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ON THE DIRICHLET KERNELS WITH RESPECT TO THE WALSH-KACZMARZ SYSTEM

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Abstract. In this paper we give a form of the Dirichlet kernels D_n^κ with respect to the Walsh-Kaczmarz system. Define the operator of Sunouchi $U^\kappa f := (\sum_{n=1}^{\infty} \frac{|S_n^\kappa f - \sigma_n^\kappa f|^2}{n})^{\frac{1}{2}}$ $(f \in L^1)$ with respect to the Walsh-Kaczmarz system. We prove that the operator U^κ is not of type (H^1, L^1).

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, respectively, that is $\mathbf{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbf{Z}_2$. The elements of G are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group .

A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

$(x \in G, n \in \mathbf{P})$. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the nullelement of G, $I_n := I_n(0) \quad (n \in \mathbf{N})$. Furthermore, let $L^p(G)$ $(1 \leq p \leq \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in G)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbf{N})(f \in L^1(G))$.

Define the Hardy space H^1 as follows. Let $f^* := \sup_{n \in \mathbf{N}} |E_n f|$ be the maximal function of $f \in L^1(G)$. Set $H^1 := \{f \in L^1(G) : f^* \in L^1(G)\}$, H^1 is a Banach space with the norm $\|f\|_{H^1} := \|f^*\|_1$. A good property of the space $H^1(G)$ is the atomic structure ([SWS]). A function $a \in L^\infty(G)$ is called an atom if either $a = 1$ or a satisfies the following three properties:

(i.) *supp* $a \subseteq I_a$,
(ii.) $\|a\|_\infty \leq 1/\mu(I_a)$ and

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(iii.) $\int_{I_a} a = 0$, for some interval I_a.

A function f belongs to the Hardy space H if there exist $\lambda_i \in \mathbf{C}$ and a_i atoms ($i \in \mathbf{N}$) that $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ and $f = \sum_{i=0}^{\infty} \lambda_i a_i$. H is a Banach space with the norm $\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|$, where the infimum is taken over all decompositions of $f = \sum_{i=0}^{\infty} \lambda_i a_i$. Moreover $H^1 = H(G)$ and $\|f\|_{H^1} \sim \|f\|_H$.

For $k \in \mathbf{N}$ and $x \in G$ denote r_k the k-th Rademacher function:

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let $n \in \mathbf{N}$. Then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0,1\}$ ($i \in \mathbf{N}$), i.e. n is expressed in the number system based 2. Denote by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{N}).$$

The Walsh-Paley system can be given in the Kaczmarz enumeration as follows:

$$\begin{aligned} \kappa_n(x) &:= r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} \\ &= r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}, \end{aligned}$$

for $x \in G,\ n \in \mathbf{P}$), $\kappa_0(x) = 1 \ (x \in G)$. Let the transformation $\tau_A : G \to G$ be defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...) \in G, \quad (A \in \mathbf{N}).$$

τ_A is measure-preserving and such that $\tau_A(\tau_A(x)) = x$. We have

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

For a function f in $L^1(G)$ the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, the Fejér kernels and the maximal functions of Fejér means are defined as follows.

$$\hat{f}^\alpha(n) := \int_G f\overline{\alpha_n},\ S_n^\alpha f := \sum_{k=0}^{n-1} \hat{f}^\alpha(k)\alpha_k,\ D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

$$\sigma_n^\alpha f := \frac{1}{n}\sum_{k=1}^{n} S_k^\alpha f,\ K_n^\alpha := \frac{1}{n}\sum_{k=1}^{n} D_k^\alpha,\ \sigma^{\alpha *} f := \sup_{n \in \mathbf{P}} |\sigma_n^\alpha f|$$

and $D_0^\alpha = K_0^\alpha := 0$, where α is either ω or κ.

The behavior of the Fourier series with respect to the Walsh-Kaczmarz system was studied by a lot of authors. Skorcov proved that the Fejer means converges uniformly to the function f for continuous functions f [SK1], G. Gát proved for the integrable functions that the Fejer means converges almost everywhere to the function [Gát] and he also proved that the maximal operator $\sigma^{\kappa *}$ is of type (p,p) for all $1 < p \le \infty$, of weak type $(1,1)$ and of type (H^1, L^1) [GN]. Wo-Sang Young and F. Schipp proved that the Walsh-Kaczmarz system is a convergence system [Y], [Sch].

The Dirichlet kernels D_n^κ has of the following form stated in Theorem 1.

Theorem 1: If $n \in \mathbf{P}$, then

$$D_n^\kappa(x) = n_{|n|} D^\omega_{2^{|n|}}(x) + r_{|n|}(x) \sum_{j=1}^{|n|} n_{|n|-j} \prod_{l=0}^{j-1} r_{|n|-l}^{n_{|n|-l}}(\tau_{|n|}(x)) D^\omega_{2^{|n|-j}}(\tau_{|n|}(x)).$$

Proof of Theorem 1: n can be written in the form $n = \sum_{j=0}^{|n|} n_j 2^j$

$$\begin{aligned} D_n^\kappa(x) &= D^\kappa_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{2^{|n|}+n_{|n|-1}2^{|n|-1}-1} \kappa_k(x) + \sum_{k=2^{|n|}+n_{|n|-1}2^{|n|-1}}^{2^{|n|}+n_{|n|-1}2^{|n|-1}+n_{|n|-2}2^{|n|-2}-1} \kappa_k(x) + ... \\ &= n_{|n|} D^\omega_{2^{|n|}}(x) + n_{|n|-1} r_{|n|}(x) \omega_{2^{|n|}}(\tau_{|n|}(x)) D^\omega_{2^{|n|-1}}(\tau_{|n|}(x)) \\ &+ n_{|n|-2} r_{|n|}(x) \omega_{2^{|n|}}(\tau_{|n|}(x)) \omega_{n_{|n|-1}2^{|n|-1}}(\tau_{|n|}(x)) D^\omega_{2^{|n|-2}}(\tau_{|n|}(x)) + ... \end{aligned}$$

Thus the n-th Dirichlet kernels with respect to Walsh-Kaczmarz system can be written in the form

$$D_n^\kappa(x) = n_{|n|} D^\omega_{2^{|n|}}(x) + r_{|n|}(x) \sum_{j=1}^{|n|} n_{|n|-j} \prod_{l=0}^{j-1} r_{|n|-l}^{n_{|n|-l}}(\tau_{|n|}(x)) D^\omega_{2^{|n|-j}}(\tau_{|n|}(x)). \quad \square$$

The sum $2^{-n} \sum_{k=0}^{2^n-1} \|D_k^\omega\|_1$ was studied by N. J. Fine [F], he has shown that this is greater than cn with a constant $c > 0$ $(n \in \mathbf{N})$.

Corollary 2: If $n \in \mathbf{N}$, then

$$2^{-n} \sum_{k=1}^{2^n} \|D_k^\kappa\|_1 \ge cn,$$

with some absolute constant $c > 0$.

Proof of Corollary 2: Let $J^t_{|n|} := \{x \in G : x_{|n|-t-1} = 1, x_{|n|-t} = 0, ..., x_{|n|-1} = 0\}$ for $|n| \ge t$. For a fix n the sets $J^t_{|n|}$ $(|n| \ge t)$ are disjoint. For all $x \in J^t_{|n|}$ we have the equality $|D_n^\kappa(x)| = |\sum_{j=1}^{|n|} n_{|n|-j} \prod_{l=0}^{j-1} r_{|n|-l}^{n_{|n|-l}}(\tau_{|n|}(x)) D^\omega_{2^{|n|-j}}(\tau_{|n|}(x))|$ $(|n| \ge t)$. Using this we may write

$$\begin{aligned} 2^{-N} \sum_{k=1}^{2^N} \|D_k^\kappa\|_1 &\ge 2^{-N} \sum_{\substack{n=1 \\ n_t=1, n_{t-1}=0}}^{2^N-1} \sum_{t=1}^{|n|} \int_{J^t_{|n|}} |D_n^\kappa(x)| dx \\ &\ge 2^{-N} \sum_{t=1}^{N-1} \sum_{n=2^{t+1}}^{2^N-1} \int_{J^t_{|n|}} |D_n^\kappa(x)| dx \\ &\ge 2^{-N} \sum_{t=1}^{N-1} \sum_{\substack{n_0,\dots,n_{t-2} \\ n_{t-1}=0, n_t=1 \\ n_{t+1},\dots,n_{N-1}}} \frac{1}{4} \ge c(N-1). \end{aligned}$$

This completes the proof of Corollary 2. □

We say that the operator $T : L^1(G) \to L^0(G)$ is of type (p,p) if

$$\|Tf\|_p \le c_p \|f\|_p$$

for all $f \in L^p(G)$ $(1 \le p < \infty)$ for some constant c_p (c_p depend only on p). The operator T is said to be of type (H^1, L^1) if there exists a constant c such that

$$\|Tf\|_1 \le c\|f\|_{H^1}$$

for all $f \in H^1(G)$. Introduce the Sunouchi operator for any function f in $L^1(G)$ by

$$U^\alpha f(x) := \left(\sum_{n=1}^{\infty} \frac{|S_n^\alpha f(x) - \sigma_n^\alpha f(x)|^2}{n} \right)^{\frac{1}{2}} \quad (x \in G)$$

where α is either ω or κ. G. I. Sunouchi has proved [Su1] that U^ω is of type (p,p) for all $1 < p < \infty$ and is not of type $(1,1)$. P. Simon proved [S] that the Sunouchi operator U^ω is not of type (H^1, L^1).

Corollary 3: The operator U^κ is not of type (H^1, L^1).

Proof of Corollary 3: In order to prove Corollary 3, let

$$F_n := D_{2^{n+1}} - D_{2^n},$$

then

$$S_N^\kappa F_n - \sigma_N^\kappa F_n = \begin{cases} \sum_{k=2^n}^{2^{n+1}-1} \frac{k}{N} \kappa_k & |N| > n \\ \sum_{k=2^n}^{N} \frac{k}{N} \kappa_k & |N| = n \\ 0 & |N| < n. \end{cases}$$

This implies that

$$U^\kappa F_n = \left(\sum_{l=0}^{\infty} \sum_{k=0}^{2^l-1} \frac{|S_{2^l+k}^\kappa F_n - \sigma_{2^l+k}^\kappa F_n|^2}{2^l+k} \right)^{\frac{1}{2}} \ge \left(\sum_{k=0}^{2^n-1} \frac{1}{(2^n+k)^3} |\sum_{j=2^n}^{2^n+k} j\kappa_j|^2 \right)^{\frac{1}{2}}$$

$$\ge c2^{-n} \sum_{k=1}^{2^n} \left| D_k^\kappa + \frac{k}{2^n}(D_k^\kappa - K_k^\kappa) \right| \ge c2^{-n} \left(\sum_{k=1}^{2^n} |D_k^\kappa| - \sum_{k=1}^{2^n} \frac{k}{2^n} |K_k^\kappa| \right).$$

Thus,

$$\|U^\kappa F_n\|_1 \ge c \left(2^{-n} \sum_{k=1}^{2^n} \|D_k^\kappa\|_1 - 2^{-2n} \sum_{k=1}^{2^n} k\|K_k^\kappa\|_1 \right)$$

holds. (For more details see P. Simon's method [S].) The fact that $\|K_k^\kappa\|_1 \le C$ $(k = 1, 2, ...)$ (see [Gát]) and Corollary 2 give Corollary 3. □

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