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# SEMINORM GENERATING RELATIONS AND THEIR MINKOWSKI FUNCTIONALS

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Abstract. We show that instead of the Minkowski functionals of absorbing, balanced, convex subsets of a vector space $X$ it is more convenient to consider first the Minkowski functionals of balanced valued linear relations of $\mathbb{R}_+$ onto $X$.

## Introduction

A relation $F$ of the set $\mathbb{R}_+$ of all positive numbers onto a vector space $X$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ will be called a seminorm generating relation for $X$ if

(1) $F(r) + F(s) \subset F(r+s)$ for all $r, s \in \mathbb{R}_+$;

(2) $\lambda F(r) \subset F(tr)$ for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$.

This definition is mainly motivated by the fact that if $A$ is an absorbing, balanced, convex subset of $X$ and $F_A$ is a relation on $\mathbb{R}_+$ to $X$ such that

$$F_A(r) = r\,A$$

for all $r \in \mathbb{R}_+$, then $F_A$ is a seminorm generating relation for $X$.

Moreover, if $p$ is a seminorm on $X$ and $F_p$ and $\bar{F}_p$ are relations on $\mathbb{R}_+$ to $X$ such that

$$F_p(r) = B_r^p(0) \qquad \text{and} \qquad \bar{F}_p(r) = \bar{B}_r^p(0)$$

for all $r \in \mathbb{R}_+$, then $F_p$ and $\bar{F}_p$ are also seminorm generating relations for $X$.

If $F$ is a seminorm generating relation for $X$, then the function $p_F$ defined by

$$p_F(x) = \inf\big(F^{-1}(x)\big)$$

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for all $x \in X$, will be called the Minkowski functional of $F$. Namely, if $A$ is an absorbing, balanced, convex subset of $X$, then $p_A = p_{F_A}$ is just the usual Minkowski functional of $A$.

After establishing some easy consequences of the definition of seminorm generating relations, we shall only prove the following basic algebraic properties of the Minkowski functionals.

**Theorem 1.** *If $F$ is a seminorm generating relation for $X$, then $p_F$ is a seminorm on $X$ such that $F_{p_F} \subset F \subset \bar{F}_{p_F}$.*

**Corollary 1.** *If $A$ is an absorbing, balanced, convex subset of $X$, then $p_A$ is a seminorm on $X$ such that $B_1^{p_A}(0) \subset A \subset \bar{B}_1^{p_A}(0)$.*

**Theorem 2.** *If $p$ is a seminorm on $X$ and $F$ is a seminorm generating relation for $X$ such that $F_p \subset F \subset \bar{F}_p$, then $p = p_F$.*

**Corollary 2.** *If $p$ is a seminorm on $X$ and $A$ is an absorbing, balanced, convex subset of $X$ such that $B_1^p(0) \subset A \subset \bar{B}_1^p(0)$, then $p = p_A$.*

**Theorem 3.** *If $F$ is a seminorm generating relation for $X$, then $p_F$ is a norm if and only if $\bigcap_{r \in \mathbb{R}_+} F(r) = \{0\}$.*

**Corollary 3.** *If $A$ is an absorbing, balanced, convex subset of $X$, then $p_A$ is a norm on $X$ if and only if $\bigcap_{r \in \mathbb{R}_+} rA = \{0\}$.*

**Theorem 4.** *If $F$ is a seminorm generating relation for $X$, then $F = F_{p_F}$ if and only if $F(r) = \bigcup_{s<r} F(s)$ for all $r \in \mathbb{R}_+$.*

**Corollary 4.** *If $A$ is an absorbing, balanced, convex subset of $X$, then $A = B_1^{p_A}(0)$ if and only if $A = \bigcup_{s<r} sA$.*

**Theorem 5.** *If $F$ is a seminorm generating relation for $X$, then $F = \bar{F}_{p_F}$ if and only if $F(r) = \bigcap_{s>r} F(s)$ for all $r \in \mathbb{R}_+$.*

**Corollary 5.** *If $A$ is an absorbing, balanced, convex subset of $X$, then $A = \bar{B}_1^{p_A}(0)$ if and only if $A = \bigcap_{s>r} sA$.*

The topological properties of seminorm generating relations and their Minkowski functionals will be investigated elsewhere.

## 1. Prerequisites

A subset $F$ of a product set $X \times Y$ is called a relation on $X$ to $Y$. If in particular $X = Y$, then we simply say that $F$ is a relation on $X$. Note that if $F$ is a relation on $X$ to $Y$, then $F$ is also a relation on $X \cup Y$.

If $F$ is a relation on $X$ to $Y$, and moreover $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in X : \ (x, y) \in F\}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of $x$ and $A$ under $F$, respectively.

If $F$ is a relation on $X$ to $Y$, then the sets $D_F = \{x \in X : \ F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of $F$, respectively. If in particular $X = D_F$ (and $Y = R_F$), then we say that $F$ is a relation of $X$ into (onto) $Y$.

A relation $F$ on $X$ to $Y$ is said to be a function if for each $x \in D_F$ there exists a unique $y \in Y$ such that $y \in F(x)$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If $F$ is a relation on $X$ to $Y$, then values $F(x)$, where $x \in X$, uniquely determine $F$ since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse relation $F^{-1}$ of $F$ can be defined such that $F^{-1}(x) = \{ y \in Y : \ x \in F(y) \}$ for all $x \in X$.

Throughout in the sequel, $X$ will denote a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. And for any $\lambda \in \mathbb{K}$ and $A, B \subset X$ we write $\lambda A = \{ \lambda x : \ x \in A \}$ and $A + B = \{ x + y : \ x \in A, \ y \in B \}$.

Note that thus two axioms of a vector space may fail to hold for the family $\mathcal{P}(X)$ of all subsets of $X$. Namely, only the one-point subsets of $X$ can have additive inverses. Moreover, in general, we only have $(\lambda + \mu) A \subset \lambda A + \mu A$.

If $A$ is a subset of $X$, then we say that:

(1) $A$ is absorbing if $X = \bigcup_{r \in \mathbb{R}_+} r A$;

(2) $A$ is balanced if $\lambda A \subset A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(3) $A$ is convex if $r A + (1 - r) A \subset A$ for all $r \in \mathbb{R}_+$ with $r < 1$.

A function $p$ of $X$ into $\mathbb{R}$ is called a seminorm on $X$ if

$$p(\lambda x) \leq |\lambda| p(x) \qquad \text{and} \qquad p(x + y) \leq p(x) + p(y)$$

for all $\lambda \in \mathbb{K}$ and $x, y \in X$. A seminorm $p$ is called a norm if $p(x) = 0$ implies $x = 0$.

If $p$ is a seminorm on $X$, then for each $r \in \mathbb{R}_+$ the relations $B_r^p$ and $\bar{B}_r^p$, defined by

$$B_r^p(x) = \big\{ y \in X : \ p(x - y) < r \big\} \quad \text{and} \quad \bar{B}_r^p(x) = \big\{ y \in X : \ p(x - y) \leq r \big\}$$

for all $x \in X$, are called the $r$-sized open and closed $p$–surroundings in $X$, respectively.

Concerning the above basic concepts we shall only need here the following simple theorems.

**Theorem 1.1.** *If $A \subset X$, then the following assertions hold:*

*(1) if $A$ is convex, then $(r + s) A = r A + s A$ for all $r, s \in R_+$;*

*(2) if $A$ is balanced, then $\lambda A \subset \mu A$ for all $\lambda, \mu \in \mathbb{K}$ with $|\lambda| \leq |\mu|$.*

**Remark 1.2.** Therefore, a balanced subset $A$ of $X$ is absorbing if and only $X = \bigcup_{n=1}^{\infty} n A$.

**Theorem 1.3.** *If* $p$ *is a seminorm on* $X$ *, then*

*(1)* $p(x) \geq 0$ *for all* $x \in X$ *;*

*(2)* $p(\lambda x) = |\lambda| \, p(x)$ *for all* $\lambda \in \mathbb{K}$ *and* $x \in X$ *.*

**Remark 1.4.** Therefore, our present definition of a seminorm coincides with the usual one.

**Theorem 1.5.** *If* $p$ *is a seminorm on* $X$ *and* $r \in \mathbb{R}_+$ *, then*

*(1)* $B_r^p(x) = x + B_r^p(0)$ *for all* $x \in X$ *;*

*(2)* $B_r^p(0)$ *is an absorbing, balanced and convex subset of* $X$ *such that* $B_r^p(0) = r \, B_1^p(0)$ *.*

**Remark 1.6.** Moreover, the same statements hold for the closed surroundings $\bar{B}_r^p$ .

## 2. Seminorm generating relations

**Definition 2.1.** A relation $F$ of $\mathbb{R}_+$ onto $X$ will be called a seminorm generating relation for $X$ if

(1) $F(r) + F(s) \subset F(r+s)$ for all $r, s \in \mathbb{R}_+$ ;

(2) $\lambda F(r) \subset F(tr)$ for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$.

The above definition is mainly motivated by the following simple

**Example 2.2.** If $A$ is an absorbing, balanced, convex subset of $X$ and $F_A$ is a relation on $\mathbb{R}_+$ to $X$ such that

$$F_A(r) = r\,A$$

for all $r \in \mathbb{R}_+$ , then $F_A$ is a seminorm generating relation for $X$ .

Since $A$ is absorbing, for each $x \in X$ there exists an $r \in \mathbb{R}_+$ such that $x \in r\,A$. Hence, it is clear that $A \neq \emptyset$, and thus $\mathbb{R}_+$ is the domain of $F_A$. Moreover, since $x \in F_A(r)$ , it is clear that $X$ is the range of $F_A$.

On the other hand, if $r, s \in \mathbb{R}_+$ , then by Theorem 1.1 (1) it is clear that

$$F_A(r+s) = (r+s)\,A = rA + sA = F_A(r) + F_A(s)\,.$$

Moreover, if $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ such that $|\lambda| \leq t$, then by Theorem 1.1 (2) it is clear that

$$\lambda F_A(r) = \lambda(r\,A) = r(\lambda A) \subset r(t\,A) = (tr)\,A = F_A(tr)\,.$$

Now, as an important particular case of Example 2.2, we can also state

**Example 2.3.** If $p$ is a seminorm on $X$ and $F_p$ and $\bar{F}_p$ are relations on $\mathbb{R}_+$ to $X$ such that

$$F_p(r) = B_r^p(0) \qquad \text{and} \qquad \bar{F}_p(r) = \bar{B}_r^p(0)$$

for all $r \in \mathbb{R}_+$, then $F_p$ and $\bar{F}_p$ are seminorm generating relations for $X$.

From Theorem 1.5 (2) we know that $A = B_1^p(0)$ is an absorbing, balanced, convex subset of $X$ such that

$$F_p(r) = B_r^p(0) = r\, B_1^p(0) = r\, A = F_A(r)$$

for all $r \in \mathbb{R}_+$. Therefore, $F_p = F_A$, and thus $F_p$ is a seminorm generating relation for $X$ by Example 2.2.

The fact that $\bar{F}_p$ is also a seminorm generating relation for $X$ can be proved quite similarly by using Remark 1.6 and Example 2.2.

In the sequel, beside Definition 2.1, we shall only need the following obvious

**Theorem 2.4.** *If $F$ is a seminorm generating relation for $X$, then*

*(1)* $0 \in F(r)$ *for all* $r \in \mathbb{R}_+$ *;*

*(2)* $r\, F(s) \subset F(r\, s)$ *for all* $r, s \in \mathbb{R}_+$ *;*

*(3)* $F(r) \subset F(s)$ *for all* $r, s \in \mathbb{R}_+$ *with* $r \le s$*.*

*Proof.* Since the assertions (1) and (2) are immediate from the homogenity property 2.1 (2) of $F$, we need only note that

$$F(r) = F(r) + \{0\} \subset F(r) + F(s - r) \subset F(s)$$

for all $r, s \in \mathbb{R}_+$ with $r < s$. Therefore, the assertion (3) also holds.

However, as a converse to Example 2.2, we can also easily prove the following

**Theorem 2.5.** *If $F$ is a seminorm generating relation for $X$, then there exists an absorbing, balanced, convex subset $A$ of $X$ such that $F = F_A$.*

*Proof.* If $r, s \in \mathbb{R}_+$, then by the homogenity property 2.4 (2) of $F$ we have

$$r\, F(s) \subset F(r\, s).$$

Hence, by writing $r^{-1}$ in place of $r$, and $r\, s$ in place of $s$, we can see that

$$r^{-1} F(r\, s) \subset F(s).$$

This implies that $F(r\, s) \subset r\, F(s)$. Therefore, the equality

$$F(r\, s) = r\, F(s)$$

is also true. Hence, under the notation $A = F(1)$, it follows that

$$F(r) = r\, F(1) = r\, A$$

for all $r \in \mathbb{R}_+$ .

Threfore, it remains only to prove that $A$ is an absorbing, balanced and convex subset of $X$ . For this, note that if $x \in X$ , then since $F$ is onto $X$ there exists an $r \in \mathbb{R}_+$ such that $x \in F(r) = r\,A$ . Therefore, $A$ is absorbing. Moreover, if $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$ , then from the homogenity property 2.1 (2) of $F$ we can at once see that

$$\lambda\,A = \lambda\,F(1) \subset F(1) = A\,.$$

Therefore, $A$ is balanced. Moreover, if $0 < t < 1$ , then by the homogenity and additivity properties of $F$ it is clear that

$$t\,A + (1-t)\,A = t\,F(1) + (1-t)\,F(1) = F(t) + F(1-t) \subset F(1) = A\,.$$

Therefore, $A$ is convex.

Now, in addition to Theorem 2.4, we can also easily state the following

**Theorem 2.6.** *If $F$ is a seminorm generating relation for $X$ , then*

*(1)* $F(r\,s) = r\,F(s)$ *for all* $r, s \in \mathbb{R}_+$ *;*

*(2)* $F(r+s) = F(r) + F(s)$ *for all* $r, s \in \mathbb{R}_+$ *;*

*(3)* $F(r)$ *is an absorbing, balanced, convex subset of* $X$ *for all* $r \in \mathbb{R}_+$ *.*

**Remark 2.7.** Note that if $F$ is a balanced valued homogeneous relation of $\mathbb{R}_+$ into $X$ , then

$$\lambda\,F(r) \subset t\,F(r) = F(t\,r)$$

for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$ . That is, the homogenity property 2.1 (2) also holds.

## 3. The Minkowski functionals of seminorm generating relations

**Definition 3.1.** If $F$ is a seminorm generating relation for $X$ , then the function $p_F$ defined by

$$p_F(x) = \inf\big(F^{-1}(x)\big)$$

for all $x \in X$ , will be called the Minkowski functional or gauge of $F$ .

**Example 3.2.** If $A$ is an absorbing, balanced, convex subset of $X$ , then we can at once see that

$$p_{F_A}(x) = \inf\big\{\, r \in \mathbb{R}_+ : \quad x \in t\,A \,\big\}$$

for all $x \in X$ . Therefore, $p_A = p_{F_A}$ is just the usual Minkowski functional of $A$ . (See, [1, p. 24].)

Therefore, it is not surprising that, as a useful reformulation of a well-known theorem on the Minkowski functionals of sets, we have the following

**Theorem 3.3.** *If $F$ is a seminorm generating relation for $X$, then $p_{_F}$ is a seminorm on $X$ such that*

$$F_{p_{_F}} \subset F \subset \bar{F}_{p_{_F}} .$$

*Proof.* If $\lambda \in \mathbb{K}$ and $x \in X$, then by the definition of $p_{_F}$ for each $\varepsilon > 0$ there exists an $r \in F^{-1}(x)$ such that $r < p_{_F}(x) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and using the homogenity property 2.1 (2) of $F$, we can infer that

$$\lambda x \in \lambda F(r) \subset F(t r),$$

and thus $t r \in F^{-1}(\lambda x)$ for all $t \in \mathbb{R}_+$ with $|\lambda| \leq t$. Hence, since $t r < t p_{_F}(x) + t \varepsilon$, it is clear that

$$p_{_F}(\lambda x) = \inf\big(F^{-1}(\lambda x)\big) < t p_{_F}(x) + t \varepsilon$$

for all $t \in \mathbb{R}_+$ with $|\lambda| \leq t$. Hence, by letting $t \to |\lambda|$ and $\varepsilon \to 0$, we can infer that

$$p_{_F}(\lambda x) \leq |\lambda| p_{_F}(x).$$

On the other hand, if $x, y \in X$, then again by the definition of $p_{_F}$ for each $\varepsilon > 0$ there exist $r \in F^{-1}(x)$ and $s \in F^{-1}(y)$ such that $r < p_{_F}(x) + \varepsilon$ and $s < p_{_F}(y) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and $y \in F(s)$, and using the additivity property 2.1 (1) of $F$, we can infer that

$$x + y \in F(r) + F(s) \subset F(r + s),$$

and thus $r + s \in F^{-1}(x + y)$. Hence, since $r + s < p_{_F}(x) + p_{_F}(y) + 2\varepsilon$, it is clear that

$$p_{_F}(x + y) = \inf\big(F^{-1}(x + y)\big) < p_{_F}(x) + p_{_F}(y) + 2\varepsilon,$$

and thus

$$p_{_F}(x + y) \leq p_{_F}(x) + p_{_F}(y).$$

Therefore, $p_{_F}$ is a seminorm on $X$.

Finally, if $r \in \mathbb{R}_+$ and $x \in F_{p_{_F}}(r) = B_r^{p_F}(0)$, i. e., $p_{_F}(x) < r$, then again by the definition of $p_{_F}$ there exists $s \in F^{-1}(x)$ such that $s < r$. Hence, by the monotonicity property 2.4 (3) of $F$, it is clear that $x \in F(s) \subset F(r)$. Therefore, $F_{p_{_F}}(r) \subset F(r)$.

On the other hand, if $r \in \mathbb{R}_+$ and $x \in F(r)$, then $r \in F^{-1}(x)$. Therefore, by the definition of $p_{_F}$, we have $p_{_F}(x) \leq r$, and hence $x \in \bar{B}_r^{p_F}(0) = \bar{F}_{p_{_F}}(r)$. Therefore, $F(r) \subset \bar{F}_{p_{_F}}(r)$ is also true.

Now, as an immediate consequence of Example 2.2 and Theorem 3.3, we can also state the following more familiar

**Corollary 3.4.** *If $A$ is an absorbing, balanced, convex subset of $X$, then $p_A$ is a seminorm on $X$ such that*

$$B_1^{p_A}(0) \subset A \subset \bar{B}_1^{p_A}(0).$$

In addition, to Theorem 3.3, it is also worth proving the following

**Theorem 3.5.** *If $p$ is a seminorm on $X$ and $F$ is a seminorm generating relation for $X$ such that*

$$F_p \subset F \subset \bar{F}_p,$$

*then $p = p_F$.*

*Proof.* If $x \in X$, then for each $r \in \mathbb{R}_+$, with $p(x) < r$, we have

$$x \in B_r^p(0) = F_p(r) \subset F(r).$$

Therefore, $r \in F^{-1}(x)$, and thus

$$p_F(x) = \inf\big(F^{-1}(x)\big) \leq r.$$

Hence, by letting $r \to p(x)$, we can infer that $p_F(x) \leq p(x)$.

On the other hand, by the definition of $p_F(x)$, for each $\varepsilon > 0$ there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, we can see that

$$x \in F(r) \subset \bar{F}_p(r) = \bar{B}_r^p(0).$$

Therefore, $p(x) \leq r < p_F(x) + \varepsilon$, and thus $p(x) \leq p_F(x)$ is also true.

**Remark 3.6.** In particular, by Theorem 3.5, we have $p = p_{F_p} = p_{\bar{F}_p}$ for every seminorm $p$ on $X$.

Moreover, as an immediate consequence of Example 2.3 and Theorem 3.5, we can also state

**Corollary 3.7.** *If $p$ is a seminorm on $X$ and $A$ is an absorbing, balanced, convex subset of $X$ such that*

$$B_1^p(0) \subset A \subset \bar{B}_1^p(0),$$

*then $p = p_A$.*

## 4. Some further properties of the Minkowski functionals

**Theorem 4.1.** *If $F$ is a seminorm generating relation for $X$, then the following assertions are equivalent:*

*(1)* $p_F$ *is a norm;* *(2)* $\bigcap_{r \in \mathbb{R}_+} F(r) = \{0\}$.

*Proof.* If $x \in F(r)$, and hence $r \in F^{-1}(x)$ for all $r \in \mathbb{R}_+$, then by the definition of $p_F$ we have $p_F(x) = 0$. Hence, if the assertion (1) holds, it follows that $x = 0$. Therefore, since $0 \in F(r)$ for all $r \in \mathbb{R}_+$, the assertion (2) also holds.

While, if $x \in X$ such that $p_F(x) = 0$, then by the definition of $p_F$ for each $r \in \mathbb{R}_+$ there exists $s \in F^{-1}(x)$ such that $s < r$. Hence, by the monotonocity property of $F$, it is clear that $x \in F(s) \subset F(r)$. Therefore, if the assertion (2) holds, then $x = 0$, and thus the assertion (1) also holds.

**Corollary 4.2.** *If $A$ is an absorbing, balanced, convex subset of $X$, then the following assertions are equivalent:*

*(1)* $p_A$ *is a norm;* *(2)* $\bigcap_{r \in \mathbb{R}_+} r\,A = \{\,0\,\}$.

**Remark 4.3.** Note that, since $A$ is balanced, we may write $\mathbb{K} \setminus \{0\}$ in place of $\mathbb{R}_+$ in the assertion (2).

**Theorem 4.4.** *If $F$ is a seminorm generating relation for $X$, then the following assertions are equivalent:*

*(1)* $F = F_{p_F}$; *(2)* $F(r) = \bigcup_{s<r} F(s)$ *for all* $r \in \mathbb{R}_+$.

*Proof.* If $r \in \mathbb{R}_+$ and $x \in F(r)$, and the assertion (1) holds, then we have $x \in F_{p_F}(r) = B_r^{p_F}(0)$. Hence, it follows that

$$\inf\big(F^{-1}(x)\big) = p_F(x) < r\,.$$

Therefore, there exists an $s \in F^{-1}(x)$ such that $s < r$. Hence, it follows that $x \in F(s)$. Therefore,

$$F(r) \subset \bigcup_{s<r} F(s)\,.$$

Now, since the converse inlusion is immediate from the monotonicity property of $F$, it is clear that the assertion (2) also holds.

While, if $r \in \mathbb{R}_+$ and $x \in F(x)$, and the assertion (2) holds, then there exists an $s < r$ such that $x \in F(s)$, and hence $s \in F^{-1}(x)$. Therefore,

$$p_F(x) = \inf\big(F^{-1}(x)\big) \le s < r\,,$$

and hence $x \in B_r^{p_F}(0) = F_{p_F}(r)$. Consequently, we have $F(r) \subset F_{p_F}(r)$. Now, since the converse inclusion is always true by Theorem 3.3, it is clear that the assertion (1) also holds.

**Corollary 4.5.** *If $A$ is an absorbing, balanced, convex subset of $X$, then the following assertions are equivalent:*

*(1)* $A = B_1^{p_A}(0)$; *(2)* $A = \bigcup_{s<1} s\,A$.

**Theorem 4.6.** *If $F$ is a seminorm generating relation for $X$, then the following assertions are equivalent:*

*(1)* $F = \bar{F}_{p_F}$; *(2)* $F(r) = \bigcap_{s>r} F(s)$ *for all* $r \in \mathbb{R}_+$.

*Proof.* If $r \in \mathbb{R}_+$, and $x \in X$ such that $x \in F(s)$, i.e., $s \in F^{-1}(x)$ for all $s > r$, then

$$p_F(x) = \inf\big(F^{-1}(x)\big) \le r\,,$$

and hence $x \in \bar{B}_r^{p_F}(0) = \bar{F}_{p_F}(r)$. Therefore, if the assertion (1) holds, then we also have $x \in F(r)$. Consequently,

$$\bigcap_{s>r} F(s) \subset F(r).$$

Hence, since the converse inclusion is immediate from the monotonicity property of $F$, it is clear that the assertion (2) also holds.

While, if $r \in \mathbb{R}_+$, and $x \in \bar{F}_{p_F}(r)$, i.e., $x \in \bar{B}_r^{p_F}(0)$, then

$$\inf\big(F^{-1}(x)\big) = p_F(x) \leq r.$$

Therefore, for each $s > r$ there exists a $t \in F^{-1}(x)$ such that $t < s$. Hence, by the monotonicity property of $F$, it is clear that $x \in F(t) \subset F(s)$. Therefore,

$$\bar{F}_{p_F}(r) \subset \bigcap_{s>r} F(s).$$

Hence, if the assertion (2) holds, it follows that $\bar{F}_{p_F}(r) \subset F(r)$. Now, since the converse inclusion is always true by Theorem 3.3, it is clear that the assertion (1) also holds.

**Corollary 4.7.** *If $A$ is an absorbing, balanced, convex subset of $X$, then the following assertions are equivalent:*

*(1)* $A = \bar{B}_1^{p_A}(0)$; *(2)* $A = \bigcap_{s>1} sA$.

**Remark 4.8.** The above corollaries are not established in the standard books on functional analysis.

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