so $x_0 \in D$ [Alternatively, it is easy to see that $R : E \times [0,1] \to Cc(E)$, given by $R(x,\lambda) = \lambda \, F \, r \, (x)$, has closed graph so it is immediate that D is closed]. In fact D is compact. To see this notice

$$D \subseteq \overline{co} \, (F \, r \, (D) \cup \{0\})$$

and it is easy to check that $\alpha(D) = 0$ (since F is condensing and r is nonexpansive). Thus D is compact (so sequentially compact). This together with $d(y_j, Q) = \frac{1}{j}$, $|\lambda_j| \leq 1$ (for $j \in \{N, N+1, ...\}$) implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Also since $y_j \in \lambda_j \, F \, r \, (y_j)$ we have, since R (defined above) : $\overline{U_N} \times [0,1] \to Cc(E)$ has closed graph, that $y^\star \in \lambda^\star \, F \, r \, (y^\star)$. Now $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. But in this case (1), with $x_j = r(y_j) \in \partial Q$ and $x = y^\star = r(y^\star)$, implies that there exists $j_0 \in \{N, N+1, ...\}$ with $\{\lambda_j \, F \, r \, (y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (2.2). Thus $B \cap Q \neq \emptyset$ i.e. there exists $x \in Q$ with $x \in F \, r \, (x) = F(x)$. $\square$

Remark 13. Of course the result in Theorem 11 holds for certain convex sets in Banach spaces where there is a nearest point retraction that is nonexpansive (or more generally α-Lipschitzian with $k = 1$).

Remark 14. If the map F in Theorem 11 is compact then the Hilbert space can be replaced by any Banach (or indeed Fréchet) space (this is immediate since all we need consider is any continuous retraction r with $r(z) \in \partial Q$ for $z \in E \backslash Q$; note such an r exists (see [7])).

Remark 15. There is an obvious analogue of Theorem 11 for ACG maps.

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Transition from decay to blow-up in a parabolic system

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Abstract. We show a locally uniform bound for global nonnegative solutions of the system $u_t = \Delta u + uv - bu$, $v_t = \Delta v + au$ in $(0, +\infty) \times \Omega$, $u = v = 0$ on $(0, +\infty) \times \partial\Omega$, where $a > 0$, $b \geq 0$ and Ω is a bounded domain in $\mathbb{R}^n$, $n \leq 2$. In particular, the trajectories starting on the boundary of the domain of attraction of the zero solution are global and bounded.

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Keywords. Blow-up, global existence, apriori estimates

1 Introduction

In many parabolic problems possessing blowing-up solutions, there also exist global bounded solutions. The large-time behavior of solutions lying on the borderline between global existence and blow-up may be quite complicated and its knowledge may be useful e.g. in the study of stationary solutions of these problems (see [8]).

Let us consider first the scalar problem

$$\left.\begin{array}{ll} u_t = \Delta u + u|u|^{p-1} + f(x,t,u,\nabla u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_o(x), & x \in \Omega, \end{array}\right\} \quad \text{(P)}$$

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where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $p > 1$ and f represents a perturbation term. If $f \equiv 0$, $0 \not\equiv U_o \geq 0$ is a smooth function, $\lambda > 0$ and $u_o = \lambda U_o$ then the solution u_λ of (P) exists globally and $u_\lambda(t) \to 0$ as $t \to +\infty$ for λ small while u_λ blows up in finite time in the $L^\infty(\Omega)$-norm if λ is large. If we put $\lambda_o = \sup\{\lambda; u_\lambda \text{ exists globally}\}$ and if we consider only radially decreasing solutions in a ball then it is known (see [4], [5]) that the solution u_{λ_o}

- is global and bounded for p subcritical, i.e. $p < (n+2)/(n-2)$ if $n > 2$,
- is global and unbounded for p critical,
- blows up in finite time for p supercritical (and $n \leq 10$).

Similarly, if $n = 1$ and $f(x,t,u,u_x) = \varepsilon(u^m)_x$, where $\varepsilon > 0$ and $m > 1$ then the solution u_{λ_o}

- is global and bounded (at least for some) $p > 2m - 1$,
- cannot be global and bounded if $p \leq 2m - 1$ and ε is "large".

Sufficient conditions for global existence and boundedness of the solution u_{λ_o} for $f \not\equiv 0$ and a more detailed discussion of the above facts can be found in [7].

In the present note we study the system

$$\left.\begin{aligned} u_t &= \Delta u + uv - bu, && x \in \Omega,\ t > 0,\\ v_t &= \Delta v + au, && x \in \Omega,\ t > 0,\\ u &= v = 0, && x \in \partial\Omega,\ t > 0,\\ u(x,0) &= u_o(x) \geq 0, && x \in \Omega,\\ v(x,0) &= v_o(x) \geq 0, && x \in \Omega, \end{aligned}\right\} \tag{S}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $n \leq 2$, $a > 0$ and $b \geq 0$. It was shown in [6] that the system (S) possesses a positive stationary solution. Moreover, any positive stationary solution $(\tilde{u}, \tilde{v})$ of (S) represents a threshold between blow-up and decay to zero provided Ω is a ball. More precisely,

- if $\lambda < \mu \leq 1$, $0 \leq u_o \leq \lambda\tilde{u}$ and $0 \leq v_o \leq \mu\tilde{v}$ then the solution of (S) exists globally and tends to zero as $t \to \infty$,
- if $\lambda, \mu > 1$, $u_o \geq \lambda\tilde{u}$ and $v_o \geq \mu\tilde{v}$ then the solution of (S) blows up in finite time.

We are interested in the behavior of all "threshold trajectories", i.e. trajectories starting on the boundary ∂D_A of the domain of attraction of the zero solution

$$\begin{aligned} D_A = \{&(u_o, v_o) \in H_0^1(\Omega)^+ \times H_0^1(\Omega)^+;\\ &\text{the solution } (u,v) \text{ of (S) exists globally and } (u(t), v(t)) \to 0 \text{ as } t \to \infty\}, \end{aligned}$$

where $H_0^1(\Omega)^+$ is the positive cone of the usual Sobolev space $H_0^1(\Omega)$. We shall prove the boundedness of any non-negative global trajectory of (S). Since the corresponding bound is locally uniform with respect to the initial values (u_o, v_o), this result implies global existence and boundedness of all trajectories starting on ∂D_A.

Our proof is based on a non-trivial generalization of *a priori* estimates for stationary solutions in [6] (based on the method of Brézis and Turner [1]) to *a priori* estimates for all global solutions of (S). Such generalization sometimes may yield satisfactory results (see e.g. the optimal result in [4] for the problem (P) with $f \equiv 0$, $u_o \geq 0$ based on the method of *a priori* estimates of Gidas and Spruck);

in general, it usually requires additional assumptions. This is also the case of our proof: the *a priori* estimates in [6] were shown for a general domain $\Omega \subset \mathbb{R}^n$ if $n \leq 3$. For technical reasons, we had to restrict ourselves to the case $n \leq 2$.

Finally let us note that the boundedness of global solutions of problems of the type (P) is well known in the case where $f(x,t,u,\nabla u)$ is independent of t and ∇u (see e.g. [2], [3] and the references therein). Then the problem has variational structure, i.e. it admits a Lyapunov functional. A perturbation result for f depending on t and ∇u can be found in [7]. Anyhow, in our situation the system (S) does not seem to be "close" to any problem with variational structure.

2 Results and proofs

Throughout the rest of this paper we shall assume that the initial couple $(u_o, v_o) \in H_0^1(\Omega)^+ \times H_0^1(\Omega)^+$ is such that the corresponding solution (u,v) of (S) exists globally (in the classical sense). Moreover, we shall assume $u_o \not\equiv 0$ and we denote by λ_1 and φ_1 the first eigenvalue and the corresponding (positive) eigenfunction of the problem $-\Delta\varphi = \lambda\varphi$ in Ω, $\varphi = 0$ on $\partial\Omega$. We denote by $\|\cdot\|_p$ and $\|\cdot\|_{H^1}$ the norm in $L^p(\Omega)$ and $H^1(\Omega)$, respectively, and we put $\|\cdot\| := \|\cdot\|_2$. We shall also briefly write $u(t)$ instead of $u(\cdot,t)$ and $\int_\Omega u\,dx$ instead of $\int_\Omega u(x,t)\,dx$. Our main result is the following theorem.

Theorem 1. *There exists a constant $C_1 = C_1(\|\nabla u_o\|, \|\nabla v_o\|)$ such that*

$$\|\nabla u(t)\| + \|\nabla v(t)\| \leq C_1 \quad \textit{for any } t \geq 0.$$

The proof of Theorem 1 will follow from the following series of lemmata (see Lemma 8 and Lemma 9).

Lemma 2. *There exists a constant $C_2 = C_2(\|u_o\|, \|v_o\|)$ such that*

$$\int_\Omega v(x,t)\varphi_1(x)\,dx \leq C_2 \quad \textit{for any } t \geq 0.$$

Proof. Multiplying the equations in (S) by φ_1 and integrating by parts yields

$$\Big(\int_\Omega u\varphi_1\,dx\Big)_t = -(\lambda_1 + b)\int_\Omega u\varphi_1\,dx + \int_\Omega uv\varphi_1\,dx, \tag{1}$$

$$\Big(\int_\Omega v\varphi_1\,dx\Big)_t = -\lambda_1\int_\Omega v\varphi_1\,dx + a\int_\Omega u\varphi_1\,dx. \tag{2}$$

Differentiating (2), using (1), (2), $au = v_t - \Delta v$ and integration by parts we get

$$\Big(\int_\Omega v\varphi_1\,dx\Big)_{tt} = -\lambda_1\Big(\int_\Omega v\varphi_1\,dx\Big)_t + a\int_\Omega (\Delta u + uv - bu)\varphi_1\,dx$$

$$\begin{aligned}
&= -\lambda_1\Big(\int_\Omega v\varphi_1\,dx\Big)_t - a(\lambda_1+b)\int_\Omega u\varphi_1\,dx + a\int_\Omega uv\varphi_1\,dx \\
&\geq -(2\lambda_1+b)\Big(\int_\Omega v\varphi_1\,dx\Big)_t - \lambda_1(\lambda_1+b)\int_\Omega v\varphi_1\,dx \\
&\quad + \frac{1}{2}\Big(\int_\Omega v^2\varphi_1\,dx\Big)_t + \frac{\lambda_1}{2}\int_\Omega v^2\varphi_1\,dx,
\end{aligned}$$

where in the last step we have used

$$\begin{aligned}
\int_\Omega (-\Delta v)v\varphi_1\,dx &= \int_\Omega \nabla v\cdot\nabla(v\varphi_1)\,dx \\
&= \int_\Omega |\nabla v|^2\varphi_1\,dx + \frac{1}{2}\int_\Omega \nabla v^2\cdot\nabla\varphi_1\,dx \geq \frac{\lambda_1}{2}\int_\Omega v^2\varphi_1\,dx.
\end{aligned}$$

Hence, denoting

$$\begin{aligned}
w &:= w(t) := \int_\Omega v(x,t)\varphi_1(x)\,dx, \\
y &:= y(t) := w'(t) + (\lambda_1+b)w(t) - \frac{1}{2}\int_\Omega v^2(x,t)\varphi_1(x)\,dx,
\end{aligned}$$

we obtain $y_t \geq -\lambda_1 y$ so that $y(t) \geq e^{-\lambda_1 t}y(0) \geq -c_0$ for some $c_0 > 0$. Since

$$\frac{1}{2}\int_\Omega v^2(x,t)\varphi_1(x)\,dx \geq c_1\int_\Omega v^2(x,t)\varphi_1^2(x)\,dx \geq c_2w^2(t) \qquad \text{for some } c_1, c_2 > 0,$$

we have

$$-c_0 \leq y \leq w' + (\lambda_1+b)w - c_2w^2 \leq w' - c_3w^2 + c_4 \qquad \text{for some } c_3, c_4 > 0,$$

hence $w' \geq c_3w^2 - (c_0+c_4)$. Since $w(t)$ exists globally, the last inequality implies $w(t) \leq \sqrt{(c_0+c_4)/c_3}$ (where $c_0 = c_0(v_o)$ and c_3, c_4 do not depend on v).

Lemma 3. *There exists a constant $C_3 = C_3(\|u_o\|, \|v_o\|)$ such that*

$$\int_\Omega u(x,t)\varphi_1(x)\,dx \leq C_3 \quad \textit{for any } t \geq 0. \tag{3}$$

Proof. Multiplying the first equation in (S) by φ_1, integrating over Ω and over $(t, t+\theta)$, using $u = \frac{1}{a}(v_t - \Delta v)$ and Lemma 2 we get

$$\begin{aligned}
\int_\Omega u\varphi_1\,dx\Big|_t^{t+\theta} &\geq -(\lambda_1+b)\int_t^{t+\theta}\int_\Omega u\varphi_1\,dx\,dt \\
&= -\frac{\lambda_1+b}{a}\int_\Omega v\varphi_1\,dx\Big|_t^{t+\theta} - \frac{\lambda_1(\lambda_1+b)}{a}\int_t^{t+\theta}\int_\Omega v\varphi_1\,dx\,dt \geq -\tilde{c},
\end{aligned}$$

where $\tilde{c} = \tilde{c}(C_2)$ does not depend on t and $\theta \in (0,1]$. Integrating the last inequality over $\theta \in (0,1)$ and using $u = \frac{1}{a}(v_t - \Delta v)$ again we obtain

$$\int_\Omega u(x,t)\varphi_1(x)\,dx - \tilde{c} \le \int_t^{t+1}\int_\Omega u\varphi_1\,dx\,dt$$
$$= \frac{1}{a}\int_\Omega v\varphi_1\,dx\Big|_t^{t+1} + \frac{\lambda_1}{a}\int_t^{t+1}\int_\Omega v\varphi_1\,dx\,dt \le C_2\frac{\lambda_1+1}{a},$$

which concludes the proof.

In what follows we shall exploit the following well known result (used also in [1] and [6]).

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain. For any $u \in H_0^1(\Omega)$, we have*

$$\|\frac{u}{\delta^r}\|_p \le C_4\|\nabla u\|, \tag{4}$$

where $\delta = \delta(x) = dist\,(x, \partial\Omega)$, $r \in [0,1]$ and $p \le \frac{2n}{n-2(1-r)}$ ($=\frac{2}{r}$ if $n = 2$).

Since $\delta(x) \le C_\varphi\varphi_1(x)$ for some $C_\varphi > 0$, it is now easy to show the next three lemmata.

Lemma 5. *There exists a constant $C_5 = C_5(\|u_o\|, \|v_o\|)$ such that*

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + b\|u\|^2 = \int_\Omega u^2 v\,dx \le C_5\|\nabla u\|^{4/3}\|\nabla v\|. \tag{5}$$

Proof. The equality in (5) can be obtained by multiplying the first equation in (S) by u and integrating over Ω. Now the Hölder inequality, Lemmata 3, 4 and any choice of $\alpha, \alpha' > 1$ with $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ imply

$$\int_\Omega u^2 v\,dx \le \Big(\int_\Omega u\delta\,dx\Big)^{2/3}\Big(\int_\Omega u^4 v^3 \delta^{-2}\,dx\Big)^{1/3}$$
$$\le (C_\varphi C_3)^{2/3}\Big(\int_\Omega\Big(\frac{u}{\delta^{1/(2\alpha)}}\Big)^{4\alpha}dx\Big)^{1/(3\alpha)}\Big(\int_\Omega\Big(\frac{v}{\delta^{2/(3\alpha')}}\Big)^{3\alpha'}dx\Big)^{1/(3\alpha')}$$
$$\le (C_\varphi C_3)^{2/3}C_4^{7/3}\|\nabla u\|^{4/3}\|\nabla v\|.$$

Lemma 6. *There exists a constant $C_6 = C_6(\|u_o\|, \|v_o\|)$ such that*

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 = a\int_\Omega uv\,dx \le C_6\|\nabla u\|^{1/2}\|\nabla v\|. \tag{6}$$

Proof. The equality in (6) follows from the second equation in (S). Now, similarly as in the proof of Lemma 5 we obtain

$$\int_\Omega uv\,dx \le \Big(\int_\Omega u\delta\,dx\Big)^{1/2}\Big(\int_\Omega uv^2\delta^{-1}\,dx\Big)^{1/2}$$
$$\le (C_\varphi C_3)^{1/2}\Big(\int_\Omega\Big(\frac{u}{\delta}\Big)^2\,dx\Big)^{1/4}\Big(\int_\Omega v^4\,dx\Big)^{1/4} \le C_6\|\nabla u\|^{1/2}\|\nabla v\|,$$

since $H^1(\Omega)$ is imbedded in $L^p(\Omega)$ for any $p\ge 1$.

Lemma 7. *There exists a constant $C_7 = C_7(\|u_o\|,\|v_o\|)$ and for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that*

$$\begin{aligned}\|u\| \le C_7\|\nabla u\|^{2/3}, \qquad \|v\| \le C_7\|\nabla v\|^{2/3},\\ \|uv\| \le C_\varepsilon(\|\nabla u\|^{2/3+\varepsilon}+1)\|\nabla v\|.\end{aligned} \tag{7}$$

Proof. Denoting $w:=u$ or $w:=v$ and $C_{23}:=\max(C_2,C_3)$ we get

$$\int_\Omega w^2\,dx \le \Big(\int_\Omega w\delta\,dx\Big)^{2/3}\Big(\int_\Omega\Big(\frac{w}{\delta^{1/2}}\Big)^4\,dx\Big)^{1/3} \le (C_\varphi C_{23})^{2/3}C_4^{4/3}\|\nabla w\|^{4/3}.$$

Putting $K_\varepsilon = \frac{2(2+\varepsilon)}{\varepsilon}$ and using $\|w\|_p \le c_p\|\nabla w\|$ for any $p\ge 1$ we obtain

$$\begin{aligned}\int_\Omega u^2v^2\,dx &\le \Big(\int_\Omega u^{2+\varepsilon}\,dx\Big)^{2/(2+\varepsilon)}\Big(\int_\Omega v^{K_\varepsilon}\,dx\Big)^{2/K_\varepsilon}\\ &\le c_{K_\varepsilon}^2\|\nabla v\|^2\Big(\int_\Omega u^2\,dx\Big)^{(2-\varepsilon)/(2+\varepsilon)}\Big(\int_\Omega u^4\,dx\Big)^{\varepsilon/(2+\varepsilon)}\\ &\le c_{K_\varepsilon}^2 c_4^{4\varepsilon/(2+\varepsilon)}C_7^{2(2-\varepsilon)/(2+\varepsilon)}\|\nabla v\|^2\|\nabla u\|^{4/3+\varepsilon'},\end{aligned}$$

where $\varepsilon' < 2\varepsilon$.

Lemma 8. *There exists a constant $C_8 = C_8(\|\nabla v_o\|,\|\nabla u_o\|)$ such that*

$$\|\nabla v(t)\| \le C_8\max_{0\le\tau\le t}\|\nabla u(\tau)\|^{1/2} \qquad \textit{for any } t\ge 0. \tag{8}$$

Proof. If $\frac{d}{dt}\|v(t)\|^2 \ge -\|\nabla v(t)\|^2$ then (6) implies

$$\|\nabla v(t)\| \le 2C_6\|\nabla u(t)\|^{1/2} \tag{9}$$

and we are done. Hence, let $\frac{d}{dt}\|v(t)\|^2 < -\|\nabla v(t)\|^2$. Then

$$\|\nabla v(t)\|^2 < -\frac{d}{dt}\|v\|^2 \le 2\|v\|\cdot\|v_t\| \le 2C_7\|\nabla v\|^{2/3}\cdot\|v_t\|,$$

so that

$$\|\nabla v\|^{4/3} \le 2C_7 \|v_t\|. \tag{10}$$

Multiplying the second equation in (S) by v_t and integrating over Ω we get

$$\|v_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 = a \int_\Omega u v_t \, dx \le \frac{1}{2}\|v_t\|^2 + \frac{a^2}{2}\|u\|^2,$$

which together with (7) yields

$$\|v_t\|^2 + \frac{d}{dt}\|\nabla v\|^2 \le a^2\|u\|^2 \le (aC_7)^2\|\nabla u\|^{4/3}. \tag{11}$$

Now (10) and (11) imply

$$\frac{1}{(2C_7)^2}\|\nabla v\|^{8/3} + \frac{d}{dt}\|\nabla v\|^2 \le (aC_7)^2\|\nabla u\|^{4/3}. \tag{12}$$

If $\|\nabla v\| \le (2aC_7^2)^{3/4}\|\nabla u\|^{1/2}$ then we are done. Otherwise the inequality (12) implies $\frac{d}{dt}\|\nabla v\|^2 < 0$ and putting

$$t_1 := \inf\{\tau > 0\, ;\, \frac{d}{dt}\|\nabla v\|^2 < 0 \ \text{ on } (\tau, t]\}$$

we have $\|\nabla v(t)\| < \|\nabla v(t_1)\|$.

If $t_1 = 0$ then $\|\nabla v(t)\| < \|\nabla v(0)\| \le C_0\|\nabla u(0)\|^{1/2}$ for some $C_0 > 0$. Hence, we may assume $t_1 > 0$.

If $\frac{d}{dt}\|v(t_1)\|^2 \ge -\|\nabla v(t_1)\|^2$ then the inequality (9) (with t replaced by t_1) implies

$$\|\nabla v(t)\| < \|\nabla v(t_1)\| \le 2C_6\|\nabla u(t_1)\|^{1/2}.$$

If $\frac{d}{dt}\|v(t_1)\|^2 < -\|\nabla v(t_1)\|^2$ then the inequality (12) (with t replaced by t_1) implies

$$\|\nabla v(t)\| < \|\nabla v(t_1)\| \le (2aC_7^2)^{3/4}\|\nabla u(t_1)\|^{1/2},$$

since the definition of t_1 implies $\frac{d}{dt}\|\nabla v(t_1)\|^2 = 0$ if $t_1 > 0$.

Lemma 9. *There exists a constant $C_9 = C_9(\|\nabla u_o\|, \|\nabla v_o\|)$ such that*

$$\|\nabla u(t)\| \le C_9 \qquad \text{for any } t \ge 0.$$

Proof. We may suppose $\|\nabla u(0)\| < \sup_{t\ge 0}\|\nabla u(t)\|$ (otherwise we are done). Let $t_o > 0$ be such that

$$\|\nabla u(t_o)\| = \max_{0\le t\le t_o}\|\nabla u(t)\|. \tag{13}$$

If $\frac{d}{dt}\|u(t_o)\|^2 \ge -\|\nabla u(t_o)\|^2$ then (5), Lemma 8 and (13) imply

$$\|\nabla u(t_o)\|^2 \le 2C_5\|\nabla u(t_o)\|^{4/3}\|\nabla v(t_o)\| \le 2C_5C_8\|\nabla u(t_o)\|^{11/6},$$

hence

$$\|\nabla u(t_o)\| \le (2C_5C_8)^6.$$

Consequently, we may assume

$$\frac{d}{dt}\|u(t_o)\|^2 < -\|\nabla u(t_o)\|^2.$$

This implies

$$\|\nabla u(t_o)\|^2 < -\frac{d}{dt}\|u\|^2 \le 2\|u\|\cdot\|u_t\| \le 2C_7\|\nabla u\|^{2/3}\|u_t\|,$$

so that

$$\|\nabla u(t_o)\|^{4/3} \le 2C_7\|u_t(t_o)\|. \tag{14}$$

Multiplying the first equation in (S) by u_t and integrating over Ω we obtain

$$\begin{aligned}\|u_t(t_o)\|^2 \le \|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 &= -b\int_\Omega uu_t\,dx + \int_\Omega uvu_t\,dx \\ &\le \frac{1}{2}\|u_t\|^2 + \|uv\|^2 + b^2\|u\|^2,\end{aligned}$$

where the inequality $\frac{d}{dt}\|\nabla u(t_o)\|^2 \ge 0$ follows from (13). Now the last inequality together with (14) and Lemmata 7, 8 imply

$$\begin{aligned}\frac{1}{(2C_7)^2}\|\nabla u(t_o)\|^{8/3} &\le \|u_t(t_o)\|^2 \le 2\|uv(t_o)\|^2 + 2b^2\|u(t_o)\|^2 \\ &\le \tilde{C}_\varepsilon(\|\nabla u(t_o)\|^{4/3+2\varepsilon}+1)(\|\nabla v(t_o)\|^2+1) \\ &\le \tilde{C}'_\varepsilon(\|\nabla u(t_o)\|^{7/3+2\varepsilon}+1),\end{aligned}$$

so that the choice $\varepsilon < 1/6$ yields the desired estimate for $\|\nabla u(t_o)\|$.

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Dynamical Systems with Several Equilibria and Natural Liapunov Functions

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Abstract. Dynamical systems with several equilibria occur in various fields of science and engineering: electrical machines, chemical reactions, economics, biology, neural networks. As pointed out by many researchers, good results on qualitative behaviour of such systems may be obtained if a Liapunov function is available. Fortunately for almost all systems cited above the Liapunov function is associated in a natural way as an energy of a certain kind and it is at least nonincreasing along systems solutions.

AMS Subject Classification. 34D20, 34A11

Keywords. Several equilibria, qualitative behaviour, Liapunov function

1 Introduction

Dynamical systems with several equilibria occur in various fields of science and engineering: electrical machines, chemical reactions, economics, biology, neural networks. These systems are models of either natural or man-made physical systems. In both cases stability properties are required for various reasons but in fact stability means always some "good behaviour" with respect to short-term disturbances. In man-made systems technological operation is connected with stability of the "operating points" i.e. of some constant solutions of the dynamical model.

Technological operation is closely connected with oriented changes from one operating point to another i.e. with transients. With respect to the new operating point (constant solution) the old operating point is a perturbed initial condidtion generating a transient motion (dynamics trajectory) that should end in the new operating point. This is clearly a stability-like property.

Stability is a property of a single solution (equilibrum) and a local one. Linear systems and systems with an almost linear behaviour have a single equilibrum that is globally asymptotically stable.

For systems with several equilibria the usual local concepts of stability are not sufficient for an adequate description. The so-called "global phase portrait" may contain both stable and nonstable equilibria. Of course each of them may be characterized separately since stability is a local concept. Nevertheless global concepts are also required for a better system description.

We consider here a single example: the case of the neural networks. The neural networks are interconnections of simple computing elements whose computational capability is increased by interconnection (emergent collective capacities). This is due to the nonlinear characteristics leading to the existence of several stable equilibria. The network achieves its computing goal if no self-sustained oscillations are present and it always achieves some steady-state (equilibrium) among a finite (while large) number of such states.

This behaviour is met in other systems also. For instance chemical systems or biological communities display several equilibria, according to the external conditions (environment). The models in macroeconomics need several equilibria since in practice this is indeed the case and economic policies (good or bad) are nothing else but "manoeuvres" that take economic systems from one stable equilibrium to another - in the same way as mechanical manoeuvres take engineering systems from one operating point to another.

2 Basic concepts and tools

The basic concepts in the field of the systems with several equilibria come from the papers of Kalman [7] from 1957 and Moser [10] from 1967. Especially the second paper relies on the following remark:

Consider the system

$$\dot{x} = -f(x), x \in \mathbb{R}^n, \tag{1}$$

where $f(x) = grad\, G(x)$ and $G : \mathbb{R}^n \to \mathbb{R}$ has the following properties:

i) $lim_{|x|\to\infty} G(x) = \infty$ and
ii) the number of the critical points is finite.

In this case any solution of (1) approaches asymptotically one of the equilibrum points (which is also a critical point of G — where the gradient i.e. f vanishes). It is only natural to call this behaviour gradient-like but there are other properties that are also important while weaker. With respect to this we shall need some basic notions. Our object will be here the system of ordinary differential equations

$$\dot{x} = f(x, t). \tag{2}$$

Definition 1. a) Any constant solution of (2) is called *equilibrium*. The set of equilibria $\mathcal{E}$ is called *stationary set*.

b) A solution of (2) is called *convergent* if it approaches asymptotically some equilibrium:

$$lim_{t\to\infty}x(t) = c \in \mathcal{E}$$

A solution is called *quasi-convergent* if it approaches asymptotically the stationary set:

$$lim_{t\to\infty}d(x(t), \mathcal{E}) = 0$$

Definition 2. System (2) is called *monostable* if every bounded solution is convergent; it is called *quasi-monostable* if every bounded solution is quasi-convergent.

Definition 3. System (2) is called *gradient-like* if every solution is convergent; it is called *quasi-gradient-like* if every solution is quasi-convergent.

Since there exist also other terms for these notions some comments are necessary. The notion of convergence still defines a solution property and was introduced by Hirsch [5,6]. Monostability has been introduced by Kalman [7] in 1957; sometimes it is called *strict mutability* (Popov [11]) while quasi-mono-stability is called by the same author *mutability* and by other *dichotomy* (Gelig, Leonov, and Yakubovich [3]). In fact for monostable (quasi-monostable) systems some kind of dichotomy occurs: their solutions are either unbounded or tend to an equilibrium (or to the stationary set); in any case self-sustained periodic or almost periodic oscillations are excluded. The quasi-gradient-like property is called sometimes *global asymptotics*.

It is obvious that while convergence is associated to solutions, monostability and gradient-like property are associated to systems. At this point we add some properties related to the stationary set (Gelig, Leonov and Yakubovith [3])

Definition 4. The stationary set $\mathcal{E}$ is *uniformly stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that for any t_0 if $d(x(t_0), \mathcal{E}) < \delta$ then $d(x(t), \mathcal{E}) < \varepsilon$ for all $t \geq t_0$.

The stationary set $\mathcal{E}$ is *uniformly globally stable* if it is uniformly Liapunov stable and the system is quasi-gradient-like (has global asymptotics).

The stationary set is *pointwise globally stable* if it is uniformly Liapunov stable and the system is gradient-like.

For autonomus (time-invariant systems) the following Liapunov-type results are available (Gelig, Leonov and Yakubovitch [3]; Leonov, Reitmann and Smirnova [9]).

Lemma 5. *Consider the nonlinear system*

$$\dot{x} = f(x) \tag{3}$$

and assume existence of a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that is nonincreasing along any solution of (3). If, additionaly, a bounded on $\mathbb{R}^+$ solution $x(t)$ for which there exists some $\tau > 0$ such that $V(x(\tau)) = V(x(0))$ is an equilibrium then the system is quasi-monostable.

Lemma 6. *If the assumptions of Lemma 5 hold and, additionaly, $V(x) \to \infty$ for $|x| \to \infty$ then system (3) is quasi-gradient-like.*

Lemma 7. *If the assumptions of Lemma 6 hold and the set $\mathcal{E}$ is discrete (i.e. it consists of isolated equilibria only) then system (3) is gradient-like.*

3 Applications from chemical kinetics

3.1. We shall consider first a model from the book of Frank-Kamenetskii [2], studied in the diffusion context by Kružkov and Peregudov [8]; here the diffusion phenomenon will be left aside. The model reads like (3) but under the following assumptions:

i) $f : Q \to \mathbb{R}^n$, $Q = \{x \in \mathbb{R}^n,\ x^i \geq 0,\ i = \overline{1,n}\}$;
ii) $f(0) = 0$;
iii) $\frac{\partial f^i}{\partial x^j} \geq 0$, $\forall x \in Q$, $i, j = \overline{1,n}$;
iv) $\sum_1^n f^i(x) \equiv 0$.

Then the following properties of the system are valid (Halanay and Răsvan [4]):

a) Q is an invariant set of the system;
b) all solutions in Q are bounded;
c) $\sum_1^n |x^i(t) - y^i(t)| \leq \sum_1^n |x^i(\tau) - y^i(\tau)|$ for all $t \geq \tau$, $x(t)$ and $y(t)$ being two solutions of (3) from Q;
d) the function $V(x) = \sum_1^n |f^i(x)|$ is nonincreasing along the solution of (3) i.e. *it is a Liapunov function*; moreover this Liapunov function cannot be constant if it is not identically zero.

We may state now

Theorem 8. *For every $M > 0$ there exist equilibria $\hat{x}$ such that $\sum_1^n \hat{x}^i = M$ and the system is gradient-like on the sets $\sum_1^n x^i = M$.*

Outline of proof: Let $x(t)$ be a solution with $\sum_1^n x^i(0) = M$. From iv) we deduce that $\sum_1^n x^i(0) \equiv M$, the solution is bounded and the ω-limit set is not empty. On the ω-limit set $V(x(t))$ is constant hence it is identically zero; from here we obtain that the ω-limit set consists only of equilibria. Since all solutions are bounded the system is quasi-gradient-like. Using c) we obtain that the system is even gradient-like.

3.2. Consider now the case of a closed chemical system subject to mass-action law and constant temperature—the formal kynetics system ([4]):

$$\begin{aligned} \dot{c}_i &= \textstyle\sum_{j=1}^n (\beta_{ij} - \alpha_{ij})(w_j^+(c) - w_j^-(c)),\ i = \overline{1,m} \\ w_j^+(c) &= k_j^+ \textstyle\prod_1^m (c_i)^{\alpha_{ij}},\ w_j^-(c) = k_j^- \textstyle\prod_1^m (c_i)^{\beta_{ij}}, \end{aligned} \tag{4}$$

where the nonnegative integers α_{ij}, β_{ij} (stoichiometric coefficients) satisfy the following assumption $(\forall j)(\exists i : \alpha_{ij}+\beta_{ij} \neq 0)$, that is each substance has to participate at least to one reaction either as reactant or as product. Using this assumption we can prove *positivity of the concentrations: if the above assumption holds then* $c_i(0) \geq 0$, $i = \overline{1,m}$ *implies that for any* $i = \overline{1,m}$ *either* $c_i(t) > 0$ *or* $c_i(t) \equiv 0$ *on the entire existence interval of the solution.* Any point c with $c_i > 0$, $i = \overline{1,m}$ is called *admissible*; the *set of the admissible points is called admissible set.* Another property of the system is existence of a *set of conservation laws* that define an *invariant hyperplane.* By writing (4) as follows

$$\dot{c} = Gw(c), \tag{5}$$

where $rank\, G = r$ we may obtain by reordering (5) the partition

$$\begin{aligned} \dot{c}^r &= G_{11}w^r(c) + G_{12}w^{m-r}(c), \\ \dot{c}^{m-r} &= G_{21}w^r(c) + G_{22}w^{m-r}(c), \end{aligned} \tag{6}$$

where $det\, G_{11} \neq 0$. Then the following linear invariant manifold is obtained

$$\mathcal{L}(c) \equiv c^{m-r} - G_{21}G_{11}^{-1}c^r = c^{m-r}(0) - G_{21}G_{11}^{-1}c^r(0) \tag{7}$$

called "substance balance hyperplane" that is in fact a linear system of conservation laws.

The equilibrium set of (4) may be quite rich but among the equilibria are of interest the *detailed-balance equilibria* defined by

$$w_j^+(c) = w_j^-(c),\ j = \overline{1,n} \tag{8}$$

and mainly those belonging to the admissible set $Q = \{c \in \mathbb{R}^m,\ c_k > 0,\ k = \overline{1,m}\}$ called *admissible detailed balance equilibria.*

The following result of Zeldovič is valid

Proposition 9. *If* (4) *has an admissible detailed balance equilibrium and in the linear manifold* $\mathcal{L}(c) = q$ *there exists an admissible point then in this manifold there exists a unique detailed balance point.*

If (4) is such that *an admissible detailed balance equilibrium exists* then the following *Liapunov function* may be associated to it:

$$V_{\hat{c}} = \sum_1^m c_k(ln(c_k/\hat{c}_k) - 1) \tag{9}$$

and the following is true (Halanay & Răsvan [4])

Theorem 10. *If an admissible detailed balance equilibrium exists, the following properties of the solutions with* $c_i(0) \geq 0$, $i = \overline{1,m}$ *are valid:*

1. *The solutions are bounded.*

2. *There are no periodic nonconstant solutions with nonnegative components.*
3. *Any equilibrium point with nonnegative components is a detailed balance point.*
4. *The ω-limit set of any solution is composed of equilibrium points only; if such a set contains an admissible detailed balance point it coincides with it being a singleton.*
5. *An admissible detailed balance point is stable in the sense of Liapunov and it is an attractor in the invariant hyperplane that contains it.*
6. *A solution such that $lim_{t\to\infty}c(t)$ exists and has all its components positive is Liapunov stable.*

Some comments are necessary. The first four properties show that, with respect to those solutions that are physically significant, *the system is quasi-gradient-like* but if among the equilibria of a given ω-limit set there is one admissible detailed balance point, the ω-limit set coincides with it; this singleton is stable in the sense of Liapunov and even asymptotically stable when the solutions are reduced to the invariant hyperplane containing this point.

The remarkable property of this system would be existence of an admissible point in the ω-limit set of any solution. In this case the ω-limit set would reduce to it and the attraction domain would coincide with the entire hyperplane. The system would be gradient-like with respect to admissible set Q. Unfortunately this is still an open question. We may nevertheless mention that some recent results for the case of two substances exist (Simon & Farkas [12]).

4 Applications from biology

A. Consider first the model of Volterra type for n species that compete for some resource:

$$\frac{dN_i}{dt} = N_i(\varepsilon_i - \sum_{j=1}^{n} \gamma_{ij} N_j),\ i = \overline{1,n} \tag{10}$$

This model has been studied intensely (e.g. Volterra[14]; Svireżev [13]) for the case ofthe so-called dissipative community: $\varepsilon_i > 0$ and there exist $\alpha_i > 0$ such that the quadratic form $\sum_1^n \sum_1^n \alpha_i \gamma_{ij} x_i x_j$ is positive definite. Here we shall consider the general case because of its similarity to mass action chemical kinetics.

We assume, as in the case of the chemical kinetics, existence of an equilibrium $\hat{N}_i,\ i = \overline{1,n}$ with all $\hat{N}_i > 0$. Associate to (10) the following function:

$$L_{\hat{N}} = \sum_1^n \hat{N}_i \left(\frac{N_i}{\hat{N}_i} - 1 - \ln \frac{N_i}{\hat{N}_i} \right) \tag{11}$$

which is of the same type as (9); with the new variables $x_i = \ln(N_i/\hat{N}_i)$ we obtain:

$$\frac{dx_i}{dt} = \varepsilon_i - \sum_{j=1}^{n} \gamma_{ij} \hat{N}_j e^{x_j}, \tag{12}$$

$$L_{\hat{N}} = \sum_{1}^{n} \hat{N}_i \left(e^{x_i} - 1 - x_i\right) \tag{13}$$

and it may be easily seen that (12) can be written as:

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} \gamma_{ij} \frac{\partial L}{\partial x_i}(x_1, \cdots, x_n) \tag{14}$$

i.e. the system is quasi-gradient-like. We have $L(x_1, x_2, \cdots, x_n) > 0$ and also

$$\frac{d}{dt} L(x_1(t), \cdots, x_n(t)) = -\sum_{1}^{n} \sum_{1}^{n} \gamma_{ij} \frac{\partial L}{\partial x_i} \frac{\partial L}{\partial x_j}$$

and if the matrix (γ_{ij}) is nonnegative definite then L is decreasing (nonincreasing along the solutions of (14). Obviously L is bounded for bounded x_i (see the previous section) hence the system is quasi-monostable. Moreover the critical point of L i.e. $x_1 = x_2 = \cdots = x_n = 0$ is globally assymptotically stable. We have also $L(x) \to \infty$ for $|x| \to \infty$ hence according to Lemma 6 the system is quasi-gradient like. Moreover the equilibria of (14) are given by

$$\sum \gamma_{ij} \frac{\partial L}{\partial x_i} = 0$$

and the structure of L shows that they are isolated. We obtained the following

Theorem 11. *If system* (10) *has an equilibrium with al components positive and the matrix* γ_{ij} *is positive definte then it is gradient-like.*

B. An example taken from a different field of biological sciences is the model of evolutionary selection of macromolecular species of Eigen and Schuster (taken from the paper of Cohen and Grossberg [1]):

$$\dot{x}_i = x_i(m_i x_i^{p-1} - q \sum_{k=1}^{n} m_k x_k^p) \tag{15}$$

Remark that if $p = 1$ a special case of (10) is obtained. In fact, as shown is the cited paper, many of the biological models may be obtained from a general neural network model that will be shown next. For this reason we do not insist here on Eigen-Schuster model.

5 Continous-time neural networks

The neural networks are structures that possess "emergent computational capabilities" that is they are interconnected simple computational elements to which interconnections confer increased computational power.

The general model considered here (Cohen and Grossberg [1]) reads

$$\dot{x}_i = a_i(x_i)[b_i(x_i) - \sum_1^n c_{ij} d_j(x_j)],\ i = \overline{1,n}, \tag{16}$$

where $c_{ij} = c_{ji}$. The following Liapunov function is associated

$$V(x) = \frac{1}{2}\sum_1^n\sum_1^n c_{ij} d_i(x_i) d_j(x_j) - \sum_1^n \int_0^{x_i} b_i(\lambda) d_i'(\lambda) d\lambda \tag{17}$$

that is much alike to the Liapunov function of the absolute stability problem.

It can be seen that (16) may be given the form

$$\dot{x} = -A(x)\, grad\, V(x), \tag{18}$$

where the items of $A(x)$ are

$$A_{ij}(x) = \frac{a_i(x_i)}{d_i'(x_i)}\delta_{ij} \tag{19}$$

Also the derivative of V along the solutions of $V(x)$ reads

$$W(x) = -\sum_1^n a_i(x_i) d_i'(x_i)\left[b_i(x_i) - \sum_1^n c_{ij} d_j(x_j)\right]^2 \leq 0$$

provided $a_i(\lambda) > 0$ and $d_i(\lambda)$ are nondecreasing. If additionally $d_i(\cdot)$ are strictly increasing the set, where $W(x) = 0$ consists only of equilibria. It follows that the system is quasi-gradient-like (Lemma 6).

Usually the property required for neural networks is gradient-like behavoiur. This property requires always specific studies since in the general case of (16) the equilibrium set may contain countably many equilibria.

6 Concluding remarks

We have presented here some models occuring in various fields of science and engineering; nevertheless they have some common features. First of all they belong to the class of so called competitive differential systems [5]. They all have many equilibria and require those qualitative concepts that were introduced for such systems (mutability, dichotomy, gradient behaviour). In obtaining the required properties the milestone is to show that the ω-limit sets of the solutions are composed of isolated equilibria only. Usually this goal is achieved using specific methods of differential topology that take into account the structure of differential equations that are competitive [5].

Existence of a suitable Liapunov function may simplify the task of showing that the ω-limit sets are composed of equilibria only; this was supposed to be the mainstream of the present paper and it illustrates that it is desirable to associate a Liapunov function, in a natural way, to any dynamical model. Of course, "guessing" a Liapunov function remains an art and a challenge.

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Boundary Layer for Chaffee-Infante Type Equation

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Abstract. This article is concerned with the nonlinear singular perturbation problem due to small diffusivity in nonlinear evolution equations of Chaffee-Infante type. The boundary layer appearing at the boundary of the domain is fully described by a corrector which is "explicitly" constructed. This corrector allows us to obtain convergence in Sobolev spaces up to the boundary.

AMS Subject Classification. 35B40, 35C20, 35Q30, 46N20, 76D10

Keywords. Boundary layers, correctors, nonlinear reaction diffusion equations, chaffee-infante equation

1 Introduction

In this article we will study the asymptotic behavior of the solutions of certain reaction diffusion equations with small diffusivity. We will focus on the Chaffee-Infante equation:

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon)^3 - u^\varepsilon = f \quad \text{in} \quad \Omega, \tag{1}$$

where Ω is a two dimensional channel

$$\Omega = (0, 2\pi) \times (0, 1), \tag{2}$$

but the methods apply to more general polynomial nonlinearities and to higher space dimensions.

The initial and boundary conditions associated with (1) and (2) are

$$u^\varepsilon = u_0 \quad \text{at} \quad t = 0, \tag{3}$$

and

$$\begin{cases} u^\varepsilon = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad 1, \\ \text{and periodicity} \ \ (2\pi) \ \ \text{for all functions} \\ \text{in the horizontal} \ \ (x) \ \ \text{direction.} \end{cases} \tag{4}$$

The corresponding "inviscid" equation is the reaction equation:

$$\frac{\partial u^0}{\partial t} + (u^0)^3 - u^0 = f \quad \text{in} \quad \Omega, \tag{5}$$

with the initial condition

$$u^0 = u_0 \quad \text{at} \quad t = 0. \tag{6}$$

We will assume that u_0 satisfies the boundary conditions (4) while f need not vanish at the wall. Thus there is a boundary layer near the wall (at $y = 0$ and $y = 1$) which is the main object under investigation in this article.

We will assume enough smoothness on u_0 and f so that all the calculations hereafter are justified. We will also consider the time T fixed and let the diffusivity ε approaches zero. This is the case since the solutions of the reaction equation (5) may develop internal layers as time approaches infinity. This would prevent us from obtaining a simple boundary layer expansion for the reaction-diffusion equation (1). The long time asymptotics will be considered elsewhere.

The difficulty of the problem lies in the disparity of the boundary conditions of (1) and (5) which makes this a singular perturbation problem. The approach that we take are the ones suggested by Lions [8], Vishik and Lyusternik [17] (see also Temam and Wang [14,15,16]), i.e. the construction and utilization of a corrector. The advantage of this approach, in terms of the common matched asymptotic expansion, is that once we have found the right corrector, the outer expansion for the corrector equation would be trivial (zero) and thus no matching is necessary at all. The other tools that we need here are maximum principle, energy estimates and anisotropic Sobolev imbeddings.

Our method can be carried over to more general reaction-diffusion type equations where the reaction term is a polynomial of odd degree and the leading coefficient positive (see for instance Temam [13]). Note however that the geometry that we consider is flat, our objectives and the type of problems we are interested in are not the same as those occuring with curved boundaries in relation in particular with the Ginzburg Landau equation (see e.g. [11], [12] and the bibliography therein).

Our main results are the following:

Theorem 1. *There exist constants K_j depending on T, u_0 and f only such that*

$$\left\| u^\varepsilon(t;x,y) - u^0(t;x,y) - M\left(t,x,\frac{y}{\sqrt{\varepsilon}}\right) - N\left(t;x,\frac{1-y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty((0,T)\times\Omega)} \le K_1\varepsilon^{1/2}, \quad (7)$$

$$\left\| u^\varepsilon(t;x,y) - u^0(t;x,y) - M\left(t,x,\frac{y}{\sqrt{\varepsilon}}\right) - N\left(t;x,\frac{1-y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty(0,T;L^2(\Omega))} \le K_2\varepsilon_1^{3/4}, \quad (8)$$

$$\left\| u^\varepsilon(t;x,y) - u^0(t;x,y) - M\left(t,x,\frac{y}{\sqrt{\varepsilon}}\right) - N\left(t;x,\frac{1-y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty(0,T;H^1(\Omega))} \le K_3\varepsilon^{1/4}, \quad (9)$$

where M and N are solutions of

$$\frac{\partial M}{\partial t} - \frac{\partial^2 M}{\partial y^2} + M^3 - M + 3g_0M^2 + 3g_0^2M = 0 \quad \text{in} \quad y>0, \quad (10)$$

$$M = 0 \quad \text{at} \quad t=0, \quad (11)$$

and

$$M = -g_0 \quad \text{at} \quad y=0, \quad M\to 0 \quad \text{as} \quad y\to+\infty, \quad (12)$$

$$\frac{\partial N}{\partial t} - \frac{\partial^2 N}{\partial y^2} + N^3 - N + 3g_1N^2 + 3g_1^2N = 0 \quad \text{in} \quad y>0, \quad (13)$$

$$N = 0 \quad \text{at} \quad t=0, \quad (14)$$

and

$$N = -g_1 \quad \text{at} \quad y=0, \quad N\to 0 \quad \text{as} \quad y\to+\infty, \quad (15)$$

where

$$g_0(t;x) = u^0|_{y=0}, \;\; g_1(t;x) = u^0|_{y=1}. \quad (16)$$

Here the spaces are defined as

$$H_p^1(\Omega) = \left\{v\in H^1(\Omega), v \text{ is periodic in } x \text{ with period } 2\pi\right\}; \quad (17)$$

$$H_{0p}^1(\Omega) = \left\{v\in H_p^1(\Omega), v=0 \text{ at } y=0 \text{ and } y=1\right\}. \quad (18)$$

The rest of the article is organized as follows. In the next section we introduce a preliminary form of the corrector and derive some useful estimates; then, in the last section, we derive the correctors (M and N) and prove the main result.

2 The Preliminary Form of the Corrector

It is obvious that u^ε cannot converge to u^0 as ε approaches zero uniformly in Ω. However it is plausible to think that the convergence is true in the interior of Ω since the diffusive coefficient is small. If this is true, $u^\varepsilon - u^0$ can be approximated by a boundary layer type function θ^ε called corrector (see Lions [8]). Considering (1) and (5) we propose that θ^ε be the solution of the following evolution equation

$$\frac{\partial \theta^\varepsilon}{\partial t} - \varepsilon \Delta \theta^\varepsilon + (\theta^\varepsilon)^3 - \theta^\varepsilon + 3u^0(\theta^\varepsilon)^2 + 3(u^0)^2\theta^\varepsilon = 0 \quad \text{in} \quad \Omega, \tag{19}$$

$$\theta^\varepsilon = 0 \quad \text{at} \quad t = 0, \tag{20}$$

$$\theta^\varepsilon = -u^0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1. \tag{21}$$

We are led to estimate $w^\varepsilon = u^\varepsilon - u^0 - \theta^\varepsilon$ which satisfies the equation

$$\frac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + (w^\varepsilon)^3 - w^\varepsilon + 3u^\varepsilon(u^0 + \theta^\varepsilon)w^\varepsilon = \varepsilon \Delta u^0 \quad \text{in} \quad \Omega, \tag{22}$$

$$w^\varepsilon = 0 \quad \text{at} \quad t = 0, \tag{23}$$

$$w^\varepsilon = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1. \tag{24}$$

Denoting K a generic constant which may depend on T, u_0 and f but is independent of ε, and which may change from place to place, we obtain:

$$\|\nabla^k u^0\|_{L^\infty((0,T)\times\Omega)} \le K \quad \text{for} \quad k = 0, 1, \dots \tag{25}$$

and by the usual maximum principle

$$\|u^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K, \tag{26}$$

$$\|\theta^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K, \tag{27}$$

$$\|M\|_{L^\infty((0,T)\times\{y>0\})} + \|N\|_{L^\infty((0,T)\times\{y>0\})} \le K. \tag{28}$$

The maximum principle applies to w^ε (equation (22)) as well. Indeed let K_1 be a constant independent of ε and larger than $3\|u^\varepsilon(u^0 + \theta^\varepsilon)\|_{L^\infty((0,T)\times\Omega)}$, and consider

$$\tilde{w}^\varepsilon = e^{-(K_1+2)t} w^\varepsilon;$$

we have

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t} - \varepsilon \Delta \tilde{w}^\varepsilon + e^{2(K_1+2)t}(\tilde{w}^\varepsilon)^3 + (K_1 + 2 + 3u^\varepsilon(u^0 + \theta^\varepsilon))\tilde{w}^\varepsilon = \varepsilon e^{-(K_1+1)t} \Delta u^0,$$

It is now easy to observe that

$$\tilde{w}^\varepsilon(t; x, y) \le \varepsilon \|\Delta u^0\|_{L^\infty((0,T)\times\Omega)} \qquad \text{for } (t; x, y) \in (0, T) \times \Omega.$$

We can derive a corresponding lower bound and thus we conclude that

$$\|w^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon. \tag{29}$$

This indicates that θ^ε is a good preliminary corrector. Furthermore, standard energy estimates for (22) yield

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq K\varepsilon, \tag{30}$$

$$\|w^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq K\varepsilon^{1/2}. \tag{31}$$

This again confirms the choice of θ^ε.

To derive $L^\infty(H^1)$ estimates on w^ε we multiply (22) by $-\Delta w^\varepsilon$ and integrate over Ω. We have, after rewriting $u^0 + \theta^\varepsilon$ as $u^\varepsilon - w^\varepsilon$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla w^\varepsilon|^2_{L^2(\Omega)} + \varepsilon|\Delta w^\varepsilon|^2_{L^2(\Omega)} + \int_\Omega 3(w^\varepsilon)^2|\nabla w^\varepsilon|^2 \\ + 3\int_\Omega (u^\varepsilon)^2|\nabla w^\varepsilon|^2 + 6\int_\Omega u^\varepsilon w^\varepsilon \nabla u^\varepsilon \cdot \nabla w^\varepsilon \\ - 6\int_\Omega u^\varepsilon w^\varepsilon |\nabla w^\varepsilon|^2 - 3\int_\Omega (w^\varepsilon)^2 \nabla u^\varepsilon \cdot \nabla w^\varepsilon \\ \leq \frac{\varepsilon}{2}|\Delta w^\varepsilon|^2_{L^2(\Omega)} + K|\nabla w^\varepsilon|^2_{L^2(\Omega)} + K\varepsilon^{3/2}.\end{aligned} \tag{32}$$

For the right-hand side of (32) we have used the following inequality, with f and u replaced by $\varepsilon \Delta u^0$ and w^ε :

$$-\int_\Omega f\Delta u = \int_\Omega \nabla f \nabla u - \int_{y=1} f\frac{\partial u}{\partial y} + \int_{y=0} f\frac{\partial u}{\partial y},$$

and hence

$$\begin{aligned}\left|\int_\Omega f\Delta u\right| &\leq |\nabla f|_{L^2(\Omega)}|\nabla u|_{L^2(\Omega)} + |f|_{L^2(\Gamma)}\left|\frac{\partial u}{\partial y}\right|_{L^2(\Gamma)} \\ &\leq |\nabla f|_{L^2(\Omega)}|\nabla u|_{L^2(\Omega)} + K|f|_{L^2(\Gamma)}|\nabla u|^{1/2}_{L^2(\Omega)}|\Delta u|^{1/2}_{L^2(\Omega)} \\ &\leq |\nabla f|_{L^2(\Omega)}|\nabla u|_{L^2(\Omega)} + \frac{\varepsilon}{2}|\Delta u|^2_{L^2(\Omega)} + |\nabla u|^2_{L^2(\Omega)} + K\varepsilon^{-1/2}|f|^2_{L^2(\Gamma)}.\end{aligned} \tag{33}$$

The treatment of inequality (32) then necessitates estimates on ∇u^ε which can be derived by multiplying (1) by $-\Delta u^\varepsilon$ integrating over Ω and applying the Uniform Gronwall inequality (see e.g. [13]). We also apply (33) with u replaced by u^ε. We find:

$$\|u^\varepsilon\|_{L^\infty(0,\infty;H^1(\Omega))} \leq K\varepsilon^{-1/4}. \tag{34}$$

Combining (26), (27), (29), (32) and (34) we deduce

$$\frac{d}{dt}|\nabla w^\varepsilon|^2_{L^2(\Omega)} + \varepsilon|\Delta w^\varepsilon|^2_{L^2(\Omega)} \leq K|\nabla w^\varepsilon|^2_{L^2(\Omega)} + K\varepsilon^{3/2} + K\varepsilon^2|\nabla u^\varepsilon|^2_{L^2(\Omega)},$$

which implies

$$\|w^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq K\varepsilon^{3/4}, \quad \|w^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq K\varepsilon^{1/4}. \tag{35}$$

By differentiating the equations in x and repeating the above procedures, we see that the above estimates remain valid for $\partial^k w^\varepsilon/\partial x^k$. This confirms our intuition that tangential derivatives are small even though the normal ones might be large.

3 The Explicit Corrector and the Proof of the Theorem

Since the tangential derivatives are small we tend to neglect them in equation (19). We also expect that θ^ε be a boundary layer type function, i.e. it decays fast in the interior of the domain, thus in terms of matched asymptotic expansions, the outer expansion should be trivial (which is easy to see) and the inner expansion matches the outer one automatically. This leads us to propose M and N defined by (10)–(16) as the inner expansions at $y = 0$ and $y = 1$ respectively. We will check that these expressions are suitable.

We first prove the decay property of M, N, and θ^ε. It is enough to prove this for θ^ε. Let $\eta \in C_0^\infty([0,1])$ be a cut-off function, $\eta \geq 0$.

Standard energy estimates yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \eta(\theta^\varepsilon)^2 + \varepsilon\int_\Omega \eta|\nabla\theta^\varepsilon|^2 + \int_\Omega \left(\eta(\theta^\varepsilon)^4 - \eta(\theta^\varepsilon)^2 + 3u^0\eta(\theta^\varepsilon)^3 + 3(u^0)^2\eta(\theta^\varepsilon)^2\right)$$
$$= -\varepsilon\int_\Omega \eta'\frac{\partial\theta^\varepsilon}{\partial y}\theta^\varepsilon = \frac{\varepsilon}{2}\int_\Omega \eta''(\theta^\varepsilon)^2. \quad (36)$$

Using a function of the form

$$\varphi^\varepsilon(t;x,y) = -g_0(t;x)\rho\left(\frac{y}{\sqrt{\varepsilon}}\right) - g_1(t,x)\rho\left(\frac{1-y}{\sqrt{\varepsilon}}\right), \quad (37)$$

with $\rho \in C^\infty([0,1]), \rho(0) = 1,$ supp $\rho \subset [0, \frac{1}{2}]$, and considering $\theta^\varepsilon - \varphi^\varepsilon$, we deduce

$$\|\theta^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq K\varepsilon^{1/4}, \quad (38)$$
$$\|\theta^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq K\varepsilon^{-1/4}. \quad (39)$$

This together with (36), implies for $\delta \in (0, \frac{1}{2})$,

$$\|\theta^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\delta))} \leq K_\delta\varepsilon^{3/4},$$
$$\|\theta^\varepsilon\|_{L^2(0,T;H(\Omega_\delta))} \leq K_\delta\varepsilon^{1/4},$$

where

$$\Omega_\delta = (0, 2\pi) \times (\delta, 1-\delta), \quad (40)$$

and K_δ is a constant depending on δ, T, f, u_0, but independent of ε.

By reiteration, we deduce

$$\|\theta^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\delta))} \leq K_\delta\varepsilon^{5/4}, \quad (41)$$
$$\|\theta^\varepsilon\|_{L^2(0,T;H^1(\Omega_\delta))} \leq K_\delta\varepsilon^{3/4}. \quad (42)$$

We could reiterate again but our aim now is to obtain estimates on higher order derivatives of θ^ε. For that purpose we multiply (19) by $-\nabla(\eta(y)\nabla\theta^\varepsilon)$ and integrate over Ω.

Notice that

$$\varepsilon\int_\Omega \Delta\theta^\varepsilon\eta'\frac{\partial\theta^\varepsilon}{\partial y} = \frac{\varepsilon}{2}\int_\Omega \eta''|\nabla\theta^\varepsilon|^2 - \varepsilon\int_\Omega \eta''\left(\frac{\partial\theta^\varepsilon}{\partial y}\right)^2,$$

$$\left|\int_\Omega \eta\nabla\theta^\varepsilon\nabla((\theta^\varepsilon)^3 + 3u^0(\theta^\varepsilon)^2 + 3(u^0)^2\theta^\varepsilon)\right| \le K\int_\Omega \eta|\nabla\theta^\varepsilon|^2 + K\varepsilon^{5/2},$$

(Thanks to (27), (41) and (42));

hence we have

$$\|\theta^\varepsilon\|_{L^\infty(0,T;H^k(\Omega_\delta))} \le K_\delta\varepsilon^{5/4}, \tag{43}$$

$$\|\theta^\varepsilon\|_{L^2(0,T;H^{k+1}(\Omega_\delta))} \le k_\delta\varepsilon^{3/4}, \quad \text{for} \quad k = 0, 1. \tag{44}$$

The procedure can be repeated for $k = 2, 3$, and with $\partial^k\theta^\varepsilon/\partial x^k$ replacing θ^ε.

Similar estimates hold for $M^\varepsilon(t, x, y) = M(t, x, \frac{y}{\sqrt{\varepsilon}})$ and also for $N^\varepsilon(t, x, y) = N(t; x, \frac{1-y}{\sqrt{\varepsilon}})$. In particular we will have for

$$C_M^\varepsilon(t; x, y) = -yM\left(t; x, \frac{1}{\sqrt{\varepsilon}}\right), \tag{45}$$

$$\|\nabla^k C_M^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon^{5/4}, \quad \text{for} \quad k = 0, 1, 2, \dots \tag{46}$$

$$\left\|\frac{\partial C_M^\varepsilon}{\partial t}\right\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon^{5/4}. \tag{47}$$

We then consider the quantity

$$q^\varepsilon = \theta^\varepsilon - M^\varepsilon - N^\varepsilon - C^\varepsilon,$$

where $C^\varepsilon = C_M^\varepsilon + C_N^\varepsilon$, $C_N^\varepsilon = -(1-y)N\left(t, x, \frac{1}{\sqrt{\varepsilon}}\right)$.

For the sake of simplicity, we now assume that $f \equiv 0$ on $y = 1$ and hence $g_1 \equiv 0$, which further implies $N \equiv 0$. Hence q^ε reduces to

$$q^\varepsilon = \theta^\varepsilon - M^\varepsilon - C_M^\varepsilon\,. \tag{48}$$

It satisfies the equation

$$\begin{aligned}\frac{\partial q^\varepsilon}{\partial t} &- \varepsilon\Delta q^\varepsilon + (\theta^\varepsilon)^3 + 3u^0(\theta^\varepsilon)^2 + 3(u^0)^2\theta^\varepsilon \\ &- (M^\varepsilon)^3 - 3g_0(M^\varepsilon)^2 - 3g_0^2M^\varepsilon - q^\varepsilon \\ &= -\frac{\partial C_M^\varepsilon}{\partial t} + \varepsilon\Delta C_M^\varepsilon + \varepsilon\frac{\partial^2 M^\varepsilon}{\partial x^2} + C_M^\varepsilon \quad \text{in} \quad \Omega,\end{aligned} \tag{49}$$

with initial and boundary conditions (thanks to $N \equiv 0$):

$$q^\varepsilon = 0 \quad \text{at} \quad t = 0, \tag{50}$$

$$q^\varepsilon = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = 1. \tag{51}$$

Notice that

$$\begin{aligned}
(\theta^\varepsilon)^3 - (M^\varepsilon)^3 &= q^\varepsilon((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2) + C_M^\varepsilon((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2),\\
3u^0(\theta^\varepsilon)^2 - 3g_0(M^\varepsilon)^2 &= 3u^0(\theta^\varepsilon + M^\varepsilon)q^\varepsilon + 3u^0(\theta^\varepsilon + M^\varepsilon)C_M^\varepsilon + 3(u^0 - g_0)(M^\varepsilon)^2,\\
3(u^0)^2\theta^\varepsilon - 3g_0^2 M^\varepsilon &= 3(u^0)^2 q^\varepsilon + 3(u^0)^2 C_M^\varepsilon + 3(u^0 + g_0)(u^0 - g_0)M^\varepsilon;
\end{aligned}$$

hence we may rewrite (49) as

$$\begin{aligned}
&\frac{\partial q^\varepsilon}{\partial t} - \varepsilon \Delta q^\varepsilon + ((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2)q^\varepsilon\\
&\qquad + 3u^0(\theta^\varepsilon + M^\varepsilon)q^\varepsilon + 3(u^0)^2 q^\varepsilon - q^\varepsilon = \tilde{f} \quad \text{in} \quad \Omega,
\end{aligned} \tag{49'}$$

where

$$\begin{aligned}
\tilde{f} = &-\frac{\partial C_M^\varepsilon}{\partial t} + \varepsilon \Delta C_M^\varepsilon + \varepsilon \frac{\partial^2 M^\varepsilon}{\partial x^2} + C_M^\varepsilon\\
&- \big((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2 + 3u^0(\theta^\varepsilon - M^\varepsilon) + 3(u^0)^2\big)C_M^\varepsilon\\
&- 3(u^0 - g_0)(M^\varepsilon)^2 - 3(u^0 + g_0)(u^0 - g_0)M^\varepsilon.
\end{aligned} \tag{52}$$

By the choice of g_0, $\frac{u^0 - g_0}{y}$ remains bounded on $(0,T) \times \Omega$. In order to obtain an L^∞ estimate on $\tilde{f}$ (sharp in terms of dependence on ε), we need to obtain an L^∞ bound on yM. Consider $(1+y)M$ which satisfies the equation

$$\begin{aligned}
&\frac{\partial((1+y)M)}{\partial t} - \frac{\partial^2}{\partial y^2}((1+y)M) + \frac{1}{(1+y)^2}((1+y)M)^3 + \frac{3g_0}{1+y}((1+y)M)^2\\
&\qquad + 3g_0^2(1+y)M - (1+y)M = -2\frac{\partial M}{\partial y},
\end{aligned} \tag{53}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial M}{\partial y}\right) - \frac{\partial^2}{\partial y^2}\left(\frac{\partial M}{\partial y}\right) + 3M^2\frac{\partial M}{\partial y} + 6g_0 M\frac{\partial M}{\partial y} + 3g_0^2\frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} = 0. \tag{54}$$

We see that $\frac{\partial M}{\partial y}$ satisfies a maximum principle and hence $(1+y)M$ too.

This combined with (27), (28), (46) and (47) yields

$$\|\tilde{f}\|_{L^\infty((0,T)\times\Omega)} \leq K\varepsilon^{1/2}. \tag{55}$$

This further implies, via a maximum principle type argument as that for w^ε,

$$\|q^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leq K\varepsilon^{1/2}. \tag{56}$$

It is also easy to check, thanks to (46), (47) and the boundedness of $\frac{u^0 - g_0}{y}$, that

$$\|\tilde{f}\|_{L^2(0,T;L^2(\Omega))} \leq K\varepsilon^{3/4}. \tag{57}$$

Thus standard energy estimates yield

$$\|q^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq K\varepsilon^{3/4}, \tag{58}$$
$$\|q^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq K\varepsilon^{1/4}. \tag{59}$$

The theorem then follows from (29), (30), (35), (46), (56), (58) and (59). This completes the proof.

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