**Alexander polynomial and Koszul resolution**

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**Abstract:** We show that a certain version the knot invariant constructed by Gomez and Sierra from the quantum Heisenberg algebra and its infinite dimensional irreducible representation is the inverse of the Alexander polynomial.

## Introduction

Let $H$ be a ribbon quasitriangular Hopf algebra, with R-matrix $R$ and $\rho : H \to End(V)$ be a (finite dimensional) irreducible representation. Let $R_\rho$ be the composition of the image of the R-matrix in the representation with the flip operator: $R_\rho = P\rho \otimes \rho(R)$. Following Turaev ([T]), if there exists an invertible element $u$ in $H$ satisfying some conditions (i.e. $\rho(u)^{\otimes 2}$ commutes with $R_\rho$ and the partial traces $Tr_2(1 \otimes \rho(u) R_\rho) = xy$ , and $Tr_2(1 \otimes \rho(u) R_\rho^{-1}) = x^{-1}y$ are non zero scalars), one constructs an invariant of links as follows. Represent your link as the closure of a braid $\alpha \in B_n$, with $B_n$ the braid group on $n$ strands. The element $R_\rho$ provides, for each $n$ a representation $\pi_n$ of $B_n$ in the tensor power $V^{\otimes n}$, commuting with the action of $H$. Denoting by $w(\alpha)$ the writhe of $\alpha$, the expression

$$x^{-w(\alpha)} y^{-n} Tr(\rho(u)^{\otimes n} \pi_n(\alpha))$$

depends only on the closure of $\alpha$ and is an invariant. The value of the invariant on the unknot is $y^{-1} Tr(\rho(u))$.

An important remark is that in fact, the partial trace $Tr_{2,\ldots,n}(1 \otimes \rho(u)^{\otimes n-1} \pi_n(\alpha))$ is an intertwiner for the action of $H : V \to V$, and so is a scalar as the representation is assumed to be irreducible. Taking the full trace on the $n$ components just amounts to multiplying this scalar by $Tr(\rho(u))$, the so-called quantum dimension of $V$.

There are at least 2 situations in which one may want to use this remark to compute the invariant from a (rescaled form) of the partial trace:

(1) if the quantum dimension of $V$ is zero; in the case of a knot, this amounts to considering the 1-1 tangle associated with it;

(2) if $V$ is infinite dimensional; but one still has to give a meaning to the partial traces.

## I. The quantum Heisenberg algebra and the Gomez-Sierra invariant [G-S]

### I.1. Definitions:

a) The quantum Heisenberg algebra is the algebra generated by elements $a, a^*, E$ and $N$, with the following relations: $E$ is central, $[N,a] = -a$, $[N,a^*] = a^*$, $[a,a^*] = \frac{(q^E - q^{-E})}{(q-q^{-1})}$.

It has a Hopf algebra structure, with coproduct given by: $E$ and $N$ are primitive, $\Delta(a) = a \otimes q^E + 1 \otimes a$, $\Delta(a^*) = a^* \otimes 1 + q^{-E} \otimes a^*$, and the four generators are in the kernel of the augmentation.

b) As shown by Gomez and Sierra, it is a quasitriangular Hopf algebra, with special element $u = 1$ (as the square of the antipode is the identity), and universal R-matrix :

$$R = q^{-(E\otimes N + N\otimes E)} exp((q-q^{-1})a \otimes a^*).$$

c) The irreducible representations are infinite dimensional, and are parametrized by the eigenvalues of $E$ and $N$. The one we shall be interested in is the following: in a suitable orthonormal basis, ($|r\rangle$, $r$ a non negative integer), $a|r\rangle = \sqrt{r}|r-1\rangle$, $a^*|r\rangle = \sqrt{r+1}|r+1\rangle$, $E$ acts by the identity, $N|r\rangle = r|r\rangle$.
So the space $V$ of the representation is naturally graded by the the nonnegative integers, this gradation being induced by the action of $N$, and as a graded vector space, is naturally isomorphic to the algebra of polynomials in one variable (or, if one prefers, to the symmetric algebra on a one-dimensional vector space).

### I.2. The invariant:

Assuming that, for $\alpha \in B_n$, the partial traces $Tr_{2,\dots,n}(\pi_n(\alpha))$ make sense in some way as a power series in $q$, Gomez and Sierra define $T(\alpha) = q^{n-1-w(\alpha)} Tr_{2,\dots,n}(\pi_n(\alpha))$, which depends only on the closure of $\alpha$ and is formally an invariant. As they show, the partial trace could also be taken with respect to the first $n-1$ factors.
By very clever computations, they check on several examples that this indeed makes sense, and that $T(\alpha)$ is equal to the inverse of the Alexander polynomial. Moreover, they conjecture that this is true in general.

Note that, besides the fact that $V$ is infinite dimensional, there is a good reason why the full trace will never make sense: $V^{\otimes n}$ contains an infinite dimensional subspace fixed by all elements of $B_n$!
Denote by $\Delta^{(m)}$ the iterated comultiplication (with $\Delta^{(2)} = \Delta$). Then, for all $k$, $\Delta^{(n)}((a^*)^k)$ commutes with all $\pi_n(\alpha)$, and as $|0\rangle^{\otimes n}$ is fixed by $\pi_n(\alpha)$, the linear span of $\Delta^{(n)}((a^*)^k)(|0\rangle^{\otimes n})$, $k \in \mathbf{N}$ is fixed.
This fact will be completely clarified when we make the connection with the Burau representation.

**I.3. Connection with the Burau representation:**

For our purposes, it will be convenient to define the Burau representation as follows:
Let $X$ an $n$-dimensional vector space, with basis $(x_1, \dots, x_n)$. The Burau representation of $B_n$ in $X$ is given by the action of its standard generators $\sigma_1, \dots, \sigma_{n-1}$:
for $i = 1, \dots, n-1$, $\sigma_i(x_j) = x_j$, if $j \neq i, i+1$ , $\sigma_i(x_i) = (1-q^{-2})x_i + q^{-1}x_{i+1}$, $\sigma_i(x_{i+1}) = q^{-1}x_i$.
Then, by functoriality, $B_n$ acts on all symmetric or exterior powers of $X$, in particular it acts on the symmetric algebra $S^{\cdot}(X)$ of $X$. If $v$ is an endomorphism of $X$, we shall denote by $S(v)$ the corresponding endomorphism of $S^{\cdot}(X)$.

**Proposition:** *Under the isomorphism of $V^{\otimes n}$ with $S^{\cdot}(X)$ sending $(a^*)^{k_1}|0\rangle \otimes \dots \otimes (a^*)^{k_n}|0\rangle$ to $x_1^{k_1} \dots x_n^{k_n}$, the representation $\pi_n$ of $B_n$ is nothing but the representation obtained by functoriality from the Burau representation.*

**Proof:** It is enough to check on the generators, and this brings back to the case $n = 2$. Denote $\pi_2(\sigma)$ as $\sigma$. Now,

$$\sigma((a^*)^k|0\rangle \otimes (a^*)^l|0\rangle) = \sigma((a^*)^k \otimes (a^*)^l)|0\rangle \otimes |0\rangle = \sigma((a^* \otimes 1))^k(1 \otimes a^*)^l)|0\rangle \otimes |0\rangle$$

$$= (\sigma(a^* \otimes 1)\sigma^{-1})^k(\sigma(1 \otimes a^*)\sigma^{-1})^l)|0\rangle \otimes |0\rangle.$$

And it is now an easy computation to see that, in the representation, the action by conjugation by $\sigma$ leaves the span of the images of $(a^* \otimes 1)$and $(1 \otimes a^*)$ invariant and acts in it via the $2 \times 2$ Burau matrix.

Note that the vector $y = x_1 + q^{-1}x_2 + \dots + q^{-(n-1)}x_n$ of $X$ is fixed by $B_n$. One introduces the reduced Burau representation $\psi$ as the representation obtained on the quotient $Y = X/ < y >$. It might be sometimes convenient to lift it to a subspace of $X$ supplement to the line generated by $y$. We choose $X' = span(x_1, \dots, x_{n-1})$.
Observe that now $V^{\otimes n}$ becomes isomorphic to $S^{\cdot}(X') \otimes S^{\cdot}(y)$, and we have another way to take a partial trace: take the trace with respect to the $S^{\cdot}(X')$ factor.

**Theorem:** *Let $\alpha \in B_n$ such that the Alexander polynomial of its closure is not zero. Then:*
*1) the partial trace of $\pi_n(\alpha)$ with respect to the $S^{\cdot}(X')$ factor makes sense, is a scalar operator and this scalar is equal to the trace of $\alpha$ acting in the symmetric algebra $S^{\cdot}(Y)$ on the reduced Burau representation;*
*2) the quantity $q^{n-1-w(\alpha)}Tr_{S^{\cdot}(Y)}(S\psi(\alpha))$ depends only on the closure of $\alpha$ and is the inverse of the Alexander polynomial of this closure.*

The Theorem will follow from the facts explained in the next parts of the paper.

**II. Recollections on the Alexander polynomial**

Although not strictly necessary, it is interesting to recall the interpretation of the Alexander polynomial in terms of the quantized enveloping algebra of $gl(1|1)$ as advocated by Kauffman and Saleur ([K-S]). It is a fermionic construction which parallels the bosonic one of Gomez and Sierra.

**II.1. Definitions:**

a) The quantized enveloping algebra of $gl(1|1)$ is the algebra generated by $\eta, \eta^*, F, N, \varepsilon$, with the following relations: $F$ is central, $[N, \eta] = -\eta$, $[N, \eta^*] = \eta^*$, $\eta^2 = 0$, $(\eta^*)^2 = 0$, $\varepsilon^2 = 1$, $\varepsilon$ commutes with $F$ and $N$, $\varepsilon\eta\varepsilon = -\eta$, $\varepsilon\eta^*\varepsilon = -\eta^*$, $\eta\eta^* + \eta^*\eta = \frac{(q^F - q^{-F})}{(q-q^{-1})}$.

It has a Hopf algebra structure, with coproduct given by: $F$ and $N$ are primitive, $\varepsilon$ is grouplike, $\Delta\eta = \eta \otimes 1 + \varepsilon q^F \otimes \eta$, $\Delta\eta^* = \eta^* \otimes q^{-F} + \varepsilon \otimes \eta^*$; in the augmentation, $\varepsilon$ goes to 1 and the other generators are in the kernel.

b) It has a quasitriangular structure, with universal R-matrix given by

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes \varepsilon + \varepsilon \otimes 1 - \varepsilon \otimes \varepsilon)(1 + (q - q^{-1})q^F \eta^* \otimes \varepsilon q^{-F} \eta) q^{-(F \otimes N + N \otimes F)},$$

and special element $\varepsilon$.

c) It has an irreducible 2 dimensional representation $W$, with basis $(|0\rangle, |1\rangle)$, on which $F$ acts as the identity, $N$ is diagonal in this basis, with eigenvalues 0 and 1, $\varepsilon$ is diagonal with eigenvalues 1 and -1, $\eta|0\rangle = 0, \eta|1\rangle = |0\rangle$, and $\eta^*$ is the transpose of $\eta$.

We denote by $\pi'_n$ the ensuing representation of $B_n$ in $W^{\otimes n}$.

**II.2. The Alexander polynomial:**

a) As the quantum dimension of the representation space is 0, Kauffman and Saleur consider, for $\alpha \in B_n$, the partial trace $Tr_{2,\ldots,n}(1 \otimes (\varepsilon)^{\otimes n-1} \pi'_n(\alpha))$ and show that a suitable normalization of it is the Alexander polynomial of the closure of $\alpha$.

b) Here, the connection with the (reduced) Burau representation is familiar:

As a *mod* 2 graded vector space, $W^{\otimes n}$ is isomorphic to the exterior algebra $\Lambda^{\cdot}(X)$ and the representation of $B_n$ on this exterior algebra, obtained by functoriality from the Burau representation is equivalent to the representation $\pi'_n$ .

Writing $\Lambda^{\cdot}(X)$ as $\Lambda^{\cdot}(X') \otimes \Lambda(y)$, taking the partial graded trace of $\pi'_n$ with respect to the first factor leads to scalar which is equal to the graded trace in the exterior algebra $\Lambda^{\cdot}(Y)$ on the reduced Burau representation, which in turn is nothing but $det(1 - \psi(\alpha))$.

c) In fact, all we need to know is that one computes the Alexander polynomial of the closure of $\alpha$ via the reduced Burau representation as: $q^{-n+1+w(\alpha)} det(1 - \psi(\alpha))$.

**III. Koszul resolution and identification of the invariant**

Recall that the symmetric and the exterior algebras are known to be Koszul algebras, dual to each other. This means that their graded tensor product, with the Koszul differential, is a resolution of the ground field: this complex is the direct sum, for each homogenous component of total degree $p$, of finite length subcomplexes: $\oplus_{i+j=p} S^j \otimes \Lambda^i$, and for $p > 0$ the subcomplex is acyclic.

The identification of the (modified) Gomez-Sierra invariant with the Alexander polynomial is now a consequence of the specialization at $t = 1$, of the following general fact: for any endomorphism $v$ of $Y$, one has an equality of formal power series in $t$:

$$(\Sigma t^p Tr S^p(v))(\Sigma(-t)^k Tr \Lambda^k(v)) = 1,$$

as a result of adding the identities: $t^p\Sigma_{k=0}^{k=p}(-1)^k TrS^{p-k}(v)Tr\Lambda^k(v) = 0$ coming from the Lefschetz principle applied to each of the short exacts sequences making the Koszul resolution.

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