The Multisegment Duality
and the Preprojective Algebras of Type A

Claus Michael Ringel

ABSTRACT. The multisegment (or Zelevinsky) duality ζ plays an important role in the representation theory of the groups GL_n over a p-adic field, but also for the quantum groups of type A. Recently, Knight and Zelevinsky have exhibited a formula which allows a direct calculation of ζ; their proof uses the representation theory of a linearly ordered quiver of type A. Some of their considerations may be interpreted homologically. It is well-known that the multisegment duality can easily be defined in terms of the corresponding preprojective algebra. The use of certain modules over the preprojective algebra seems to illuminate the considerations of Knight and Zelevinsky. We are going to outline all the essential steps, only few details are left to the reader.

Throughout the paper we fix some natural number n. We denote by S the set of pairs (i,j) where $1 \le i \le j \le n$; or also the set of corresponding *segments* $[i,j] = \{i, i{+}1, \dots, j\}$ of natural numbers. Note that we may identify S with the set of positive roots in the root system of type A_n; the one-element segment $[i,i]$ corresponds to the simple root a_i and $[i,j]$ to $a_i + a_{i+1} + \cdots + a_j$. The elements d of $\mathbb{N}_0^S$ may be considered as formal linear combinations of segments with non-negative integral coefficients, they are called *multisegments,* such an element is of the form $d = (d_{ij})_{(i,j)}$; here all the d_{ij} belong to $\mathbb{N}_0$ and the index pairs (i,j) to S.

In addition, we also fix some algebraically closed field K. All the vector spaces will be defined over K. In particular, when dealing with a quiver Δ, the representations we consider are given by K-spaces and K-linear maps; note that such representations are just the modules over the corresponding path algebra $K\Delta$. The quivers we deal with will have no multiple arrows, thus we may refer to a path $(a_0 \to a_1 \to \cdots \to a_t)$ by specifying the vertices but without labeling the arrows.

1. The linearly oriented quiver of type A_n.

We denote by $\Lambda = \Lambda(n)$ the path algebra of the following quiver: the vertices are the natural numbers $1, 2, \dots, n$, and there are the arrows $\alpha_i \colon i \to i{+}1$ for $1 \le i < n$. Gabriel's theorem asserts that the indecomposable representations correspond bijectively to elements of S. In fact, let us denote by $S(i)$ the simple representation corresponding to the vertex i. The segment $[i,j]$ corresponds to the indecomposable representation $M(i,j)$ with composition factors $S(i), S(i{+}1), \dots, S(j)$ (it is unique up to isomorphism). In this way one obtains all the isomorphism classes of indecomposable representations. According to the Krull-Remak-Schmidt theorem, any Λ-module M is isomorphic to one of the form $M(d) = \bigoplus_{ij} d_{ij} M(i,j)$. The map $d \mapsto M(d)$ yields a bijection between $\mathbb{N}_0^S$ and the set of isomorphism classes of finitely generated Λ-modules.

Given any Λ-module M and $i \le j$ we denote by $r_{ij}(M)$ the rank of the linear map corresponding to the path $i \to i{+}1 \to \cdots \to j$ (or, equivalently, of the K-linear map $M \to M$ given by multiplication by $\alpha_{j-1} \dots \alpha_{i+1}\alpha_i$). Clearly, the value $r_{ji}(M)$ only depends on the isomorphism class of M, and $r_{ii}(M)$ is equal to the Jordan-Hölder multiplicity of $S(i)$ in M. Note that these functions r_{ij} form a complete set of invariants: given all the values $r_{ij}(M)$, we recover the isomorphism class d of M, namely

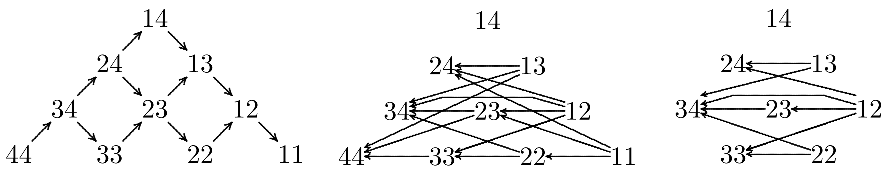
$$d_{ij} = r_{ij}(M) - r_{i-1,j}(M) - r_{i,j+1}(M) + r_{i-1,j+1}(M).$$

We denote by E the Ext-quiver of Λ, its vertices are the isomorphism classes of the indecomposable representations (or the elements of S), and there is an arrow $(i,j) \to$

(k,l) provided $\operatorname{Ext}^1(M(i,j),M(k,l)) \neq 0$ (note that in our case, these Ext-groups are 1-dimensional). It is well-known that $\operatorname{Ext}^1(M(i,j),M(k,l)) \neq 0$ if and only if $i+1 \le k \le j+1 \le l$. For the proof, denote by τ the Auslander-Reiten translation, thus $\tau M(i,j)$ is equal to $M(i+1,j+1)$ for $j < n$ and to the zero representation otherwise. Now, $\operatorname{Ext}^1(M(i,j),M(k,l))$ has the same dimension as $\operatorname{Hom}(M(k,l),\tau M(i,j))$, and the fact that $\operatorname{Hom}(M(k,l),M(i+1,j+1)) \neq 0$ is described by the mentioned inequalities.

Let (i,j) be an element of E. We write $E(i,j)$ for the full subquiver of E given by the vertices (k,l) such that $[k,l] \cap [i,j] \neq \emptyset$. Note that $E(i,j)$ is a convex subquiver of E.

Here an example, for $n = 4$. We arrange the vertices of the Ext-quiver E (shown in the middle) in the same way as we arrange the Auslander-Reiten quiver (shown to the left); we also exhibit the quiver $E(2,3)$ (to the right):



$\Gamma(\Lambda)$ $\qquad$ E $\qquad$ $E(2,3)$

2. The preprojective algebra of type A_n.

The preprojective algebra $\overline{\Lambda}$ of type A_n can be described by a quiver and relations as follows: as vertices take the natural numbers $1,2,\dots,n$, take arrows $\alpha_i\colon i \to i+1$ and $\alpha_i^*\colon i+1 \to i$, for $1 \le i < n$, and the relations $\alpha_1^*\alpha_1 = 0$, $\alpha_i^*\alpha_i = \alpha_{i-1}\alpha_{i-1}^*$ for $1 < i < n$ and $\alpha_{n-1}\alpha_{n-1}^* = 0$ (the general definition involves signs for the relations, but since we deal with a tree type, these signs do not matter). Note that $\overline{\Lambda}$ contains Λ as a subalgebra. Also, we may consider the subalgebra of $\overline{\Lambda}$ generated by the paths of length zero and the arrows α_i^*, we denote it by Λ°. Observe that Λ° is again the path algebra of a linearly oriented A_n-quiver. The algebras Λ and Λ° have the same simple modules and as above we denote by $M^\circ(i,j)$ an indecomposable Λ°-modules with composition factors $S(i), S(i+1),\dots,S(j)$; and for $d \in \mathbb{N}_0^S$, let $M^\circ(d) = \bigoplus d_{ij}M^\circ(i,j)$. Thus again we identify the isomorphism classes of the indecomposable Λ°-modules with S, those of arbitrary Λ°-modules with $\mathbb{N}_0^S$.

Given any $\overline{\Lambda}$-module N, we denote by $\pi(N)$ the space N considered as a Λ-module, similarly, the space N considered as a Λ°-module will be denoted by $\pi^\circ(N)$.

As for Λ-modules, we also consider ranks of linear maps for $\overline{\Lambda}$-modules and for Λ°-modules: Given any $\overline{\Lambda}$-module N and $i < j$ we denote by $r_{ij}(N)$ the rank of the linear map corresponding to the path $i \to i+1 \to \cdots \to j$ and by $r^\circ_{ij}(N)$ that corresponding to the path $i \leftarrow i+1 \leftarrow \cdots \leftarrow j$; for a Λ°-module M', we similarly define $r^\circ_{ij}(M')$. Of course, for any $\overline{\Lambda}$-module N, we have

$$r_{ij}(\pi(N)) = r_{ij}(N), \quad \text{and} \quad r^\circ_{ij}(\pi^\circ(N)) = r^\circ_{ij}(N).$$

The Λ-modules $M(i,j)$, where $(i,j) \in S$, can be considered as $\overline{\Lambda}$-modules. Let us write down the groups $\operatorname{Ext}^1_{\overline{\Lambda}}(M(i,j),M(k,l))$:

$$\operatorname{Ext}^1_{\overline{\Lambda}}(M(i,j),M(k,l)) = \begin{cases} K & \text{if } k+1 \le i \le l+1 \le j, \\ K & \text{if } i+1 \le k \le j+1 \le l, \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, the corresponding exact sequences split when we apply π, in the second, they split when we apply π°. Of course, in the second case, the extension groups

$\mathrm{Ext}^1_{\overline{\Lambda}}(-,-)$ are just the extension groups $\mathrm{Ext}^1_{\Lambda}(-,-)$ as discussed in the previous section, in particular the non-vanishing of these groups is described by the Ext-quiver E. Let us denote by $\mathrm{Ext}^1_{(\overline{\Lambda},\Lambda)}(N,N')$ the set of extensions which split when we apply π; of course, these are the extension groups with respect to a relative homological structure on the category of all $\overline{\Lambda}$-modules. Since

$$\mathrm{Ext}^1_{(\overline{\Lambda},\Lambda)}(M(i,j),M(k,l)) = \begin{cases} K & \text{if } k+1 \le i \le l+1 \le j, \\ 0 & \text{otherwise,} \end{cases}$$

we see that the quiver E^* opposite to E describes the non-vanishing of $\mathrm{Ext}^1_{(\overline{\Lambda},\Lambda)}$. This will be the Ext-quiver we are interested in, and we have to deal with the paths in this quiver. Instead of considering paths in E^* (as we should) we can equally well work with paths in E, we only have to remember that the individual steps in such a path mention first the submodule and then the factor module of the corresponding extension.

3. The multisegment duality.

The following observation by Zelevinsky [Z,KZ] (and later also by Lusztig [L]) can be used as a definition of the multisegment duality ζ. For every Λ-module, let $\mathcal{Z}(M)$ be the set of $\overline{\Lambda}$-modules N with $\pi(N) = M$; note that this is a K-variety. *Let d be a multisegment. The K-variety $\mathcal{Z}(M(d))$ contains an open and dense subset $\mathcal{O}$ such that for all $N \in \mathcal{O}$, the Λ°-modules $\pi^\circ(N)$ are isomorphic, say isomorphic to $M^\circ(\zeta d)$.*

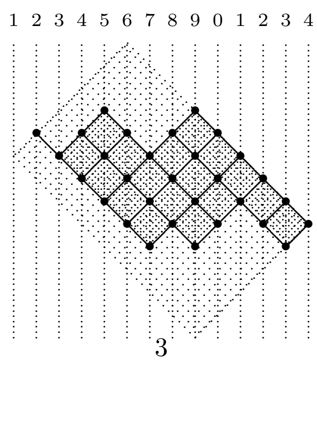
Here is the recipe proposed by Knight and Zelevinsky [KZ] in order to calculate ζd. Determine the maximal value of $r^\circ_{ij}(N)$ for all $\overline{\Lambda}$-modules N with $\pi(N) = M(d)$. This maximal value is just $r^\circ_{ij}(M^\circ(\zeta d))$. But remember: Λ° is the path algebra of a directly oriented A_n-quiver, thus the functions r°_{ij} form a complete set of invariants and allow to recover ζd easily.

4. Laminae and paths in E.

Denote by $P(i)$ the indecomposable projective $\overline{\Lambda}$-modules with top $S(i)$. Note that $\pi(P(i)) = \bigoplus_{t=1}^{i} M(t, n-i+t)$ and $\pi^\circ(P(i)) = \bigoplus_{t=1}^{n-i+1} M^\circ(t, i-1+t)$. Of importance is the following fact: *For every i, the submodules of $P(i)$ form a distributive lattice, thus there are only finitely many submodules.* Proof: Every semisimple subquotient is multiplicity-free.

An indecomposable subquotient of any $P(i)$ is said to be a *lamina,* direct sums of laminae are said to be *laminated modules.* For $n \le 3$, all modules are laminated, for $n \ge 4$ not; an example of a module which is not laminated will be presented below.

Here is a typical example of a lamina L. We take $r = 14$ and exhibit a subquotient of $P(6)$. The upper row labels the composition factors (but only the last digit is shown: it starts on the left with $1, 2, \dots$ and ends with $\dots, 13, 14$). The dotted frame shows the shape of $P(6)$. The bullets indicate base vectors, the lines the multiplication by α_i (in south-east direction) and α_i^* (in south-west direction).



We have

$$\pi(L) = M(2,7) \oplus M(4,9) \oplus M(5,10) \oplus M(8,13) \oplus M(9,14).$$

Theorem 1. *For every path $w = (a_0, a_1, \dots, a_p)$ in E, there exists up to isomorphism a unique indecomposable $\overline{\Lambda}$-module $L(w)$ with $\pi(L(w)) = \bigoplus_i M(a_i)$. It is a lamina and every lamina is obtained in this way.*

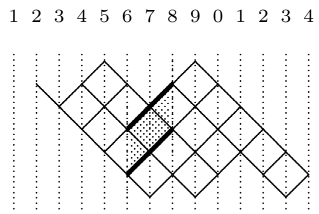
Remark: The paths to be considered are actually those in E^*, not in E, but of course, there is no real difference. Given a lamina L, the decomposition of $\pi(L)$ into indecomposables clearly gives a path in the relative Ext-quiver E^*. It is easy to see that also all the paths are obtained in this way. In fact, the relative global dimension of the pair $(\overline{\Lambda}, \Lambda)$ is at most 1.

Let us stress that Theorem 1 in conjunction with the definition of a lamina shows well the benefit of invoking preprojective algebras: The theorem provides a module theoretical realization of a combinatorial object, namely of paths in the quiver E. The advantage of working with $\overline{\Lambda}$ stems from the fact that there is no need for a cumbersome construction of the modules $L(w)$ using iterated extensions of serial modules, it suffices to identify them with subfactors of indecomposable projective modules, a much easier task.

Theorem 2. *Let w be a path in $E(i,j)$ of length p. If $p \geq j - i$, then $r^\circ_{ij}(L(w)) = p - j + i + 1$, otherwise $r^\circ_{ji}(L(w)) = 0$.*

More generally, we may consider an arbitrary path w in E. Let us denote by w_{ij} its part inside $E(i,j)$, this is again a path since $E(i,j)$ is convex in E. Then $r^\circ_{ij}(L(w)) = r^\circ_{ij}(L(w_{ij}))$.

Example: Let $r = 14$, let $i = 6, j = 8$ and consider again the lamina L exhibited above:



Here, w is the path

$$w = \big((2,7) \to (4,9) \to (5,10) \to (8,13) \to (9,14)\big),$$

and

$$w_{ij} = \big((2,7) \to (4,9) \to (5,10) \to (8,13)\big),$$

The path w_{ij} has length 3, and we have $j - i = 8 - 6 = 2$. Thus $r^\circ_{ij}(L(w)) = l(w_{ij}) - j + i + 1 = 2$.

5. Laminated $\overline{\Lambda}$-modules.

Let Δ be a directed quiver without multiple arrows. Define the p-capacity $\kappa_p(\Delta)$ of Δ as the maximum of

$$\sum_{w \in \mathcal{W}} \big(l(w) - p + 1\big)$$

where $\mathcal{W}$ is a set of vertex-disjoint paths w of length $l(w) \geq p$.

Given a quiver $\Delta = (\Delta_0, \Delta_1)$ and a function $d \colon \Delta_0 \to \mathbb{N}_0$, we denote by Δ^d the quiver obtained from Δ by replacing any vertex x of Δ_0 by vertices (x,i) with $1 \leq i \leq d(x)$, and with arrows $(\alpha, i, j) \colon (x,i) \to (y,j)$ for any arrow $\alpha \colon x \to y$ and $1 \leq i \leq d(x)$, $1 \leq j \leq d(y)$.

Theorem 3. *Let d be a multisegment and let $i < j$. Then there exists a laminated module N with $\pi(N) = M(d)$ such that $r^\circ_{ij}(N) = \kappa_{j-i}(E(i,j)^d)$.*

Proof: Let $p = j - i$. By definition of $c = \kappa_{j-i}(E(i,j)^d)$, we get a set $\mathcal{W}$ of vertex-disjoint paths w of length at least p such that $c = \sum_{w\in\mathcal{W}}(l(w) - p + 1)$. But any path w just yields a lamina $L(w)$ and $r^\circ_{ij}(L(w)) = l(w) - p + 1$. As module N we take the direct sum of these laminae.

6. The p-capacity of a directed quiver.

Given a quiver Δ, the quiver $\Delta[p]$ is constructed as follows: take $p+1$ copies of Δ_0, say vertices (x, i) with x a vertex of Δ and $0 \le i \le p$, and draw an arrow $(x, i-1) \to (y, i)$ provided there is an arrow $x \to y$. This is a p-network (see [FF] or also [KZ]), the length of a path in a p-network is at most p, and the maximal number of vertex-disjoint paths is called its capacity and denoted by $\kappa(\Delta[p])$. Of course, this capacity is just the p-capacity of $\Delta[p]$

Proposition. $\kappa_p(\Delta) = \kappa(\Delta[p])$.

Proof: The inequality $\kappa_p(\Delta) \le \kappa(\Delta[p])$ is obvious: Let us assume that there is given a set $\mathcal{W}$ of vertex disjoint paths of length at least p. Any path $x_0 \to x_1 \to \cdots \to x_{p+t}$ in Δ with $t \ge 0$ yields $t+1$ disjoint paths in $\Delta[p]$, namely the paths $(x_i, 0) \to (x_{i+1}, 1) \to \cdots \to (x_{p+i}, p)$ for $0 \le i \le t$.

The reverse inequality follows from Poljak [P], see [KZ], Theorem 2.3.

7. The Knight-Zelevinsky formula.

The essential observation of Knight-Zelevinsky [KZ] is the following: *Let d be a multisegment, let $i < j$. Then*

$$(*) \qquad r^\circ_{ij}(M^\circ(\zeta d)) = \kappa_{j-i}(E(i,j)^d).$$

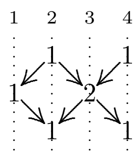
(The paper uses as right side $\kappa(E(i,j)^d[j-i])$, but as we have noted in the last section, this is the same.) This formula reduces the calculation of the rank function r°_{ij} to a purely combinatorial problem, and Knight and Zelevinsky use for the right side the "maximal flow = minimal cut" theorem for networks in order to arrive at a formula for r°_{ij} which concerns a minimum.

On the basis of $(*)$, we may reformulate Theorem 3 as follows:

Theorem 4. *Let d be a multisegment and let $i < j$. Then there exists a laminated module N with $\pi(N) = M(d)$ such that $r^\circ_{ij}(N) = r^\circ_{ij}(M^\circ(\zeta d))$.*

This result shows that the optimal value for r°_{ij} can be achieved using laminated modules. However, let us stress the following: *If we deal only with laminated modules, it can be impossible that all the r°_{ij} become optimal at the same time!*

Example: Consider the case $n = 4$ and let d be given by $d_{12} = d_{24} = d_{33} = d_{44} = 1$ and $d_{ij} = 0$ otherwise. We exhibit an indecomposable $\overline{\Lambda}$-module N with $\pi(N) = M(d)$ which is not laminated (we write down the dimension vector of a module $\widetilde{N}$ for the universal covering of $\overline{\Lambda}$ which is sent under the covering functor to N; due to the fact that the square has to be commutative and that there is a zero relation on the right arms, $\widetilde{N}$ is unique up to isomorphism and shift under the Galois group):



and we see that $r^\circ_{12}(N) = 1 = r^\circ_{24}(N)$.

Consider any laminated $\overline{\Lambda}$-module N' with $\pi(N') = M(d)$ and $r^\circ_{24}(N') = 1$. In order to have $r^\circ_{24}(N') = 1$, the module N' must have a direct summand N'' which is a subquotient of $P(3)$ or $P(4)$. Since $\pi(N'')$ must be a direct summand of $\pi(N)$, we see the only possibility is $\pi(N'') = M(1,2) \oplus S(3) \oplus S(4)$. But then $r^\circ_{12}(N'') = 0$. Also, if $N' = N'' \oplus N'''$, then $\pi(N''') = M(2,4)$ and also $r^\circ_{12}(N''') = 0$, thus $r^\circ_{12}(N) = 0$.

On the other hand, the results presented here may also be used in order to get a concise proof of $(*)$. Indeed, as we have seen, there exists a laminated module N with $\pi(N) = M(d)$ such that $r^\circ_{ij}(N) = \kappa_{j-i}(E(i,j)^d)$. Since $r^\circ_{ij}(M(\zeta d)) \geq r^\circ_{ij}(N)$, we obtain the inequality

$$r^\circ_{ij}(M^\circ(\zeta d)) \geq \kappa_{j-i}(E(i,j)^d).$$

The converse inequality follows directly from part of the work of Poljak [P], see [KZ], and actually this is the easy part of [P].

Acknowledgement. The author is grateful to Steffen König for drawing his attention to the paper of Knight and Zelevinsky. In a seminar at Bielefeld organized by him and Thomas Brüstle in 1996, the Knight-Zelevinsky paper, its background and applications have been discussed. The author is indebted to all the participants of the seminar, besides the organizers in particular to Robert Marsh, Sergej Ovsienko and Changchang Xi. A first version of the paper has been presented at the Geiranger conference, August 1996; the final one in a lecture at the Mathematical Institute, Chinese Academy of Science, Beijing, October 1999.

References

[DR] V. Dlab, C.M. Ringel: The preprojective algebra of a modulated graph. Springer LNM 832 (1980), 216–231.

[FF] L.R. Ford, D.R. Fulkerson: Flows in networks. Princeton University Press. Princeton NJ. (1962)

[L] G. Lusztig: Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc. 3 (1990), 447-498.

[KZ] H. Knight, A. Zelevinsky: Representations of quivers of type A and the multisegment duality. Advances Math. 117 (1996), 273-293.

[P] S. Poljak: Maximum rank of powers of matrix given by a pattern. Proc. Amer. Math. Soc. 106 (1989), 1137-1144.

[R] C.M. Ringel: The preprojective algebra of a quiver. Can. Math. Soc. Conference Proceedings 24. (1998), 467-480.

[Z] A. Zelevinsky: A p-adic analog of the Kazhdan-Lusztig conjecture. Funct. Anal. Appl. 15 (1981), 83-92.

Fakultät für Mathematik, Universität Bielefeld
POBox 100 131, D-33 501 Bielefeld, Germany
e-mail: ringel@mathematik.uni-bielefeld.de

[AMA - Algebra Montpellier Announcements - 01-1999] [November 1999]
Received October 21, 1999.