COUNTING EQUIVALENCE CLASSES OF IRREDUCIBLE REPRESENTATIONS

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Abstract. Let n be a positive integer, and let R be a (possibly infinite dimensional) finitely presented algebra over a computable field of characteristic zero. We describe an algorithm for deciding (in principle) whether R has at most finitely many equivalence classes of n-dimensional irreducible representations. When R does have only finitely many such equivalence classes, they can be effectively counted (assuming that $k[x]$ posesses a factoring algorithm).

1. Introduction

Let n be a positive integer, fixed throughout. In [5] we observed that the existence of n-dimensional irreducible representations of finitely presented noncommutative algebras can be algorithmically decided. In this note we outline a procedure for effectively "counting" the number of such irreducible representations, up to equivalence, in characteristic zero. Our approach combines standard computational commutative algebra with results from [1] and [9].

1.1. Assume that k is a computable field of characteristic zero, and that $\overline{k}$ is the algebraic closure of k.

Henceforth, let

$$R = k\{X_1, \dots , X_s\}/\langle f_1, \dots , f_t\rangle,$$

for some fixed choice of $f_1, \dots , f_t$ in the free associative k-algebra $k\{X_1, \dots , X_s\}$. In a slight abuse of notation, "X_ℓ" will also denote its image in R, for $1 \leq \ell \leq s$.

By an *n-dimensional representation of R* we will always mean a unital k-algebra homomorphism from R into the k-algebra $M_n(\overline{k})$ of $n\times n$ matrices over $\overline{k}$. Representations $\rho, \rho' \colon R \to M_n(\overline{k})$ are *equivalent* if there exists a matrix $Q \in GL_n(\overline{k})$ such that

$$\rho'(X) = Q\rho(X)Q^{-1},$$

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for all $X \in R$.

We will say that the representation $\rho\colon R \to M_n(\overline{k})$ is *irreducible* when $\overline{k}\rho(R) = M_n(\overline{k})$ (cf. [1, §9]). Observe that ρ is irreducible if and only if $\rho \otimes 1\colon R \otimes_k \overline{k} \to M_n(\overline{k})$ is surjective, if and only if $\rho \otimes 1$ is irreducible in the more common use of the term. (In particular, our approach below will use calculations over the computable field k to study representations over the algebraically closed field $\overline{k}$.)

1.2. The existence of an n-dimensional representation of R depends only on the consistency of a system of algebraic equations, over k, in $(t.n^2)$-many variables. Consequently, the existence of n-dimensional representations of R is decidable (in principle) using Groebner basis methods. This idea is extended in [5] to give a procedure for deciding the existence of n-dimensional irreducible representations. On the other hand, posessing a nonzero finite dimensional representation is a Markov property, and so the existence – in general – of a finite dimensional representation of R cannot be effectively decided, by [3].

We now state our main result; the proof will be presented in §2.

Theorem. *Having at most most finitely many equivalence classes of irreducible n-dimensional representations is an algorithmically decidable property of R.*

1.3. Assume that $k[x]$ is equipped with a factoring algorithm. If it has been determined that R has at most finitely many equivalence classes of n-dimensional irreducible representations, these equivalence classes can (in principle) be effectively counted; see (2.9).

2. Proof of Theorem

2.1. (i) Set

$$B = k[x_{ij}(\ell) : 1 \leq i, j \leq n, 1 \leq \ell \leq s].$$

For $1 \leq \ell \leq s$, let $\mathbf{x}_\ell$ denote the $n{\times}n$ generic matrix $(x_{ij}(\ell))$, in $M_n(B)$. For $g \in k\{X_1, \dots, X_s\}$, let $g(\mathbf{x})$ denote the image of g, in $M_n(B)$, under the canonical map

$$k\{X_1, \dots, X_s\} \xrightarrow{X_\ell \longmapsto \mathbf{x}_\ell} M_n(B).$$

Identify B with the center of $M_n(B)$.

(ii) Let $\operatorname{Rel}(M_n(B))$ be the ideal of $M_n(B)$ generated by $f_1(\mathbf{x}), \dots, f_t(\mathbf{x})$.

(iii) Let $\operatorname{Rel}(B)$ denote the ideal of B generated by the entries of the matrices $f_1(\mathbf{x})$, $\dots$, $f_t(\mathbf{x}) \in M_n(B)$. Note that

$$\operatorname{Rel}(B) = \operatorname{Rel}(M_n(B)) \cap B.$$

(iv) Let

$$A = k\{\mathbf{x}_1, \dots, \mathbf{x}_s\},$$

the k-subalgebra of $M_n(B)$ generated by the generic matrices $\mathbf{x}_1, \dots, \mathbf{x}_s$. Set

$$\operatorname{Rel}(A) \;=\; \operatorname{Rel}(M_n(B)) \cap A.$$

2.2. Every n-dimensional representation of R can be written in the form

$$R \xrightarrow{X_\ell \longmapsto \mathbf{x}_\ell + \mathrm{Rel}(A)} \left(\frac{A}{\mathrm{Rel}(A)}\right) \xrightarrow{\text{inclusion}} \left(\frac{M_n(B)}{\mathrm{Rel}(M_n(B))}\right) \longrightarrow M_n(\overline{k}),$$

and every k-algebra homomorphism

$$M_n(B)/\operatorname{Rel}(M_n(B)) \;\to\; M_n(\overline{k})$$

is completely determined by the induced map

$$B/\operatorname{Rel}(B) \to \overline{k}.$$

For each representation $\rho\colon R \to M_n(\overline{k})$, let $\chi_\rho\colon B \to \overline{k}$ be the homomorphism (with $\mathrm{Rel}(B) \subseteq \ker \chi_\rho$) given by this correspondence.

2.3. Let T be the k-subalgebra of B generated by the coefficients of the characteristic polynomials of elements in A. (Since the characteristic of k is zero, T is in fact generated by the traces, as $n{\times}n$ matrices, of the elements in A.) Set

$$\mathrm{Rel}(T) = \mathrm{Rel}(B) \cap T.$$

Note, when $\rho, \rho'\colon R \to M_n(\overline{k})$ are equivalent representations, that the restrictions of χ_ρ and $\chi_{\rho'}$ to T will coincide.

2.4. Let $\mathrm{simple}_n(R)$ denote the set of equivalence classes of irreducible n-dimensional representations of R. By (2.3) there is a well-defined function

$$\Phi : \mathrm{simple}_n(R) \;\longrightarrow\; V(\mathrm{Rel}(T)),$$

where $V(\mathrm{Rel}(T))$ denotes the $\overline{k}$-affine algebraic set of points on which the polynomials in $\mathrm{Rel}(T)$ vanish. It follows from [1, pp. 558–559] that Φ is injective.

2.5. (i) Recall the *mth standard identity*

$$s_m \;=\; \sum_{\sigma \in S_m} (\operatorname{sgn}\sigma) Y_{\sigma(1)} \cdots Y_{\sigma(m)} \;\in\; \mathbb{Z}\{Y_1, \dots, Y_m\}.$$

If Λ is a commutative ring, then the Amitsur-Levitzky Theorem ensures that $M_n(\Lambda)$ satisfies s_m if and only if $m \geq 2n$; see, for example, [6, 13.3.2, 13.3.3].

(ii) Let S denote the finite subset of T ($\subseteq B$) comprised of

$$\operatorname{trace}\left(M_0 \cdot s_{2(n-1)}(M_1, \dots, M_{2(n-1)})\right),$$

for all monic monomials $M_0, \dots, M_{2(n-1)}$, in the generic matrices $\mathbf{x}_1, \dots, \mathbf{x}_s$, of length less than

$$p = n\sqrt{2n^2/(n-1) + 1/4} + n/2 - 2.$$

(The choice of p will follow from [7]; see [5, 2.2].) Let $\rho\colon R \to M_n(\overline{k})$ be a representation. It now follows from [5, §2] that ρ is irreducible if and only if

$$S \;\not\subseteq\; \ker \chi_\rho.$$

(Other sets of polynomials can be substituted for S; see [5, 2.6vi,vii].)

2.6. (i) Set

$$W = V(\mathrm{Rel}(T)) \setminus V(S).$$

Combining (2.4) with (2.5ii), we obtain a bijection

$$\Phi : \mathrm{simple}_n(R) \longrightarrow W.$$

(ii) Set

$$J = \mathrm{ann}_B \left(\frac{\mathrm{Rel}(B) + B.S}{\mathrm{Rel}(B)} \right), \quad \text{and} \quad I = J \cap T = \mathrm{ann}_T \left(\frac{\mathrm{Rel}(T) + T.S}{\mathrm{Rel}(T)} \right).$$

A finite generating set for J can be specified, using standard methods, and we can identify T/I with its image in B/J. Since $V(I)$ is the Zariski closure of W, to prove the theorem it suffices to find an effective procedure for determining whether or not T/I is finite dimensional. (When not indicated otherwise, "dimension" refers to "dimension as a k-vector space.")

2.7. (i) For the generic matrices $\mathbf{x}_1, \dots, \mathbf{x}_s$, set Trace =

$$\left\{ \mathrm{trace}(\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_u) : \mathbf{y}_1, \dots, \mathbf{y}_u \in \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \text{ and } 1 \leq u \leq n^2 \right\}.$$

In [9] (cf. [4, p. 54]) it is shown that $T = k[\mathrm{Trace}]$. (A larger finite generating set for T was established in [8].)

(ii) By (2.6ii), to prove the theorem it remains to find an algorithm for deciding whether the monomials in Trace ($\subseteq B$) are algebraic over k, modulo J. We accomplish this task using a variant of the subring membership test (cf., e.g., [2, p. 270]): Let C be a commutative polynomial ring, over k, in m variables. Let L be an ideal – equipped with an explicitly given list of generators – in C. Choose $f \in C$. Observe that f is algebraic over k, modulo L, if and only if $L \cap k[f] \neq \{0\}$. Next, embed C, in the obvious way, as a subalgebra of the polynomial ring $C' = k(t) \otimes_k C$. Observe that $L \cap k[f] \neq \{0\}$ if and only if 1 is contained in the ideal $(t - f).C' + L.C'$ of C'. Hence, the decidability of ideal membership in C' implies the decidability of algebraicity modulo L in C.

The proof of the theorem follows.

2.8. Roughly speaking, the complexity of the procedure described in (2.1 – 2.7) varies according to the degrees of the polynomials involved in deciding the algebraicity of Trace modulo J. Note, for example, that the degrees of the members of S can be as large as p^{2n-1}, for p as in (2.5ii).

2.9. Assume that it has already been determined that the number (equal to $|W|$) of equivalence classes of irreducible n-dimensional representations of R is finite. Further assume that $k[x]$ is equipped with a factoring algorithm. We conclude our study by sketching a procedure for calculating – in principal – this number.

Set $D = T/I$, and identify D with the (finite dimensional) k-subalgebra of B/J generated by the image of Trace. Since B/J can be given a specific finite presentation, finding a k-basis E for D amounts to solving systems of polynomial equations in B, and this task can be accomplished employing elimination methods. Next, using the regular representation of D, and the finite presentation of B/J, we can algorithmically specify E as a set of commuting $m\times m$ matrices over k, for some m. Furthermore, the nilradical $N(D)$ will be precisely the set of elements of D whose traces, as $m\times m$ matrices, are zero. Consequently, we can effectively compute the dimension of $D/N(D)$. This dimension is equal to $|W|$.

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